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If you want to run, run a mile. If you want to experience
a different life, run a marathon.
Émil Zatopek[1].

[1]Famous runner of twentieth century

Dedication

This thesis is dedicated to

My parents, whose sincerely raised me with their caring and offered me unconditional love, a very special thank for the myriad of ways in which, throughout my life, you have actively supported me in my determination to find and realize my potential.

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Chapter 1

Introduction

Control theory is an interdisciplinary branch of engineering and mathematics that deals with influence behavior of dynamical systems. Controllability is one of the fundamental concepts in mathematical control theory. This is a qualitative property of dynamical control systems and it is of particular importance in control theory. Systematic study of controllability was started at the beginning of sixties in the last century, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out.

Roughly speaking, controllability generally means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. It should be mentioned, that in the literature there are many different definitions of controllability, which strongly depend on one hand on a class of dynamical control systems and on the other hand on the form of admissible controls.

In recent years various controllability problems for different types of linear semilinear and nonlinear dynamical systems have been considered in many publications and monographs. Moreover, it should be stressed, that the most literature in this direction has been mainly concerned with different controllability problems for dynamical systems with unconstrained controls and without delays in the state variables or in the controls.

The main purpose of the chapter is to present without mathematical proofs a review of recent controllability problems for a wide class of dynamical systems. Moreover, it should be pointed out, that exact mathematical descriptions of controllability criteria can be found for example in the following publications [49, 46]

Controllability significance

Controllability plays an essential role in the development of modern mathematical control theory. There are various important relationships between controllability, stability and stabilizability of linear both finite-dimensional and infinitedimensional control systems. Controllability is also strongly related with the theory of realization and so called minimal realization and canonical forms for linear time-invariant control systems such as the Kalmam canonical form, the Jordan canonical form or the Luenberger canonical form. It should be mentioned, that for many dynamical systems there exists a formal duality between the concepts of controllability and observability. Moreover, controllability is strongly connected with the minimum energy control problem for many classes of linear finite dimensional, infinite dimensional dynamical systems, and delayed systems both deterministic and stochastic.

Therefore, controllability criteria are useful in the following branches of mathematical control theory:

• stabilizability conditions, canonical forms, minimum energy control and minimal realization for positive systems,

•stabilizability conditions, canonical forms, minimum energy control and minimal realization for fractional systems,

•minimum energy control problem for a wide class of stochastic systems with delays in control and state variables,

• duality theorems, canonical forms and minimum energy control for infinite dimensional systems,

• controllability, duality, stabilizability, mathematical modeling and optimal control of quantum systems.

Controllability has many important applications not only in control theory and systems theory, but also in such areas as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospce engineering and recently in quantum systems theory.

Systematic study of controllability was started at the beginning of the sixties in the 20-th century, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out. The extensive list of these publications can be found for example in the monographs [43] and [42] or in the survey papers [44] and [45].

During last few years quantum dynamical systems have been discussed in many publications. This fact is motivated by possible applications in the theory of quantum informatics [27, 95]. Quantum control systems are either defined in finite-dimensional complex space or in the space of linear operators over finite-dimensional complex space. In the first case the quantum states are called state vectors and in the second density operators.

Control system description of a quantum closed system is described by bilinear ordinary differential state equation in the form of Schrödinger equation for state vectors and Liouville [18, 24] equation for density matrices. Therefore, controllability investigations require using special mathematical methods as Lie groups and Lie algebras.

Traditional controllability concept can be extended for so called structural controllability, which may be more reasonable in case of uncertainties [43, 42]. It should be pointed out, that in practice most of system parameter values are difficult to identify and are known only to certain approximations. Thus structural controllability, which is independent of a specific value of unknown parameters are of particular interest. Roughly speaking, linear system is said to be structurally controllable if one can find a set of values for the free parameters such that the corresponding system is controllable in the standard sense [43, 42]. Structural controllability of linear control system is strongly related to numerical computations of distance from a given controllable switched linear control system to the nearest an uncontrollable one [43, 42]. First of all let us observe, that from algebraic characterization of controllability and structural controllability immediately follows that controllability is a generic property in the space of matrices defining such systems [43, 42]. Therefore, the set of controllable switched systems is an open and dense subset. Hence, it is important to know how far a controllable linear system is from the nearest uncontrollable linear system. This is especially important for linear systems with matrices whose coefficients are given with some parameter uncertainty. An explicit bound for the distance between a controllable linear control system to the closed set of uncontrollable switched linear control system can be obtained using special norm defined for the set of matrices and singular value decomposition for controllability matrix [43, 42].

Nonlinear and semilinear dynamical systems

The last decades have seen a continually growing interest in controllability theory of dynamical systems. This is clearly related to the wide variety of theoretical results and possible applications. Up to the present time the problem of controllability for continuous-time and discrete-time linear dynamical systems has been extensively investigated in many papers (see e.g. [43, 42, 44, 93] for extensive list of references). However, this is not true for the nonlinear dynamical systems especially with different types of delays in control and state variables, and for nonlinear dynamical systems with constrained controls. Similarly, only a few papers concern constrained controllability problems for continuous or discrete semi-linear dynamical systems. It should be pointed out, that in the proofs of controllability results for nonlinear and semi-linear dynamical systems linearization methods and generalization of open mapping theorem [8, 76] are extensively used. The special case of nonlinear dynamical systems are semi-linear systems. Let us recall that semi-linear dynamical control systems contain linear and pure nonlinear parts in the differential state equations [38, 8, 73, 87].

Infinite-dimensional systems

Infinite-dimensional dynamical control systems plays a very important role in mathematical control theory. This class consists of both continuous-time systems and discrete-time systems [43, 42, 44, 45, 93]. Continuous-time infinite-dimensional systems include for example, a very wide class of so-called distributed parameter systems described by numerous types of partial differential equations defined in bounded or unbounded regions and with different boundary conditions.

For infinite-dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability [43, 42]. It follows directly from the fact that in infinite-dimensional spaces there exist linear subspaces which are not closed. On the other hand, for nonlinear dynamical systems there exist two fundamental concepts of controllability; namely local controllability and global controllability [43, 42]. Therefore, for nonlinear abstract dynamical systems defined in infinite-dimensional spaces the following four main kinds of controllability are considered: local approximate controllability, global approximate controllability, local exact controllability and global exact controllability [43, 42, 44, 45].

Controllability problems for finite-dimensional nonlinear dynamical systems and stochastic dynamical systems have been considered in many publications; see e.g. [43, 42, 45, 57], and [58], for review of the literature. However, there exist only a few papers on controllability problems for infinitedimensional nonlinear systems [73, 29].

Among the fundamental theoretical results, used in the proofs of the main results for nonlinear or semi-linear dynamical systems, the most important include:

•generalized open mapping theorem,

•spectral theory of linear unbounded operators,

• linear semi-groups theory for bounded linear operators,

•Lie algebras and Lie groups,

•fixed-point theorems such as Banach, Schauder, Schaefer and Nussbaum theorems,

• theory of completely positive trace preserving maps,

•mild solutions of abstract differential and evolution equations in Hilbert and Banach spaces.

Nonlinear neutral impulsive integrodifferential evolution systems in Banach spaces.

In various fields of science and engineering, many problems that are related to linear viscoelasticity, nonlinear elasticity and Newtonian or non- Newtonian fluid mechanics have mathematical models which are described by differential or integral equations or integrodifferential equations. This part of the paper centers around the controllability for dynamical systems described by the integrodifferential models. Such systems are modelled by abstract delay differential equations. In particular abstract neutral differential equations arise in many areas of applied mathematics and, for this reason, this type of equation has been receiving much attention in recent years and they depend on the delays of state and their derivative. Related works of this kind can be found in [72, 77].

The study of differential equations with traditional initial value problem has been extended in several directions. One emerging direction is to consider the impulsive initial conditions. The impulsive initial conditions are combinations of traditional initial value problems and short-term perturbations, whose duration can be negligible in comparison with the duration of the process. Several authors [72, 77] have investigated controllability of the impulsive differential equations.

As far as the controllability problems associated with finite-dimensional systems modelled by ordinary differential equations are concerned, this theory has been extensively studied during the last decades. In the finite-dimensional context, a system is controllable if and only if the algebraic Kalman rank condition is satisfied. According to this property, when a system is controllable for some time, it is controllable for all time. But this is no longer true in the context of infinite-dimensional systems modelled by partial differential equations.

The large class of scientific and engineering problems modelled by partial differential and integrodifferential equations can be expressed in various forms of differential and integrodifferential equations in abstract spaces. It is interesting to study the controllability problem for such models in Banach spaces. The controllability problem for first and second order nonlinear functional differential and integrodifferential systems in Banach spaces has been studied by many authors by using semigroup theory, cosine family of operators and various fixed point theorems for nonlinear operators [73] and [87] such as Banach theorem, Nussbaum theorem, Schaefer theorem, Schauder theorem, Monch theorem or Sadovski theorem.

In recent years, the theory of impulsive differential equations has provided a natural frame work for mathematical modelling of many real world phenomena, namely in control, biological and medical domains. In these models, the investigated simulating processes and phenomena are subjected to certain perturbations whose duration is negligible in comparison with the total duration of the process. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. These process tend to be more suitably modelled by impulsive differential equations, which allow for discontinuities in the evolution of the state.

On the other hand, the concept of controllability is of great importance in mathematical control theory. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where the initial and final states may vary over the entire space. Many authors have studied the controllability of nonlinear systems with and without impulses, see for instance [52, 70, 87].

In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [7, 72, 59] and the nonlocal initial condition which in many cases, has much better effect in applications then the traditional initial condition. The nonlocal initial value problems can be more useful than the standard initial value problems to describe many physical phenomena of dynamical systems. It should be pointed out, that the study of Volterra-Fredholm integrodifferential equations plays an important role for abstract formulation of many initial, boundary value problems of perturbed differential partial integro-differential equations.

Recently, many authors have studied about mixed type integrodifferential systems without (or with) delay conditions. Moreover, controllability of impulsive functional differential systems with nonlocal conditions has been studied by using the measures of noncompactness and Monch fixed point theorem and some sufficient conditions for controllability have been established.
It should be mentioned, that without assuming the compactness of the evolution system the existence, uniqueness and continuous dependence of mild solutions for nonlinear mixed type integrodifferential equations with finite delay and nonlocal conditions has been also established.. The results were obtained by using Banach fixed point theorem and semi-group theory. More recently, the existence of mild solutions for the nonlinear mixed type integro-differential functional evolution equations with nonlocal conditions was derived and the results were achieved by using Monch fixed point theorem and fixed point theory.
To the best of our knowledge, up to now no work reported on controllability of impulsive mixed Volterra- Fredholm functional integrodifferential evolution differential system with a finite delay and nonlocal conditions.

Stochastic systems

Classical control theory generally is based on deterministic approaches. However, uncertainty is a fundamental characteristic of many real dynamical systems. Theory of stochastic dynamical systems is now a well-established topic of research, which is still in intensive development and offers many open problems. Important fields of application are economics problems, decision problems, statistical physics, epidemiology, insurance mathematics, reliability theory, risk theory and others methods based on stochastic equations. Stochastic modelling has been widely used to model the phenomena arising in many branches of science and industry such as biology, economics, mechanics, electronics and telecommunications. The inclusion of random effects in differential equations leads to several distinct classes of stochastic equations, for which the solution processes have differentiable or non-differentiable sample paths. Therefore, stochastic differential equations and their controllability require many different method of analysis.

The general theory of stochastic differential equations both finite-dimensional and infinite-dimensional and their applications to the field of physics and technique can be found in the many mathematical monographs and related papers. This theory formed a very active research topic since provides a natural framework for mathematical modelling of many physical phenomena.

Controllability, both for linear or nonlinear stochastic dynamical systems, has recently received the attention of many researchers and has been discussed in several papers and monographs, in which where many different sufficient or necessary and sufficient conditions for stochastic controllability were formulated and proved [56, 57, 58, 66, 67]. However, it should be pointed out that all these results were obtained only for unconstrained admissible controls, finite dimensional state space and without delays in state or control.

Stochastic controllability problems for stochastic infinitedimensional semi-linear impulsive integrodifferential dynamical systems with additive noise and with or without multiple time-varying point delays in the state variables are also discussed in the literature. The proofs of the main results are based on certain theorems taken from the theory of stochastic processes, linearization methods for stochastic dynamical systems, theory of semi-groups of linear operators, different fixed-point theorems as Banach, Schauder, Schaefer, or Nussbaum fixed-point theorems and on so-called generalized open mapping theorem presented and proved in the survey paper [59, 67].

Delayed systems

Up to the present time the problem of controllability in continuous and discrete time linear dynamical systems has been extensively investigated in many papers (see e.g. [43, 42, 44, 41, 56, 40, 33]). However, this is not true for the nonlinear or semi-linear dynamical systems, especially with delays in control and with constrained controls. Only a few papers concern constrained controllability problems for continuous or discrete nonlinear or semi-linear dynamical systems with constrained controls [40, 39].

Dynamical systems with distributed [52] delays in control and state variable were also considered. Using some mapping theorems taken from functional analysis and linear approximation methods sufficient conditions for constrained relative and absolute controllability will be derived and proved.

Let us recall that semi-linear dynamical control systems with delays may contain different types

of delays, both in pure linear and pure nonlinear parts, in the differential state equations. Sufficient conditions for constrained local relative controllability near the origin in a prescribed finite time interval for semi-linear dynamical systems with multiple variable point delays or distributed delays in the control and in the state variables, which nonlinear term is continuously differentiable near the origin are presented in [40] and [39].
In the above mentioned papers it is generally assumed that the values of admissible controls are in a given convex and closed cone with vertex at zero, or in a cone with nonempty interior. The proof of the main result are based on a so called generalized open mapping theorem presented in the paper [76]. Moreover, necessary and sufficient conditions for constrained global relative controllability of an associated linear dynamical system with multiple point delays in control are also discussed.

Positive systems

In recent years, the theory of positive dynamical systems has become a natural frame work for mathematical modelling of many real world phenomena, namely in control, biological and medical domains. Positive dynamical systems are of fundamental importance to numerous applications in different areas of science such as economics, biology, sociology and communication. Positive dynamical systems both linear and nonlinear are dynamical systems with states, controls and outputs belonging to positive cones in linear spaces. Therefore, in fact positive dynamical systems are nonlinear systems. Among many important developments in control theory over last two decades, control theory of positive dynamical systems [33] has played an essential role.

Controllability, reachability and realization problems for finite dimensional positive both continuous-time and discretetime dynamical systems were discussed for example in monograph [33] and paper [50], using the results taken directly from the nonlinear functional analysis and especially from the theory of semi-groups of bounded operators and general theory of unbounded linear operators.

Fractional systems

The development of controllability theory both for continuous- time and discrete-time dynamical systems with fractional derivatives and fractional difference operators has seen considerable advances since the publication of papers [34, 35] and monograph [71]. Although classic mathematical models are still very useful, large dynamical systems prompt the search for more refined mathematical models, which leads to better understanding and approximations of real processes.

The general theory of fractional differential equations and fractional impulsive integrodifferential equations and their applications to the field of physics and technique can be found in the monograph [71]. This theory formed a very active research topic since provides a natural framework for mathematical modelling of many physical phenomena. In particular, the fast development of this theory has allowed to solve a wide range of problems in mathematical modelling and simulation of certain kinds of dynamical systems in physics and electronics. Fractional derivative techniques provide useful exploratory tools, including the suggestion of new mathematical models and the validation of existing ones.

Mathematical fundamentals of fractional calculus and fractional differential and difference equations are given in the monographs [71], and in the related papers [33, 35]. Most of the earliest work on controllability for fractional dynamical systems was related to linear continuous-time or discretetime systems with limited applications of the real dynamical systems. In addition, the earliest theoretical work concerned time-invariant processes without delays in state variables or in control.

Using the results presented for linear fractional systems and applying linearization method the sufficient conditions for local controllability near the origin are formulated and proved in the paper [50]. Moreover, applying generalized open mapping theorem in Banach spaces [76] and linear semi-group theory in the paper [93] the sufficient conditions for approximation controllability in finite time with conically constrained admissible controls are formulated and proved.

Quantum dynamical systems

Fast recent development of quantum information field in both theory and experiments caused increased interest in new methods of quantum systems control. Various models for open-loop and closed-loop control scenarios for quantum systems have been developed in recent years [27, 24].

Quantum systems can be classified according to their interaction with the environment. If a quantum system exchange neither information nor energy with its environment it is called closed and its time evolution is described completely by a Hamiltonian and its respective unitary operator. On the other hand if the exchange of information or energy occurs, the system is called open.

Due to the destructive nature of quantum measurement in some models one has to be constrained to open-loop control of a quantum system. This fact means that during the time evolution of the quantum system it is physically impossible to extract any information about the state of the system.

In the simplest case open-loop control of the closed quantum system is described by the bi-linear model. In this case the differential equation of the evolution is described by the sum of the drift Hamiltonian and the control Hamiltonians. The parameters of the control Hamiltonians may be constrained in various ways due to physical constraints of the system. Many quantum systems can be only controlled locally, which means that control Hamiltonians act only on one of the Hilbert spaces that constitute larger tensor product Hilbert space of the system.

The control constrained to local operations is of a great interest in various applications, especially in quantum computation and spin graph systems. Other possible constraints, such as constrained energy or constrained frequency, are possible. They are very important in the scope of optimal control of quantum systems.

In the most generic case open quantum systems are not controllable with coherent, unitary control due to the fact that the action of the generic completely positive trace preserving map cannot be reversed unitarily. For example Markovian dynamics of finite-dimensional open quantum system is not coherently controllable. However, many schemes of incoherent control of open quantum systems have been described. Some of these schemes are based on the technique known as quantum error correcting codes. In incoherent control schemes quantum unitary evolution together with quantum measurements is used to drive the system to the desired state even if quantum noise is present in the system.

Controllability problems for different types of dynamical systems require the application of numerous mathematical concepts and methods taken directly from differential geometry, functional analysis, topology, matrix analysis and theory of ordinary and partial differential

equations and theory of difference equations. The state-space models of dynamical systems provides a robust and universal method for studying controllability of various classes of systems.
Finally, it should be stressed, that there are numerous open problems for controllability concepts for special types of dynamical systems. For example, it should be pointed out, that up to present time the most literature on controllability problems has been mainly concerned with unconstrained controls and without delays in the state variables or in the controls.

In this thesis in the second chapter we recall some definitions and properties of the theory of stochastic calculus the third chapter is concerning about some controllability conditions via one of the fixed point methods, namely the contraction mapping principle. In this method, assuming controllability of the associated linear system under some natural conditions, we proved the controllability of semi-linear stochastic system, and we used the generalized implicit function theorem to show the local null controllability of non-linear stochastic system.
In the fourth chapter we talk about the controllability of stochastic dynamical system in general case, and we study the controllability of fractional stochastic dynamical systems with delays in control, this chapter is divide in three sections in the first section we define the Reimann-Liouville fractional operators, Caputo fractional derivative, Mittag-Leffler function..., in the second section we study a relative controllability of fractional stochastic dynamical systems with multiple delays in control, in the third section we study a global relative controllability of fractional stochastic dynamical systems with distributed delays in control.
The last chapter is about controllability of fractional stochastic dynamical systems without delays in control, we study the relative controllability of semilinear fractional stochastic control systems in Hilbert spaces.

Chapter 2

Preliminary Background

2.1 Basic Definitions

In this section the basic notations of the theory of stochastic calculus are considered. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_s\}$ satisfying the *usual conditions*:

- $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t$ for all $s \geq 0$;
- All $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ are contained in $\mathcal{F}_t$.

A family $(X(t), t \geq 0)$ of $\mathbb{R}^d$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *stochastic process*, this process is adapted if all $X(t)$ are $\mathcal{F}_t$-measurable. Denoting $\mathcal{B}$, the Borel $\sigma-$field on $[0, \infty)$. The process X is measurable if $(t, \omega) \mapsto X(t, \omega)$ is a $\mathcal{B} \bigotimes \mathcal{F}-$measurable mapping. We say that $(X(t), t \geq 0)$ is *continuous* if the *trajectories* $t \mapsto X(t, \omega)$ are continuous for all $\omega \in \Omega$.

2.2 Brownian Motion

2.2.1 Definition and Properties

Definition 2.2.1. *A stochastic process* $(W_t)_{t \in \mathbb{R}_+}$ *is called a standard Brownian motion if it satisfies the following conditions:*

1. $\mathbb{P}[W_t(\omega) = 0] = 1$, *for all* $\omega \in \Omega$,

2. Independent increments. *For each* $0 \leq t_1 < t_2 < \ldots < t_m$, *the real valued*

$$W(t_1), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1}),$$

are independent.

3. Stationary increments. *For each* $0 \leq s < t, W(t) - W(s)$ *is a centered real valued normally distributed with variance* $(t-s)$, *i.e.*,

$$W(t) - W(s) \sim \mathcal{N}(0, t-s).$$

4. $\mathbb{P}(\omega \in \Omega, t \to W_t(\omega) \quad \text{is} \quad \text{continuous}) = 1.$

Remark 2.2.1. 1. *Notice that the natural filtration of the Brownian motion is* $\mathcal{F}_t^W = \sigma(W_s, s \leq t)$.

2. *We can define the Brownian motion without the last condition of continuous paths, because with a stochastic process satisfying the second and the third conditions, by applying the Kolmogorov's continuity theorem, there exists a modification of* $(W_t)_{t \in \mathbb{R}_+}$ *which has continuous paths a.s.*

3. *A Brownian motion is also called a Wiener process since, it is the canoncial process defined on the Wiener space.*

Proposition 2.2.1. *The Brownian motion* $(W_t)_{t \in \mathbb{R}_+}$ *is a Gaussian process with mean* 0 *and covariance function* $Cov(W_t, W_s) = s \wedge t$.

Proof. We have that $W_t = W_t - W_0$. Thus $W_t \sim \mathcal{N}(0, t)$ by definition. Moreover, without loss of generality, we assume $s < t$. Hence, we have

$$E(W_s W_t) = E(W_s(W_t - W_s) + W_s^2) = 0 + s = s. \qquad \square$$

Note that since the Brownian motion is a continuous Gaussian process, the proposition 2.2.1

characterizes uniquely the Brownian motion.

We will give here some properties of the standard Brownian motion.

Properties 2.2.1.1. *Let $W(t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion*

1. *Self-similarity. For any $T > 0$, $\{T^{-1/2}W(Tt)\}$ is Brownian motion.*
2. *Symmetry. $\{-W(t), t \geq 0\}$ is also a Brownian motion.*
3. *$\{tW(1/t), t > 0\}$ is also a Brownian motion.*
4. *If $W(t)$ is a Brownian motion on $[0,1]$, then $(t+1)W(1/t+1) - W(1)$ is a Brownian motion on $[0,\infty)$.*

2.2.2 Quadratic variation and Brownian motion

Proposition 2.2.2. *Let $W(t)_{t\in\mathbb{R}+}$ be a Brownian motion. For $t \geq 0$, for all sequence of subdivisions $\Delta_n[0,t]$, such that $\lim_{n\to\infty} |\Delta_n[0,t]| = 0$ we have*

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(W_{\frac{it}{2^n}} - W_{\frac{(i-1)t}{2^n}} \right)^2 = t, \qquad p.s.$$

2.2.3 Brownian paths

Proposition 2.2.3. *A Brownian motion has its paths a.s., locally γ-Hölder continuous for $\gamma \in [0, 1/2)$.*

Proposition 2.2.4. *The Brownian motion's sample paths are a.s., nowhere differentiable.*

2.2.4 Brownian motion and martingales

As a stochastic process, we could ask, knowing all well properties of martingales, if the brownian motion is one.

Proposition 2.2.5. *Let $(W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion. Then the following processes are $(\mathcal{F}_t^W)$-martingales:*

1. $W(t)$,

2. $W^2(t) - t$,

3. For any u, $e^{uW(t)-\frac{u^2}{2}t}$.

2.3 Stochastic Integration with respect to Brownian Motion

This section is devoted to the study of an integration where the integrator is a Brownian motion. In fact, we would like to define

$$\int_{\mathbf{T}} f_s dW_s, \tag{2.1}$$

where f_s is a stochastic process.

2.3.1 Wiener integral

The Wiener integral is an integral where we have deterministic integrands and a Brownian motion (or more generally a Gaussian process) as an intergrator. First of all, we will define it for step functions.

Integrands as step functions

Let us denote by $\mathcal{E}$ the set of step functions. For $f \in \mathcal{E}$, i.e., $f = \sum_{i=1}^{n} f_{i-1}\mathbf{1}_{(t_{i-1},t_i]}$, where $t_0 = a$ and $t_n = b$, we define the Wiener Integral as follows

$$I(f) = \int_a^b f(t)dW_t := \sum_{i=1}^{n} f_{i-1}(W_{t_i} - W_{t_{i-1}}). \tag{2.2}$$

Proposition 2.3.1. *For $f \in \mathcal{E}$, we have that $I(f)$ is a gaussian random variable with mean zero and variance*

$$\mathbb{E}(I(f)^2) = \int_a^b f(t)^2 dt. \tag{2.3}$$

Integrands as square integrable function

Let $f \in L^2([a,b])$ and $(f_n)_{n\in N} \in \mathcal{E}$ such that $f_n \to f$ in $L^2([a,b])$. By the Proposition 2.3.1, $I(f_n)_{n\in N}$ is a Cauchy sequence in $L^2(\Omega)$. Because it is a Hilbert space, it is complete and thus $I(f_n)_{n\in N}$ converges in $L^2(\Omega)$. Therefore, let us define the Wiener Integral as the L^2-limit of the sequence $I(f_n)_{n\in N}$, i.e.,

$$I(f) := \lim_{n\to\infty} I(f_n), \quad \text{in} \quad L^2(\Omega). \tag{2.4}$$

Definition 2.3.1. *For $f \in L^2([a,b])$, we define the Wiener integral of f by*

$$I(f) := \int_a^b f(t)dW_t := \lim_{n\to\infty} I(f_n) := \lim_{n\to\infty} f_n(t)dW_t. \tag{2.5}$$

2.3.2 Itô integral

Here we will study the simplest stochastic integral, where the integrand and the integrator are random variable. The first who defined this integral was K. Itô in 1944 . Therefore we named this integral after him. In fact, the integrand will be an adapted stochastic process w.r.t the natural filtration of the Brownian motion. Let us start with the simplest case of random integrands.

Integrands as stochastic step processes

Let us denote by $\mathcal{E}$ the set of simples $(\mathcal{F}_t)$-predictables processes $(H_t)_{t\in\mathbb{R}_+}$, i.e.,

$$H_t(\omega) = \sum_{i=1}^n h_i(\omega)\mathbf{1}(t_{i-1}, t_i](t), \quad t \in T.$$

with $0 \le t_0 \le t_1 \le \ldots \le t_n$ and h_i a $(\mathcal{F}_{t_{i-1}})$-measurable random variable which belongs to $L^2(\Omega)$. Then we can define the integral for $H \in \mathcal{E}$ w.r.t a Brownian motion by

$$I(H) = (H.W)_s = \int_T H_s dW_s := \begin{cases} \sum_{i=1}^n h_i(W_{t_i} - W_{t_{i-1}}) & if \quad \mathbf{T} = \mathbb{R}_+, \\ \sum_{i=1}^n h_i(W_{t_i\wedge T} - W_{t_{i-1}\wedge T}) & if \quad \mathbf{T} = [0,T]. \end{cases} \tag{2.6}$$

Clearly, if our integrand, namely H, is a constant, in the sense it is not a random variable, then we come back to the above definition of the Wiener integral.

Integrands as square integrable stochastic processes

The idea is to extend, by density of $\mathcal{E}$ in $L^2(\Omega)$, the definition of $I(H)$ in (2.6) to larger processes, i.e., processes in $L^2(\Omega)$ as the limit of processes in $\mathcal{E}$, like we did for the Wiener integral. Indeed, by density, we have for each $(H_t)_{t\in\mathbb{R}_+} \in L^2(\Omega)$ there exists a sequence $((H_{t,n})_{t\in\mathbb{R}_+} \in L^2(\Omega))_{n\in\mathbb{N}} \in \mathcal{E}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} \mathbb{E}(|H_t - H_{t,n}|)dt = 0.$$

However, our integrands, as the L^2 limit of processes in $\mathcal{E}$, must satisfy certain constraints to be well-defined. Therefore, we will take as the space of integrands $L^2_{ad}(\Omega \times \mathbf{T}, (\mathcal{F}_t)_{t\in\mathbf{T}})$. Obviously we have $\mathcal{E} \in L^2_{ad}(\Omega \times \mathbf{T}, (\mathcal{F}_t)_{t\in\mathbf{T}})$ and $\overline{\mathcal{E}} = L^2_{ad}(\Omega \times \mathbf{T}, (\mathcal{F}_t)_{t\in\mathbf{T}})$. In this way, we have the following theorem which defines the so-called Itô integral.

Theorem 2.3.2.1. *There exists a unique linear application*

$$\mathcal{I} : L^2_{ad}(\Omega \times \boldsymbol{T}, (\mathcal{F}_t)_{t\in\boldsymbol{T}}) \to L^2(\Omega, (\mathcal{F}), \mathbb{P}) \tag{2.7}$$

such that: 1. For

$$Ht(\omega) = \sum_{i=1}^{n} h_i(\omega)1(t_{i-1}, t_i](t) \in \mathcal{E},$$

$$I(H) = \begin{cases} \sum_{i=1}^{n} h_i(W_{t_i} - W_{t_{i-1}}) & if \quad \boldsymbol{T} = \mathbb{R}_+, \\ \sum_{i=1}^{n} h_i(W_{t_i\wedge t} - W_{t_{i-1}\wedge t}) & if \quad \boldsymbol{T} = [0, t]. \end{cases} \tag{2.8}$$

2. For

$$\tilde{H} \in L^2_{ad}(\Omega \times \boldsymbol{T}, (\mathcal{F}_t)_{t\in\boldsymbol{T}}),$$

$$E(\mathcal{I}(\tilde{H})^2) = E\left(\int_{\boldsymbol{T}} \tilde{H}_s^2 ds\right). \tag{2.9}$$

Chapter 3

Controllability of non-linear stochastic systems

The problem of controllability of a linear stochastic system

$$\begin{aligned} &dx(t) = [Ax(t) + Bu(t)]dt + \overline{\sigma}(t)dw(t), \quad t \in [0,T], \\ &x(0) = x_0, \end{aligned} \tag{3.1}$$

has been studied by various authors (see [25]). In this paper, we examine the controllability of a semi-linear stochastic system

$$\begin{aligned} &dx(t) = [Ax(t) + Bu(t) + f(t,x(t))]dt + \sigma(t,x(t))dw(t), \\ &x(0) = x_0 \in \mathbb{R}^n, \end{aligned} \tag{3.2}$$

and a non-linear stochastic system

$$\begin{aligned} &dx(t) = F(t,x(t),u(t))dt + \Sigma(t,x(t),u(t))dw(t), \\ &x(0) = x_0, \end{aligned} \tag{3.3}$$

where A and B are matrices of dimension $n \times n$ and $n \times m$ respectively,$\overline{\sigma} : [0,T] \to \mathbb{R}^{n\times n}$, $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times n}$, $F : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$, $\Sigma : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^{n\times n}$ and w is a n-dimensional Wiener process. Deterministic analogue of these problems has been examined by several authors (see [69, 96]) using one of the following methods : methods based on stability theory of Lyapunov, methods for systems defined on a manifold, methods which are geometrical in nature and fixed point methods.

For fixed ε, p [94] and [57] gave conditions for $0 \in \mathcal{A}_\varepsilon^p(T, x_0)$ via the Lyapunov approach for several types of non-linear stochastic systems, where $\mathcal{A}_\varepsilon^p(T, x_0)$ is the set of non-random (ε, p)-attainable points from x_0 in time T defined by $\mathcal{A}_\varepsilon^p(T, x_0) = \{h \in \mathcal{R}^n : \exists u \in U_{ad}, \mathbb{P}(\|x(T) - h\|^2 \leq \varepsilon) \geq p\}$ [23] studied the sample controllability for non-linear random differential equations. In this chapter, we study the controllability and the local null controllability of the systems (3.2) and (3.3) respectively.

3.1 Definitions

In this section, we adopt the following notation:

• $\{\mathcal{F}_t | t \in [0, T]\}$: the filtration generated by $\{w(s) : 0 \leq s \leq t\}$.

• $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ the Hilbert space of all $\mathcal{F}_T$-measurable square integrable variables with values in $\mathbb{R}^n$

• $L_P^{\mathcal{F}}([0, T], \mathbb{R}^n)$: the Banach space of all p-integrable and $\mathcal{F}_t$-measurable processes with values in $\mathbb{R}^n$ for $p \geq 2$

• H_2 : the Banach space of all square integrable and $\mathcal{F}_t$-adapted processes $\varphi(t)$ with norm

$$\|\varphi\|^2 = \sup_{t \in [0,T]} \mathbb{E}\|\varphi(t)\|^2$$

• $\mathcal{L}(X, Y)$: the space of all linear bounded operators from a Banach space X to a Banach space Y.

• $\phi(t) = exp(At), U_{ad} = L_2^{\mathcal{F}}([0, T], \mathbb{R}^m)$ or $U_{ad} = L_4^{\mathcal{F}}([0, T], \mathbb{R}^m)$ (In section 3)

Now let usintrodu ce the following operators and sets.

1. The operator

$$L_0^T \in \mathcal{L}(L_2^{\mathcal{F}}([0, T], \mathbb{R}^m), L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n))$$

is defined by

$$L_0^T u = \int_0^T \phi(T - s) B u(s) ds$$

and set of all states attainable from x_0 in time $t > 0$

$$\mathcal{R}_t(x_0) = \{x(t, x_0, u) : u(.) \in L_2^{\mathcal{F}}([0, T], \mathbb{R}^m)\}$$

where $x(t,x_0,u)$ is the solution of (3.3) corresponding to $x_0 \in \mathbb{R}^n, u(.) \in L_2^{\mathcal{F}}([0,T],\mathbb{R}^m)$. Clearly the adjoint $(L_0^T)^* : L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n) \to (L_2^{\mathcal{F}}([0,T],\mathbb{R}^m)$ is defined by

$$(L_0^T)^*z = B^*\phi^*(T-t)\mathbb{E}\{z|\mathcal{F}_t\}$$

2. The controllability operator Π_0^T associated with (1.)

$$\Pi_0^T\{.\} = L_0^T(L_0^T)^*\{.\} = \int_0^T \phi(T-t)BB^*\phi^*(T-t)\mathbb{E}\{.|\mathcal{F}_t\}$$

which belongs to $\mathcal{L}(L_2^{\mathcal{F}}([0,T],\mathbb{R}^m), L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n))$ and the controllability matrix $\Gamma_s^T \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$

$$\Gamma_s^T = \int_s^T \phi(T-t)BB^*\phi^*(T-t)dt$$

In what follows, we will use the following definitions.

Definition 3.1.1. *The system (3.3) is completely controllable on [0,T] if*

$$\mathcal{R}_T(x_0) = L_2(\Omega,\mathcal{F}_t,\mathbb{R}^n)$$

that is, all the points in $L_2(\Omega,\mathcal{F}_t,\mathbb{R}^n)$ can be reached from the point x_0 at time T.

Definition 3.1.2. *Let any trajectory $x(.,x_0,u^0) = x^0(.)$ of (3.3) with $u^0 \in U_{ad}$ and $x^0(0) = x_0$ such that $x^0(T) = 0$ be given. Then the system (3.3) is approximate controllable on $[0,T]$ if there is a neighborhood $N(x_0)$ of x_0 in $\mathbb{R}^n$ such that for any x_* in $N(x_0)$, there exists an admissible control u^* such that $x(T,x_*,u^*) = 0$.*

The following lemma givesa formula for a control transferring the state x_0 to an arbitrary state x_T.

Lemma 3.1.1. *Assume that the operator Π_0^T is invertible. Then for arbitrary $x_T \in L_2(\Omega,\mathcal{F}_t,\mathbb{R}^n$, $f(.) \in L_2^{\mathcal{F}}([0,T],\mathbb{R}^n)$, $\sigma(.) \in L_2^{\mathcal{F}}([0,T],\mathbb{R}^{n\times n})$, the control*

$$\begin{aligned} u(t) &= B^*\phi^*(T-t)\mathbb{E}\left\{(\Pi_0^T)^{-1} \times \left(x_T - \phi(T)x_0 - \int_0^T \phi(T-s)f(s)ds \right.\right. \\ &\quad \left.\left. - \int_0^T \phi(T-s)\sigma(s)dw(s)\right)|\mathcal{F}_t\right\} \end{aligned} \tag{3.4}$$

transfers the system

$$x(t) = \phi(t)x_0 + \int_0^t \phi(t-s)[Bu(s)+f(s)]ds + \int_0^t \phi(t-s)\sigma(s)dw(s) \tag{3.5}$$

from $x_0 \in \mathbb{R}^n$ *to* x_T *at time* T *and*

$$\begin{aligned} x(t) &= \phi(t)x_0 + \Pi_0^t \left[\phi^*(T-s)(\Pi_0^T)^{-1} \times \left(x_T - \phi(T)x_0 - \int_0^T \phi(T-r)f(r)\, dr \right.\right. \\ &\quad \left.\left. - \int_0^T \phi(T-r)\sigma(r)dw(r) \right) \right] + \int_0^t \phi(t-s)f(s)ds + \int_0^t \phi(t-s)\sigma(s)dw(s) \end{aligned} \tag{3.6}$$

Proof. By substituting (3.4) in (3.5), one can easily obtain

$$\begin{aligned} x(t) &= \phi(t)x_0 + \Pi_0^t \left[\phi^*(T-s)(\Pi_0^T)^{-1} \times \left(x_T - \phi(T)x_0 - \int_0^T \phi(T-r)f(r)\, dr \right.\right. \\ &\quad \left.\left. - \int_0^T \phi(T-r)\sigma(r)dw(r) \right) \right] + \int_0^t \phi(t-s)f(s)ds + \int_0^t \phi(t-s)\sigma(s)dw(s) \end{aligned} \tag{3.7}$$

Writing $t = T$ in (3.6), we see that the control $u(.)$ transfers the system (3.5) from x_0 to x_T.

□

3.2 Controllability via contraction mapping principle

In thiss ection, we derive controllability conditions for the semi-linear stochastic system (3.2) using the contraction mapping principle.

We impose the following conditions on data of the problem:

(A1) (f, σ) satisfies the Lipschitz condition with respect to x

$$\|f(t, x_1) - f(t, x_2)\| + \|\sigma(t, x_1) - \sigma(t, x_2)\| \leq L\|x_1 - x_2\|$$

(A2) (f, σ) is continuous on $[0, T] \times \mathbb{R}^n$ and satisfies

$$\|f(t, x)\| + \|\sigma(t, x)\| \leq L(\|x\| + 1)$$

(A3) The linear system (3.1) is completely controllable.

Remark: In [64], it is shown that complete controllability and approximate controllability of the system (3.1) coincide. That is why we study the complete controllability of the semi-linear stochastic system (3.2).

By a solution of the system (3.2), we mean a solution of the non-linear integral equation

$$x(t) = \phi(t)x_0 + \int_0^t \phi(t-s)[Bu(s) + f(s, x(s))]ds + \int_0^t \phi(t-s)\sigma(s, x(s))dw(s) \tag{3.8}$$

It is obvious that, under the conditions (A1) and (A2); for every $u(.) \in U_{ad}$ the integral equation (3.8) has a unique solution in H_2. To apply the contraction mapping principle, we define the non-linear operator $\mathbb{T}$ from H_2 to H_2 as follows:

$$(\mathbb{T}x)(t) = \phi(t)x_0 + \int_0^t \phi(t-s)[Bu(s) + f(s,x(s))]ds + \int_0^t \phi(t-s)\sigma(s,x(s))dw(s) \quad (3.9)$$

where

$$\begin{aligned} u(t) &= B^*\phi^*(T-t)\mathbb{E}\left\{(\Pi_0^T)^{-1} \times \left(x_T - \phi(T)x_0 - \int_0^T \phi(T-s)f(s,x(s))ds \right.\right. \\ &\quad \left.\left. - \int_0^T \phi(T-s)\sigma(s,x(s))dw(s)\right) |\mathcal{F}_t\right\} \end{aligned} \quad (3.10)$$

From precedent Lemma, the control (3.10) transfers the system (3.8) from the initial state x_0 to the final state x_T provided that the operator $\mathbb{T}$ has a fixed point. So, if the operator $\mathbb{T}$ has a fixed point then the system (3.8) is completely controllable.

Now for convenience, let usintr oduce the notation

$$l_1 = \max\{\|\phi(t)\|^2 : t \in [0,T]\}, \qquad l_2 = \|B\|^2$$

$$l_3 = \mathbb{E}\|x_T\|^2, \qquad M = \max\{\|\Gamma_s^T\|^2 : s \in [0,T]\}$$

Lemma 3.2.1. *For evry $z \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ there exists a process $\varphi(.) \in L_2^{\mathcal{F}}([0,T], \mathbb{R}^{n\times n})$ such that*

$$z = \mathbb{E}z + \int_0^T \varphi(s)dw(s) \quad (3.11)$$

$$\Pi_0^T z = \Gamma_0^T \mathbb{E}z + \int_0^T \Gamma_0^T \varphi(s)dw(s) \quad (3.12)$$

Moreover

$$\begin{aligned} \mathbb{E}\|\Pi_0^t z\|^2 &\le M\mathbb{E}\|\mathbb{E}\{z|\mathcal{F}_t\}\|^2 \\ &\le M\mathbb{E}\|z\|^2, \quad \forall z \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \end{aligned} \quad (3.13)$$

Proof. For the proof of (3.11) see [60], for (3.12) see [64]. One can easily obtain the boundedness of Π_0^t from (3.11) and (3.12) in such a way that

$$\begin{aligned} \mathbb{E}\|\Pi_0^t z\|^2 &= \|\Gamma_0^t \mathbb{E}z\|^2 + \mathbb{E}\int_0^t \|\Gamma_s^t \varphi(s)\|^2 ds \\ &\le M\left(\|\mathbb{E}z\|^2 + \int_0^t \mathbb{E}\|\varphi(s)\|^2 ds\right) \\ &= M\mathbb{E}\|\mathbb{E}\{z|\mathcal{F}_t\}\|^2 \\ &\le M\mathbb{E}\|z\|^2 \end{aligned} \quad (3.14)$$

□

Note that if the assumption (A3) holds, then for some $\gamma > 0$

$$\mathbb{E}\langle \Pi_0^t z, z\rangle \geq \gamma \mathbb{E}\|z\|^2, \quad \text{for all} z \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$$

see [64] and consequently

$$\|(\Pi_0^T)^{-1}\| \leq \frac{1}{\gamma} = l_4$$

Theorem 3.2.1. *Assume that the conditions (A1), (A2) and (A3) hold. If the inequality*

$$4l_1 L(Ml_1 l_4 + 1)(T + 1)T < 1 \tag{3.15}$$

holds, then the system (3.8) is completely controllable.

Proof. See [67]

Remark: Obviously hypothesis (3.15) is fulfilled if L is sufficiently small.

3.3 The local null controllability

In this section, we use the generalized implicit function theorem to consider the local null controllability of the non-linear stochastic system (3.3) via a suitable associated linearized system.

Theorem 3.3.1. *([4]): Let X be a topological space, Y and Z be Banach spaces. W be a neighbourhood of (x_0, y_0) in $X \times Y$, G be a mapping from W to Z, $G(x_0, y_0) = z_0$. If*

- *The mapping $x \to G(x_0, y_0)$ is continuous at x_0*
- *There exists a linear bounded operator $\Lambda : Y \to Z$ such that for $\varepsilon > 0$ there exist $\delta > 0$ and a neighbourhood $N(x_0)$ of x_0 such that*

$$\|G(x, y') - G(x, y'') - \Lambda(y' - y'')\| < \varepsilon \|y' - y''\| \tag{3.16}$$

- $\Lambda Y = Z$

Then there exists $K > 0$, a neighbourhood $N(x_0, z_0)$ of (x_0, z_0) in $X \times Z$ and a function $\varphi : N(x_0, z_0) \to Y$ such that

1. $G(x, \varphi(x, z)) = z$.

2. $\|\varphi(x, z) - y_0\| \leq K\|G(x, y_0) - z\|$.

We impose the following conditions on problem of the data:

(B1) (F, Σ) is continuously differentiable with respect to (x,u).

(B2) There exists M > 0 such that

$$\|F_x(t, x, u)\| + \|\Sigma_x(t, x, u)\| + \|F_u(t, x, u)\| + \|\Sigma_u(t, x, u)\| \leq M$$

where $F_x(\Sigma_x)$ denotes the derivative of $F(\Sigma)$ with respect to x, $F_u(\Sigma_u)$ denotes the derivative of $F(\Sigma)$ with respect to u.

(B3) The system (3.17) defined below is completely controllable. Associate (3.3) with the linear stochastic system

$$\begin{aligned} dz(t) &= [A(t)z(t) + B(t)v(t)]dt + [C(t)z(t) + D(t)v(t)]dw(t) \\ z(0) &= h_0 \end{aligned} \tag{3.17}$$

where

$$v \in U_{ad} = L_4^{\mathcal{F}}([0, T], \mathbb{R}^m), A(t) = F_x(t, x^0(t), u^0(t))$$

$$B(t) = F_u(t, x^0(t), u^0(t)), C(t) = \Sigma_x(t, x^0(t), u^0(t))$$

$$D(t) = \Sigma_u(t, x^0(t), u^0(t))$$

Theorem 3.3.2. *Let $x^0(.) = x(., x_0, u^0)$ and $x^\varepsilon(.) = x(., x_0, u^\varepsilon)$ be the solutions of (3.3) corresponding to $u^0, u^\varepsilon = u^0 + \varepsilon v$ respectively, where $v \in U_{ad}$ and $\varepsilon > 0$ then*

$$\max_{0 \leq t \leq T} \mathbb{E}\|x^\varepsilon(t) - x^0(t)\|^2 \to 0 \quad as \quad \varepsilon \to 0 \tag{3.18}$$

$$\max_{0 \leq t \leq T} \mathbb{E}\|x^\varepsilon(t) - x^0(t)\|^4 \to 0 \quad as \quad \varepsilon \to 0 \tag{3.19}$$

Proof. By the definitions of $x^0(.)$ and $x^\varepsilon(.)$

$$\begin{aligned} x^{\varepsilon}(t)-x^{0}(t) &= \int_0^t [F(x^{\varepsilon}(s),u^0(s)+\varepsilon v(s))-F(x^0(s),u^0(s))]ds \\ &+\int_0^t [\Sigma(x^{\varepsilon}(s),u^0(s)+\varepsilon v(s))-\Sigma(x^0(s),u^0(s))]dw(s) \\ &= \int_0^t [F(x^{\varepsilon}(s),u^0(s)+\varepsilon v(s))-F(x^0(s),u^0(s)+\varepsilon v(s))]ds \\ &+\int_0^t [F(x^0(s),u^0(s)+\varepsilon v(s))-F(x^0(s),u^0(s))]ds \\ &+\int_0^t [\Sigma(x^{\varepsilon}(s),u^0(s)+\varepsilon v(s))-\Sigma(x^0(s),u^0(s)+\varepsilon v(s))]dw(s) \\ &+\int_0^t [\Sigma(x^0(s),u^0(s)+\varepsilon v(s))-\Sigma(x^0(s),u^0(s))]dw(s) \end{aligned} \tag{3.20}$$

Taking norm of both sides of the above equality and using the Lipschitz condition (which follows from (B2)) we infer that there exist $L_1 > 0$ and $L_2 > 0$ such that

$$\mathbb{E}\|x^{\varepsilon}(t)-x^0(t)\|^2 \leq L_1\int_0^t \mathbb{E}\|x^{\varepsilon}(s)-x^0(s)\|^2 ds+\varepsilon^2 L_2\int_0^t \mathbb{E}\|v(s)\|^2 ds \tag{3.21}$$

By the Gronwall lemma

$$\mathbb{E}\|x^{\varepsilon}(t)-x^0(t)\|^2 \leq \left[exp\left(\int_0^T L_1 ds\right)\right]\times\varepsilon^2 L_2\int_0^T \mathbb{E}\|v(s)\|^2 ds \tag{3.22}$$

By tending ε to zero, we obtain (3.18). The limit (3.19) can be proved in a similar way.

□

Corollary 3.3.1.

$$\max_{0\leq t\leq T}\mathbb{E}\left\|\frac{x^{\varepsilon}(t)-x^0(t)}{\varepsilon}\right\|^{2n} \leq \left[exp\left(\int_0^T L_1 ds\right)\right]\times L_2\int_0^T \mathbb{E}\|v(s)\|^2 ds, \qquad n=1,2 \tag{3.23}$$

Proof. For case $n = 1$ see [67]. Case of $n = 2$ can be similarly be proved.

Theorem 3.3.3.

$$\max_{0\leq t\leq T}\mathbb{E}\left\|\frac{x^{\varepsilon}(t)-x^0(t)}{\varepsilon}-z(t)\right\|^2 \to 0 \tag{3.24}$$

as $\varepsilon \to 0$ where $z(.)$ is the solution to (3.17) with $h_0 = 0$

Proof. see [67].

Lemma 3.3.1. *[67]*

Let $z^{\varepsilon}(.)$ *be a solution of*

$$dz^{\varepsilon}(t) = [A^{\varepsilon}(t)z^{\varepsilon}(t) + B^{\varepsilon}(t)v(t)]dt + [C^{\varepsilon}(t)z^{\varepsilon}(t) + D^{\varepsilon}(t)v(t)]dw(t)$$

$$z^{\varepsilon}(0) = 0$$

and $z(.)$ *be a solution of (3.17) with* $h_0 = 0$, $v \in L_4^{\mathcal{F}}(0, T, \mathbb{R}^m)$ *Suppose that there exists* $M > 0$ *such that for each* $t \in [0, T]$

$$\|A^{\varepsilon}(t)\|^2 + \|B^{\varepsilon}(t)\|^2 + \|C^{\varepsilon}(t)\|^2 + \|D^{\varepsilon}(t)\|^2 \leq M$$

and

$$\mathbb{E}\|A^{\varepsilon}(t) - A(t)\|^4 + \mathbb{E}\|B^{\varepsilon}(t) - B(t)\|^4 + \mathbb{E}\|C^{\varepsilon}(t) - C(t)\|^4 + \mathbb{E}\|D^{\varepsilon}(t) - D(t)\|^4$$

tends to zero uniformly in t *as* $\varepsilon \to 0$. *Then*

$$\max\left\{\left(\max_{0\leq t\leq T} \mathbb{E}\|z^{\varepsilon}(t) - z(t)\|^2\right)^{1/2}, \mathbb{E}\int_0^T \|v\|^4 dt \leq 1\right\} \to 0 \quad as \quad \varepsilon \to 0$$

Theorem 3.3.4. *Suppose that (B1), (B2). Then if the system (3.17) is controllable then the system (3.3) is approximately controllable.*

Proof. Assume that there is an admissible control $u^0(.)$ such that the solution $x(., x_0, u^0)$ of (3.3) satisfies $x(T, x_0, u^0) = 0$. In others words, $G(x_0, u^0)(T) = 0$ where G maps $\mathbb{R}^n \times L_4^{\mathcal{F}}([0, T], \mathbb{R}^m)$ to $L_2^{\mathcal{F}}([0, T], \mathbb{R}^n)$ and is defined as

$$G(x, u)(t) = x + \int_0^t F(s, x(s), u(s))ds + \int_0^t \Sigma(s, x(s), u(s))dw(s)$$

where the process $x(.)$ is the solution to equation (3.3) corresponding to the control $u(.)$ and initial condition x. By Theorem 3.3.3, G is differentiable at $u^0(.)$ with derivative $z(.)$ which is the solution of (3.17) with $h_0 = 0$, and

$$G_u(x_0, u^0)(T)(v) = z(T)$$

On the other hand by precedent Lemma $G_u(x_0, u^0)(T)$ is continuous at $u^0(.)$. So, $\Lambda = G_u(x_0, u^0)(T)$ satisfies the inequality (3.16) (see [4]). It is known that the system (3.17) is controllable if and only if

$$ImG_u(x_0, u^0)(T) = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$$

By theorem 3.3.1 with $z_0 = 0$ there exists a neighbourhood $N(x_0) \subset \mathbb{R}^n$ and a function $\varphi : N(x_0) \to L_4^{\mathcal{F}}([0,T], \mathbb{R}^m)$ such that $G(x_*, \varphi(x_*))(T) = 0$ for every $x_* \in N(x_0)$.
Thus for every $x_* \in N(x_0)$, there exist $u^*(.) = \varphi(x_*) \in L_4^{\mathcal{F}}([0,T], \mathbb{R}^m)$ such that $x(T, x_*, u^*) = 0$. Therefore the system (3.3) is locally null controllable.

□

Chapter 4

Controllability of fractional stochastic dynamical systems with delays in control

This chapter is concerned with the global relative controllability of fractional stochastic dynamical systems with multiple delays in control for finite dimensional spaces and global relative controllability of linear and nonlinear fractional stochastic dynamical systems with distributed delays in control for finite dimensional spaces. Sufficient conditions for controllability results are obtained using Banach fixed point theorem and the controllability Grammian matrix which is defined by the Mittag-Leffler matrix function.

4.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathcal{P}$-null sets). Let $\alpha, \beta > 0$, with $n-1 < \alpha < n, n-1 < \beta < n$ and $n \in \mathbb{N}, D$ is the usual differential operator. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space,$\mathbb{R}_+ = [0, \infty)$ and suppose $f \in L^1(\mathbb{R}_+)$ The following definitions and properties are well known, for α, β and f as a suitable function (see, for instance, [35]):

(a) Riemann-Liouville fractional operators:

$$(I_{0^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,$$
$$(D_{0^+}^{\alpha} f)(x) = D^n (I_{0^+}^{n-\alpha} f)(x)$$

(b) Caputo fractional derivative:

$$({}^cD_{0+}^{\alpha}f)(x) = (I_{0+}^{n-\alpha}D^n f)(x),$$

in particular $I_{0+}^{\alpha}\ {}^cD_{0+}^{\alpha}f(t) = f(t) - f(0)(0 < \alpha < 1)$.

The following is a well known relation, for finite interval $[a, b] \in \mathbb{R}_+$

$$(D_{a^+}^{\alpha}f)(x) = ({}^cD_{a^+}^{\alpha}f)(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(1+j-\alpha)}(x-a)^{j-\alpha}, n = \mathcal{R}(\alpha) + 1$$

The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\{{}^cD_{0+}^{\alpha}f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+)s^{\alpha-1-k}.$$

The Riemann-Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann-Liouville sense require initial conditions of special form lacking physical interpretation. To overcome this difficulty Caputo introduced a new definition of fractional derivative but in general, both the Riemann-Liouville and the Caputo fractional operators possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. Due to this fact, the concept of sequential fractional differential equations are discussed in [35].

(c) Linear Sequential Derivative:

For $n \in \mathbb{N}$, the sequential fractional derivative for suitable function $y(x)$ is defined by

$$y^{(\alpha k)} := (D^{k\alpha}y)(x) = (D^{\alpha}D^{(k-1)\alpha}y)(x),$$

where $k = 1, ..., n$, $(D^{\alpha}y)(x) = y(x)$, and D^{α} is any fractional differential operator, here we mention it as ${}^cD_{0+}^{\alpha}$.

(d) Mittag-Leffler Function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \qquad for \quad \alpha, \beta > 0.$$

The general Mittag-Leffler function satisfies

$$\int_0^{\infty} e^{-t}t^{\beta-1}E_{\alpha,\beta}(t^{\alpha}z)dt = \frac{1}{1-z}, \quad for \quad |z| < 1$$

The Laplace transform of $E_{\alpha,\beta}(t^{\alpha}z)$ follows from the integral

$$\int_0^{\infty} e^{-st}t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})dt = \frac{s^{\alpha-\beta}}{s^{\alpha} \mp a}.$$

That is

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha} \mp a},$$

In particular, for $\beta = 1$,

$$E_{\alpha,1}(\lambda z^{\alpha}) = E_{\alpha}(\lambda z^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \lambda, z \in \mathbb{C}$$

have the interesting property ${}^cD^{\alpha}E_{\alpha}(\lambda t^{\alpha}) = \lambda E_{\alpha}(\lambda t^{\alpha})$ and

$$\mathcal{L}\{E_{\alpha}(\pm at^{\alpha})\} = \frac{s^{\alpha-1}}{s^{\alpha} \mp a} \quad for \beta = 1.$$

For brevity of notation let us take I^q_{0+} as I^q and ${}^cD^q_{0+}$ as ${}^cD^q$ and the fractional derivative is taken as Caputo sense.

(e) Solution representation:

Consider the linear fractional stochastic differential equation of the form

$$\begin{aligned} &{}^cD^q x(t) = Ax(t) + \sigma(t)\tfrac{dw(t)}{dt}, \quad t \in [0,T], \\ &x(0) = x_0, \end{aligned} \tag{4.1}$$

where $0 < q < 1$, $x \in \mathbb{R}^n$ and A is an $n \times n$ matrix, w(t) is given l-dimensional Winer process with filtration $\mathcal{F}_t$ generated by $w(s), 0 \le s \le t$ and $\sigma : [0,T] \longrightarrow \mathbb{R}^{n\times l}$ is appropriate function. In order to find the solution, apply Laplace transform on both sides and use the Laplace transform of Caputo derivative, we get

$$s^qX(s) - s^{q-1}x(0) = AX(s) + \Sigma(s)\frac{dw(s)}{ds}.$$

Apply inverse Laplace transform on both sides (see [10]) we have

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{s^{q-1}(s^qI - A)^{-1}\}x_0 + \mathcal{L}^{-1}\left\{\Sigma(s)\frac{dw(s)}{ds}\right\} * \mathcal{L}^{-1}\{(s^qI - A)^{-1}\}.$$

Finally, substituting Laplace transformation of the Mittag-Leffler function, we get the solution of the given system

$$x(t) = E_q(At^q)x_0 + \int_0^t (t-s)^{q-1}\left(\int_0^{\tau} \sigma(\theta)dw(\theta)\right) E_{q,q}(A(t-s)^q)ds$$

Where $E_q(At^q)$ is the matrix extension of the mentioned Mittag-Leffler functions with the following representation:

$$E_q(At^q) = \sum_{k=0}^{\infty} \frac{A^k t^{kq}}{\Gamma(1+kq)}$$

with the property ${}^cD^qE_q(At^q) = AE_q(At^q)$.

4.2 Relative controllability of fractional stochastic dynamical systems with multiple delays in control

Let $L^2_{\mathcal{F}_t}(J \times \Omega, \mathbb{R}^n)$ be the Banach space of all $\mathcal{F}_t$-measurable square integrable processes $x(t)$ with norm $\|x\|^2_{L^2} = \sup_{t\in J} \mathbb{E}\|x(t)\|^2$, where $\mathbb{E}(.)$ denotes the expectation with respect to the measure $\mathbb{P}$. Let $C = C([0,T]; L^2_{\mathcal{F}_t})$ be the Banach space of continuous maps from [0, T] into $L^2_{\mathcal{F}_t}(J \times \Omega, \mathbb{R}^n)$ satisfying $\sup_{t\in J} \mathbb{E}\|x(t)\|^2 < \infty$. Consider the linear fractional stochastic dynamical system with multiple delays in control represented by the fractional stochastic differential equation of the form

$$\begin{aligned} {}^cD^qx(t) &= Ax(t) + \sum_{k=1}^{M} B_k u(h_k(t)) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in J := [0,T], \\ x(0) &= x_0, \end{aligned} \tag{4.2}$$

Where $0 < q < 1, x(t) \in \mathbb{R}^n, u \in \mathbb{R}^l, A$ is an $n \times n$ matrix, B_k are $n \times l$ matrices, for $k = 0, 1, ..., M$, $w(t)$ is a given l-dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s)$, $0 \le s \le t$ and $\sigma : [0,T] \to \mathbb{R}^{n\times l}$ is appropriate function.

Let us assume the following assumptions

(i) Assume the function $h_k : J \to \mathbb{R}, k = 0, 1, ..., M$ are twice continuously differentiable and strictly increasing in J. Moreover,

$$h_k(t) \le t \quad \text{for } t \in J, i = 0, 1, ..., M. \tag{4.3}$$

(ii) Introduce the time lead functions $r_k(t) : [h_k(0), h_k(T)] \to J, k = 0, 1, ..., M$ such that $r_k(h_k(t)) = t$ for $t \in J$ Further assume that $h_0(t) = t$ and for t=T, the following inequalities hold

$$h_M(T) \le h_{M_1}(T) \le ...h_{m+1}(T) \le 0 = h_m(T) < h_{m-1}(T) = ...h_1(T) = h_0(T) = T. \tag{4.4}$$

(iii) Let $h > 0$ be given. For functions $u : [-h, T] \to \mathbb{R}^l$ and $t \in J$, we use the symbol u_t to denote the function on $[-h, 0]$, defined by $u_t(s) = u(t+s)$ for $s \in [-h, 0)$.

The following definitions of complete state of the system (4.2) at time t and relative controllability are assumed.

Definition 4.2.1. *The set $\phi(t) = \{x(t), u_t\}$ is the complete state of the system (4.2) at time t.*

Definition 4.2.2. *System (4.2) is said to be globally relatively controllable on J if for every complete state $\phi(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on J such that the corresponding trajectory of the system (4.2) satisfies $x(T) = x_1$.*

Note that the solution of system (4.2) can be expressed in the following form

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \sum_{k=0}^{M} B_k u(h_k(s))ds \\ &+ \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta)dw(\theta) \right) E_{q,q}(A(t-s)^q)ds. \end{aligned}$$

Taking into account the time lead functions $r_k(t)$, this solution can be further changed into

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{M} \int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u(s)ds \\ &+ \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta)dw(\theta) \right) E_{q,q}(A(t-s)^q)ds. \end{aligned} \tag{4.5}$$

Using the inequalities (4.4), the above equation becomes,

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{m} \int_{h_k(0)}^{0} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u_0(s)ds \\ &+ \sum_{k=0}^{m} \int_0^t (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u(s)ds \\ &+ \sum_{k=m+1}^{M} \int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u_0(s)ds \\ &+ \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta)dw(\theta) \right) E_{q,q}(A(t-s)^q)ds. \end{aligned} \tag{4.6}$$

For brevity, let us introduce the following notation:

$$\begin{aligned} \varphi(t) &= \sum_{k=0}^{m} \int_{h_k(0)}^{0} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u_0(s)ds \\ &+ \sum_{k=m+1}^{M} \int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u_0(s)ds. \end{aligned} \tag{4.7}$$

and

$$\chi(t) = \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(t-s)^q) ds.$$

Recall the controllability Grammian matrix

$$\psi_0^T = \sum_{k=0}^{m} \int_0^T (T - r_k(s))^{q-1} \left[E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s) \right] \left[E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s) \right]^* ds$$

Theorem 4.2.1. *The linear stochastic control system (4.2) is relatively controllable on* $[0, T]$ *if and only if the controllability Grammian matrix* ψ_0^T *is positive definite for some* $T > 0$.

Proof. Since is positive definite, it is non-singular and therefore its inverse is well defined. Define the control function as,

$$u(t) = [B_k^* E_{q,q}(A^*(T - r_k(t))^q) r_k'(t)] \psi^{-1} [x_1 - E_q(At^q)x_0 - \varphi(T) - \chi(T)], \quad k = 0, 1, ..., m. \tag{4.8}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily. Inserting (4.8) in (4.6) and using (4.7) we get

$$\begin{aligned} x(T) &= E_q(A(T)^q)x_0 + \varphi(T) + \sum_{k=0}^{m} \int_0^T (T - r_k(s))^{q-1} \left[E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s) \right] \\ &\quad \times \left[B_k^* E_{q,q}(A^*(T - r_k(s))^q) r_k'(s) \right] \psi^{-1} [x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)] ds \\ &\quad + \int_0^T (T-s)^{q-1} \left(\int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(T-s)^q) ds \\ &= x_1. \end{aligned}$$

Thus the control $u(t)$ transfers the initial state $\phi(0)$ to the desired vector $x_1 \in \mathbb{R}^n$ at time T. Hence the system (4.2) is controllable.

On the other hand, if it is not positive definite, there exists a nonzero ϕ such that $\phi^* \psi \phi = 0$, that is

$$\phi^* \sum_{k=0}^{m} \int_0^T (T - r_k(s))^{q-1} \left[E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s) \right] \left[E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s) \right]^* \phi ds = 0$$

$$\phi^* \sum_{k=0}^{m} (T - r_k(s))^{q-1} \left[E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s) \right] = 0, \quad \text{on } [0, T].$$

Let $x_0 = [E_q(A(T)^q)]^{-1} \phi$. By assumption, there exists a control u such that it steers the complete initial state $\phi(0) = \{x(0), u_0(s)\}$ to the origin in the interval $[0, T]$. It follows that

$$
\begin{aligned}
x(T) &= E_q(A(T)^q)x_0 + \varphi(T) + \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}\left[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)\right] \\
&\quad \times \left[B_k^* E_{q,q}(A^*(T-r_k(s))^q)r_k'(s)\right]\psi^{-1}[x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)]ds \\
&\quad + \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\
&= \phi + \varphi(T) + \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}\left[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)\right] \\
&\quad \times \left[B_k^* E_{q,q}(A^*(T-r_k(s))^q)r_k'(s)\right]\psi^{-1}[x_1 - E_q(AT^q)x_0 - \varphi(T) - \chi(T)]ds \\
&\quad + \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\
&= 0.
\end{aligned}
$$

Thus,

$$
0 = \phi^*\phi + \sum_{k=0}^{m}\int_0^T \phi^*(T-r_k(s))^{q-1}\left[E_{q,q}(A(T-r_k(s))^q)B_k r_k'(s)\right]u(s)ds + \phi^*(\varphi(T)+\chi(T))
$$

But the second and third term are zero leading to the conclusion $\phi^*\phi = 0$. This is a contradiction to $\phi \neq 0$. Thus ψ is positive definite. Hence the desired result.

□

Consider a nonlinear fractional stochastic dynamical system with multiple delays in control represented by the fractional stochastic differential equation of the form

$$
\begin{aligned}
{}^cD^q x(t) &= Ax(t) + \sum_{k=1}^{M} B_k u(h_k(t)) + f(t,x(t))\sigma(t,x(t))\frac{dw(t)}{dt}, \quad t \in J := [0,T], \\
x(0) &= x_0,
\end{aligned}
\tag{4.9}
$$

Where $0 < q < 1, x(t) \in \mathbb{R}^n, u \in \mathbb{R}^l, A, B_k$ are defined as above and $f : J \times \mathbb{R}^n \to \mathbb{R}^n$,$\sigma : J \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are appropriate functions.Then the solution of the system (4.9) can be expressed in the following form

$$
\begin{aligned}
x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\sum_{k=0}^{M} B_k u(h_k(s))ds \\
&\quad + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds + \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))d\omega(\theta)\right) \\
&\quad \times E_{q,q}(A(t-s)^q)ds.
\end{aligned}
$$

using the time leader functions $r_k(t)$ the solution becomes,

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \sum_{k=0}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k^{'}(s)u(s)ds \\ &+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds + \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))d\omega(\theta)\right) \\ &\times E_{q,q}(A(t-s)^q)ds. \end{aligned} \tag{4.10}$$

Now using the inequalities (4.4), the above equation for t = T can be expressed as

$$\begin{aligned} x(T) &= E_q(A(T)^q)x_0 + \sum_{k=0}^{m}\int_{h_k(0)}^{0}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k^{'}(s)u_0(s)ds \\ &+ \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k^{'}(s)u(s)ds \\ &+ \sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_k r_k^{'}(s)u_0(s)ds \\ &+ \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\ &+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))d\omega(\theta)\right)E_{q,q}(A(t-s)^q)ds. \end{aligned} \tag{4.11}$$

For brevity, let us introduce the following notation using (4.7)

$$\begin{aligned} \Upsilon(\phi(0),x_1,x) &= x_1 - E_q(A(T)^q)x_0 - \varphi(T) - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\ &- \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))d\omega(\theta)\right)E_{q,q}(A(t-s)^q)ds. \end{aligned} \tag{4.12}$$

Now let us define the controllability Grammian matrix and the control function

$$\psi_0^T = \sum_{k=0}^{m}\int_0^T (T-r_k(s))^{q-1}[E_{q,q}(A(T-r_k(s))^q)B_k r_k^{'}(s)u(s)][E_{q,q}(A(T-r_k(s))^q)B_k r_k^{'}(s)u(s)]^* ds \tag{4.13}$$

$$u(t) = [B_k^* E_{q,q}(A^*(T-r_k(t))^q)r_k^{'}(t)]\psi^{-1}\Upsilon(\phi(0),x_1,x) \quad \text{for} \quad k=0,1,...,m \tag{4.14}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and * denotes the matrix transpose. Inserting (4.14) in (4.11) by using (4.12) and (4.13), it is easy to verify that the control u(t) transfers the initial complete state $\phi(0)$ to the desired vector x_1 at time T for each fixed x. Now observing (4.12) and substituting (4.14) in (4.10), we have

$$
\begin{aligned}
x(t) \;=\; & E_q(A(t)^q)x_0 + \sum_{k=0}^{m} \int_{h_k(0)}^{0} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k^{'}(s) u_0(s) ds \\
& + \sum_{k=0}^{m} \int_{0}^{t} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k^{'}(s) \\
& \times B_k^* E_{q,q}(A^*(T-r_k(s))^q) r_k^{'}(s) \psi^{-1} \Upsilon(\phi(0), x_1, x) ds \\
& + \sum_{k=m+1}^{M} \int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k^{'}(s) u_0(s) ds \\
& + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s, x(s)) ds \\
& + \int_0^t (t-s)^{q-1} \left(\int_0^{\tau} \sigma(\theta, x(\theta)) d\omega(\theta) \right) E_{q,q}(A(t-s)^q) ds.
\end{aligned}
\tag{4.15}
$$

Now, we impose the following conditions on data of the problem:

(**iv**) The linear fractional stochastic dynamical system (4.2) is globally relatively controllable.

(**v**) f and σ satisfy Lipschitz and linear growth conditions. That is, there exists some constants $N, \widetilde{N}, L, \widetilde{L} > 0$ such that

$$\|f(t,x) - f(t,y)\|^2 \le N\|x-y\|^2, \|f(t,x)\|^2 \le \widetilde{N}(1+\|x\|^2)$$

$$\|\sigma(t,x) - \sigma(t,y)\|^2 \le L\|x-y\|^2, \|\sigma(t,x)\|^2 \le \widetilde{L}(1+\|x\|^2).$$

For our convenience, let us introduce the following notations.

$$a_1 = \max\left\{\|E_q(At^q)\|^2; t \in J\right\}, \quad a_2 = \max\left\{\|u_0(t)\|^2; t \in J\right\}, \quad r_k = \max\left\{\|r_k^{'}(t)\|^2; t \in J\right\}$$

$$b_k = \max\left\{\|E_{q,q}(A(t-r_k(s))^q)\|^2; s \in [0,T]\right\}, c_k = \int_0^T (T-r_k(s))^{2(q-1)} ds$$

$$\widetilde{c}_k = \int_{h_k(0)}^{0} (T-r_k(s))^{2(q-1)} ds; \widehat{c}_k = \int_{h_k(0)}^{h_k(T)} (T-r_k(s))^{2(q-1)} ds$$

We claim that if (**iv**) holds, the operator ψ_0^T is strictly positive definite and thus the inverse linear operator $(\psi_0^T)^{-1}$ is bounded, say, by l, (see [6] for more details).

Theorem 4.2.2. *Under the conditions (**iv**) and (**v**), the nonlinear system (4.9) is globally relatively controllable on J.*

Proof. Firstly, from the definition (4.14) we can write the control function u as

$$\begin{aligned}
u(t) &= [B_k^* E_{q,q}(A^*(T-r_k(t))^q) r_k'(t)]\psi^{-1} \\
&\times \Bigg[x_1 - E_q(A(T)^q)x_0 - \sum_{k=0}^{m} \int_{h_k(0)}^{0} (T-r_k(s))^{q-1} E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s) u_0(s) ds \\
&+ \sum_{k=0}^{m} \int_0^t (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) \\
&- \sum_{k=m+1}^{M} \int_{h_k(0)}^{h_k(t)} (T-r_k(s))^{q-1} E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s) u_0(s) ds \\
&- \int_0^T (T-s)^{q-1} E_{q,q}(A(T-s)^q) f(s,x(s)) ds \\
&- \int_0^T (T-s)^{q-1} \left(\int_0^\tau \sigma(\theta, x(\theta)) d\omega(\theta) \right) E_{q,q}(A(T-s)^q) ds \Bigg].
\end{aligned}$$

Secondly, we define the operator $\mathcal{P} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\begin{aligned}
\mathcal{P}(x)(t) &= E_q(A(T)^q)x_0 + \sum_{k=0}^{m} \int_{h_k(0)}^{0} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u_0(s) ds \\
&+ \sum_{k=0}^{m} \int_0^t (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) \\
&\times B_k^* E_{q,q}(A^*(T-r_k(s))^q) r_k'(s) \psi^{-1} \Upsilon(\phi(0), x_1, x) ds \\
&+ \sum_{k=m+1}^{M} \int_{h_k(0)}^{h_k(t)} (t-r_k(s))^{q-1} E_{q,q}(A(t-r_k(s))^q) B_k r_k'(s) u_0(s) ds \\
&+ \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s,x(s)) ds \\
&+ \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta, x(\theta)) d\omega(\theta) \right) E_{q,q}(A(t-s)^q) ds.
\end{aligned}$$

In order to prove the global relative controllability of the system (4.9) it is enough to show that $\mathcal{P}$ has a fixed point in $\mathcal{C}$. To do this, we can employ the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps $\mathcal{C}$ into itself. We have

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)\|^2 \leq\ & 6a_1\mathbb{E}\|x_0\|^2 + 6\sum_{k=0}^{m}\mathbb{E}\left\|\int_{h_k(0)}^{0}(T-r_k(s))^{q-1}E_{q,q}(A(T-r_k(s))^q)B_kr_k^{'}(s)u_0(s)ds\right\|^2\\ &+6\sum_{k=0}^{m}\mathbb{E}\left\|\int_0^t(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr_k^{'}(s)\right.\\ &\left.\times B_k^*E_{q,q}(A^*(T-r_k(s))^q)r_k^{'}(s)\psi^{-1}\Upsilon(\phi(0),x_1,x)ds\right\|^2\\ &+6\sum_{k=m+1}^{M}\mathbb{E}\left\|\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_kr_k^{'}(s)u_0(s)ds\right\|^2\\ &+6\mathbb{E}\left\|\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds\right\|^2\\ &+6\mathbb{E}\left\|\int_0^t(t-s)^{q-1}\left(\int_0^\tau\sigma(\theta,x(\theta))d\omega(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2.\end{aligned}$$

It follows from Lemma 2.5, in [84], and the above notation that:

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)\|^2 \leq\ & 6a_1\mathbb{E}\|x_0\|^2 + 6a_2\left(\sum_{k=0}^{m}\widetilde{c}_kb_kr_k\|B_k\|^2+\sum_{k=m+1}^{M}\widehat{c}_kb_kr_k\|B_k\|^2\right)\\ &+6b\frac{t^{2q-1}}{2q-1}\int_0^t\mathbb{E}\|f(s,x(s))\|^2ds+6l^2\sum_{k=0}^{m}c_kb_k^2r_k^2\|B_k\|^4\int_0^t\mathbb{E}\|\Upsilon(\phi(0),x_1,x)\|^2ds\\ &+6Lb\frac{t^{2q-1}}{2q-1}\int_0^t\left(\int_0^\tau\mathbb{E}\|\sigma(\theta,x(\theta))\|^2d\theta\right)ds.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)\|^2 \leq\ & 6a_1\mathbb{E}\|x_0\|^2 + 6a_2\left(\sum_{k=0}^{m}\widetilde{c}_kb_kr_k\|B_k\|^2+\sum_{k=m+1}^{M}\widehat{c}_kb_kr_k\|B_k\|^2\right)\\ &+6b\frac{t^{2q-1}}{2q-1}\int_0^t\mathbb{E}\|f(s,x(s))\|^2ds+6l^2\sum_{k=0}^{m}c_kb_k^2r_k^2\|B_k\|^4\int_0^t\mathbb{E}\|\Upsilon(\phi(0),x_1,x)\|^2ds\\ &+6Lb\frac{t^{2q-1}}{2q-1}\int_0^t\left(\int_0^\tau\mathbb{E}\|\sigma(\theta,x(\theta))\|^2d\theta\right)ds.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)\|^2 \leq\ & 6l^2\eta\mathbb{E}\|x_1\|^2+6a_1\mathbb{E}\|x_0\|^2(1+l^2\eta)+6a_2\beta(1+l^2\eta)\\ &+6b\frac{T^{2q-1}}{2q-1}\widetilde{N}(1+l^2\eta)(1+\|x\|_{L^2}^2)+6L_\sigma\widetilde{L}b\frac{T^{2q-1}}{2q-1}(1+l^2\eta)(1+T\|x\|_{L^2}^2).\end{aligned}$$

It follows from the above inequality and the condition (v) that there exists $c>0$ such that

$$\mathbb{E}\|\mathcal{P}(x)(t)\|^2 \leq c(1+\|x\|^2_{L^2}).$$

Therefore $\mathcal{P}$ maps $\mathcal{C}$ into itself.

Secondly, we claim that $\mathcal{P}$ is a contraction mapping on $\mathcal{C}$. For $x, y \in \mathcal{C}$,

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2 \ &\leq 3\sum_{k=0}^{m}\mathbb{E}\Bigg\|\int_0^t (t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q B_k r_k^{'} \\ &\times B_k^* E_{q,q}(A^*(T-r_k(s))^q) r_k^{'}(s)\psi^{-1}\left[\Upsilon(\phi(0),x_1,x)-\Upsilon(\phi(0),x_1,y)\right] ds\Bigg\|^2 \\ &+3\mathbb{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)(f(s,x(s))-f(s,y(s)))ds\right\|^2 \\ &+3\mathbb{E}\left\|\int_0^t (t-s)^{q-1}\left(\int_0^\tau (\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta)))d\omega(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2.\end{aligned}$$

Using Lemma 2.5, in [84], condition (v), and the above notations we get

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2 \ &\leq \ 3l^2\frac{T^{2q-1}}{2q-1}b\sum_{k=0}^{m}c_k b_k^2 r_k^2\|B_k\|^4\Bigg[\int_0^T \mathbb{E}\|f(s,x(s))-f(s,y(s))\|^2 ds \\ &+L_\sigma\int_0^\tau \mathbb{E}\|\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta))\|^2 d\theta\Bigg] \\ &+3\frac{T^{2q-1}}{2q-1}b\int_0^t \mathbb{E}\left\|f(s,x(s))-f(s,y(s))\right\|^2 ds \\ &+3\frac{T^{2q-1}}{2q-1}bL_\sigma\int_0^t\left(\int_0^\tau \mathbb{E}\|\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta))\|^2 d\theta\right)ds. \\ &\leq \ 3l^2 b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]\int_0^T \mathbb{E}\|x(s)-y(s)\|^2 ds \\ &+3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]\int_0^T \mathbb{E}\|x(s)-y(s)\|^2 ds.\end{aligned}$$

It results that

$$\sup_{t\in[0,T]}\mathbb{E}\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\|^2 \leq \left[3l^2 b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]+3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]\right]\sup_{t\in[0,T]}\mathbb{E}\|x(t)-y(t)\|^2.$$

Therefore we conclude that if $\left(3l^2 b\eta\frac{T^{2q-1}}{2q-1}[N+LL_\sigma]+3b\frac{T^{2q-1}}{2q-1}[N+TLL_\sigma]\right) < 1$, then $\mathcal{P}$ is a contraction mapping on $\mathcal{C}$, implies that the mapping $\mathcal{P}$ has a unique fixed point $x(.) \in \mathcal{C}$.

Hence we have

$$
\begin{aligned}
x(t) \;=\;& E_q(A(t)^q)x_0 + \sum_{k=0}^{m}\int_{h_k(0)}^{0}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k^{'}(s)u_0(s)ds \\
&+\sum_{k=0}^{m}\int_{0}^{t}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k^{'}(s)u(s)ds \\
&+\sum_{k=m+1}^{M}\int_{h_k(0)}^{h_k(t)}(t-r_k(s))^{q-1}E_{q,q}(A(t-r_k(s))^q)B_k r_k^{'}(s)u_0(s)ds \\
&+\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\
&+\int_0^t (t-s)^{q-1}\left(\int_0^{\tau}\sigma(\theta,x(\theta))d\omega(\theta)\right)E_{q,q}(A(t-s)^q)ds.
\end{aligned}
$$

Thus $x(t)$ is the solution of the system (4.9), and it is easy to verify that $x(T) = x_1$. Further the control function $u(t)$ steers the system (4.9) from initial complete state $\phi(0)$ to x_1 on J. Hence the system (4.9) is globally relatively controllable on J.

□

Example

In this example, we apply the results obtained in the previous section for the following stochastic fractional dynamical systems with multiple delays in control which involves sequential Caputo derivative

$$
\begin{aligned}
&{}^cD^q x(t) = Ax(t) + B_1u(t) + B_2u(t-h) + f(t,x(t)) + \sigma(t,x(t))\tfrac{d\omega(t)}{dt};\, 0<q<1, t\in[0,T] \\
&x(0) = x_0
\end{aligned}
\tag{4.16}
$$

where

$$
A = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}, \qquad B_1 = B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

$$
f(t,x(t)) = \begin{pmatrix} x_1(t)\cos x_2(t) + 3x_2(t) \\ x_2(t)\sin x_1(t) + 2x_1(t) \end{pmatrix}, \qquad \sigma(t,x(t)) = \begin{pmatrix} (2t^2+1)x_1(t)e^{-t} & 0 \\ 0 & x_2(t)e^{-t} \end{pmatrix}.
$$

Let us introduce the variables $x_1(t) = x(t)$ and $x_2(t) = {}^cD^{\frac{q}{2}}x_1(t)$.Then

$${}^cD^{\frac{q}{2}}x_1(t) = {}^cD^{\frac{q}{2}}x(t) = x_2$$

The Mittag-Leffler matrix of the given system is given by

$$E_q(At^q) = \begin{pmatrix} E_q(-t^q) & 0 \\ 3E_q(-t^q) - 3E_q(-2t^q) & E_q(-2t^q) \end{pmatrix}.$$

Further

$$E_{q,q}(A(T-s)^q) = \begin{pmatrix} E_{q,q}(-(T-s)^q) & 0 \\ 3E_{q,q}(-(T-s)^q) - 3E_{q,q}(-2(T-s)^q) & E_{q,q}(-2(T-s)^q) \end{pmatrix},$$

$$E_{q,q}(A(T-(s+h))^q) = \begin{pmatrix} E_{q,q}(-(T-(s+h))^q) & 0 \\ 3E_{q,q}(-(T-(s+h))^q) - 3E_{q,q}(-2(T-(s+h))^q) & E_{q,q}(-2(T-(s+h))^q) \end{pmatrix}.$$

By simple matrix calculation one can see that the controllability matrix

$$\begin{aligned} \psi_0^T &= \sum_{k=0}^{m} \int_0^T (T - r_k(s))^{q-1} [E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s)][E_{q,q}(A(T-r_k(s))^q) B_k r_k'(s)]^* ds \\ &= \int_0^T \left[(T-s)^{q-1} \begin{pmatrix} a^2 & ac \\ ac & b^2 + c^2 \end{pmatrix} + (T-(s+h))^{q-1} \begin{pmatrix} \bar{a}^2 & \bar{a}\bar{c} \\ \bar{a}\bar{c} & \bar{b}^2 + \bar{c}^2 \end{pmatrix} \right] ds. \end{aligned}$$

is positive definite for any $T > h$, where

$$\begin{aligned} &a = E_{q,q}(-(T-s)^q), \qquad b = E_{q,q}(-2(T-(s+h))^q), \\ &c = 3E_{q,q}(-(T-s)^q) - 3E_{q,q}(-2(T-s)^q), \qquad \bar{a} = E_{q,q}(-(T-(s+h))^q \\ &\bar{b} = E_{q,q}(-2(T-(s+h))^q), \qquad \bar{c} = 3E_{q,q}(-(T-(s+h))^q) - 3E_{q,q}(-2(T-(s+h))^q). \end{aligned}$$

Further the functions $f(t, x(t))$ and $\sigma(t, x(t))$ satisfies the hypothesis mentioned in Theorem 4.2.2, and so the fractional system (4.16) is globally relatively controllable on $[0, T]$.

4.3 Global relative controllability of fractional stochastic dynamical systems with distributed delays in control

Let $L^2_{\mathcal{F}_t}(J \times \Omega, \mathbb{R}^n)$ be a Banach space of all $\mathcal{F}_t$ mesurable square processes $x(t)$ with norm $\|x\|^2_{L^2} = \sup_{t \in J} \mathbb{E}\|x(t)\|^2$, where $\mathbb{E}(.)$ denotes the expectation with respect to the mesure $\mathbb{P}$. Let $C = C([0, T]; L^2_{\mathcal{F}_t})$ be the Banach space of continuous maps from $[0, T]$ into $L^2_{\mathcal{F}_t}(J \times \Omega, \mathbb{R}^n)$ satisfying $\sup_{t \in J} \mathbb{E}\|x(t)\|^2 < \infty$.

Consider the linear fractional stochastic dynamical system with distributed delays in control represented by the fractional stochastic differential equation of the form

$$
\begin{aligned}
&{}^cD^q x(t) = Ax(t) + \int_{-h}^{0} d_\tau B(t,\tau)u(t+\tau) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in J := [0,T], \\
&x(0) = x_0,
\end{aligned} \tag{4.17}
$$

Where $0 < q < 1, x(t) \in \mathbb{R}^n$, and the second integral term is in the Lebesgue-Stieltjes sense with respect to τ . Let $h > 0$ be given. For function $u : [-h, T] \to \mathbb{R}^m$ and $t \in J$, we use the symbol u_t to denote the function on $[-h, 0]$, defined by $u_t(s) = u(t+s)$ for $s \in [-h, 0)$. A is an $n \times n$ matrix, $B(t, \tau)$ is an $n \times m$ matrix continuous in t for fixed τ and is of bounded variation in τ on $[-h, 0]$ for each $t \in J$ and continuous from left in τ on the interval $(-h, 0)$. Here $\omega(t)$ is a given m-dimensional Wiener process with the filtration $\mathbb{F}_t$ generated by $\omega(s)$, $0 \le s \le t$ and $\sigma : [0, T] \to \mathbb{R}^{n \times m}$.

The following definitions of complete state of the system (4.17) at time t and relative controllability are assumed

Definition 4.3.1. *The set $\phi(t) = \{x(t), u_t\}$ is the complete state of the system (4.17) at time t.*

Definition 4.3.2. *System (4.17) is said to be globally relatively controllable on J if for every complete state $\phi(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on J such that the corresponding trajectory of the system (4.17) satisfies $x(T) = x_1$.*

Note that the solution of system (4.17) can be expressed in the following form

$$
\begin{aligned}
x(t) \;=\; & E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \left[\int_{-h}^{0} d\tau B(s,\tau)u(s+\tau) \right] ds \\
& + \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(t-s)^q) ds.
\end{aligned}
$$

where $E_q(A(t)^q)$ is the Mittag Leffler matrix function. Now using the well known result of unsymmetric Fubini theorem [19] and change of order of integration to the last term, we have

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_{-h}^{0} dB_\tau \left[\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)u(s+\tau)B(s,\tau)\right] ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &= E_q(A(t)^q)x_0 + \int_{-h}^{0} dB_\tau \left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] \\ &+ \int_{-h}^{0} dB_\tau \left[\int_0^{t+\tau} (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u(s)ds\right] \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &= E_q(A(t)^q)x_0 + \int_{-h}^{0} dB_\tau \left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] \\ &+ \int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)u(s)ds\right] \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds, \end{aligned} \tag{4.18}$$

where

$$B_t(s,\tau) = \begin{cases} B(s,\tau), & s \le t \\ 0, & s > t \end{cases} \tag{4.19}$$

and dB_τ denotes the integration of Lebesgue Stieltjes sense with respect to the variable τ in the function $B(t,\tau)$.

For brevity, let us introduce the following notations:

$$\varphi(t,s) = \int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau), \tag{4.20}$$

and

$$\chi(t) = \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds. \tag{4.21}$$

Recall the controllability Grammian matrix

$$\psi_0^T = \int_0^T \varphi(T,s)\varphi^*(T,s)ds$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and the $*$ denotes the matrix transpose.

Theorem 4.3.1. *The linear stochastic control system (4.17) is relatively controllable on* $[0,T]$ *if and only if the controllability Grammian matrix* ψ_0^T *is positive definite for some* $T > 0$.

Proof : Since ψ is positive definite, it is non-singular and therefore its inverse is well defined. Define the control function as,

$$\begin{aligned} u(t) &= \varphi^*(T,t)\psi^{-1}\Big(x_1 - E_q(At^q)x_0 - \int_{-h}^{0} dB_\tau\Big[(T-(s-\tau))^{q-1} \\ &\quad \times E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\Big] - \chi(T)\Big). \end{aligned} \tag{4.22}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily. Inserting (4.22) in (4.18) and using (4.20) we get

$$\begin{aligned} x(T) &= E_q(A(T)^q)x_0 + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0} (T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right. \\ &\quad + \int_0^T \left[\int_{-h}^{0}(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)d_\tau B_T(s-\tau,\tau)\right] \\ &\quad \times \left[\int_{-h}^{0}(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)d_\tau B_T(s-\tau,\tau)\right]^* \psi^{-1} \\ &\quad \times \Big(x_1 - E_q(AT^q)x_0 - \int_{-h}^{0} dB_\tau\Big[(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q) \\ &\quad \times B(s-\tau,\tau)u_0(s)ds\Big] - \chi(T)\Big)d\tau \\ &\quad + \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\ &= x_1 \end{aligned} \tag{4.23}$$

Thus the control $u(t)$ transfers the initial state $\phi(0)$ to the desired vector $x_1 \in \mathbb{R}^n$ at time T . Hence the system (4.17) is controllable.

On the other hand, if it is not positive definite, there exists a nonzero ϕ such that $\phi^*\psi\phi = 0$, that is

$$\phi^*\int_0^T \varphi(T,s)\varphi^*(T,s)\phi ds = 0$$

$$\phi^*\varphi(T,s) = 0, \qquad \text{on [0,T]}.$$

Let $x_0 = [E_q(AT^q)]^{-1}\phi$. By assumption, there exists a control u such that it steers the complete initial state $\phi(0) = \{x(0), u_0(s)\}$ to the origin in the interval $[0,T]$. It follows that

$$
\begin{aligned}
x(T) &= E_q(A(T)^q)x_0 + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0}(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right. \\
&+ \int_0^T \left[\int_{-h}^{0}(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)d_\tau B_T(s-\tau,\tau)\right] u(s)ds \\
&+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right) E_{q,q}(A(T-s)^q)ds \\
&= \phi + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0}(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right. \\
&+ \int_0^T \left[\int_{-h}^{0}(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)d_\tau B_T(s-\tau,\tau)\right] u(s)ds \\
&+ \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right) E_{q,q}(A(T-s)^q)ds \\
&= 0.
\end{aligned}
$$

Thus

$$
\begin{aligned}
0 &= \phi^*\phi + \int_0^T \phi^*\varphi(T,s)u(s)ds \\
&+\phi^*\left(\int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0}(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] + \chi(T)\right).
\end{aligned}
$$

Then, taking into account that both of the terms
$\phi^*\phi + \int_0^T \phi^*\varphi(T,s)u(s)ds$ and

$$
\phi^*\left(\int_{-h}^{0} dB_\tau \left[\int_{\tau}^{0}(T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] + \chi(T)\right)
$$

are zero leading to the conclusion $\phi^*\phi = 0$. This is a contradiction to $\phi \neq 0$. Thus ψ is positive definite. Hence the desired result.

□

Consider a nonlinear fractional stochastic dynamical system with distributed delays in control represented by the fractional stochastic differential equation of the form

$$
\begin{aligned}
&{}^cD^q x(t) = Ax(t) + \int_{-h}^{0} d_\tau B(t,\tau)u(t+\tau) + f(t,x(t)) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in J := [0,T], \\
&x(0) = x_0,
\end{aligned} \tag{4.24}
$$

Where $0 < q < 1, x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ A and B are as above, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : J \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ and $w(t)$ is a given m-dimensional Wiener process with the filtration

$\mathbb{F}_t$ generated by $w(s)$. Then the solution of the system (4.24) can be expressed in the following form [22]

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\left[\int_{-h}^0 d\tau B(t,\tau)u(t+\tau)\right]ds. \end{aligned}$$

Using the well known result of unsymmetric Fubini theorem [19] and change of order of integration to the last term, we have

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &+ \int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] \\ &+ \int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)\right]u(s)ds \end{aligned} \tag{4.25}$$

where

$$B_t(s,\tau) = \begin{cases} B(s,\tau), & s \le t \\ 0, & s > t \end{cases} \tag{4.26}$$

and dB_τ denotes the integration of Lebesgue Stieltjes sense with respect to the variable τ in the function $B(t,\tau)$.

For brevity, let us introduce the following notations:

$$\begin{aligned} \Upsilon(\phi(0),x_1;x) &= x_1 - E_q(A(T)^q)x_0 - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\ &- \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\ &- \int_{-h}^0 dB_\tau\left[\int_\tau^0 (T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right]. \end{aligned} \tag{4.27}$$

Define the control function

$$u(t) = \varphi^*\psi^{-1}\Upsilon(\phi(0),x_1;x), \tag{4.28}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and $*$ denotes the matrix transpose.

Now, we impose the following conditions on data of the problem:

i. The linear fractional stochastic dynamical system (4.17) is globally relatively controllable.

ii. f and σ satisfy Lipschitz and linear growth conditions. That is, there exists some constants $N, \widetilde{N}, L, \widetilde{L} > 0$ such that

$$\|f(t,x) - f(t,y)\|^2 \leq N\|x-y\|^2, \quad \|f(t,x)\|^2 \leq \widetilde{N}(1+\|x\|^2)$$
$$\|\sigma(t,x) - \sigma(t,y)\|^2 \leq L\|x-y\|^2, \quad \|\sigma(t,x)\|^2 \leq \widetilde{L}(1+\|x\|^2).$$

For our convenience, let us introduce the following notations.

$$\begin{aligned}
a_1 &= \max\left\{\|E_q(At^q)\|^2; t \in J\right\}, & a_2 &= \max\left\{\|E_{q,q}(A(t-s)^q)\|^2; t \in J\right\} \\
a_3 &= \max\left\{\|E_q(A(t-(s-\tau))^q)\|^2; t \in J\right\}, & c_1 &= \max\left\{\|u_0(t)\|^2; t \in J\right\} \\
c_2 &= \int_{-h}^{0} (t-(s-\tau))^{2(q-1)} ds, & c_3 &= \int_{-\tau}^{0} (t-(s-\tau))^{2(q-1)} ds \\
M_B &= \max\left\{\|B(s-\tau,\tau)\|^2; 0 \leq \tau < s \leq T\right\}, & M &= \max\left\{\|\varphi(t,s)\|^2; 0 \leq s < t \leq T\right\}.
\end{aligned}$$

We claim that if **i.** holds, the operator ψ_0^T is strictly positive definite and thus the inverse linear operator $(\psi_0^T)^{-1}$ is bounded, say, by l, (see [54] for more details)

Theorem 4.3.2. *Under the conditions **i.** and **ii.**, the nonlinear system (4.24) is globally relatively controllable on J .*

Proof: Firstly, from the definition of the control function (4.28), we can write u as

$$\begin{aligned}
u(t) &= \varphi^*(T,t)\psi^{-1}\Upsilon(\phi(0), x_1; x) \\
&= \varphi^*(T,t)\psi^{-1}\Bigg(x_1 - E_q(A(T)^q)x_0 - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\
&\quad - \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta, x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\
&\quad - \int_{-h}^0 dB_\tau\left[\int_\tau^0 (T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right]\Bigg).
\end{aligned}$$

Secondly, we define the operator $\mathcal{P}: C \to C$ by

$$\begin{aligned}
\mathcal{P}(x)(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\
&\quad + \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\
&\quad + \int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] \\
&\quad + \int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)\right]u(s)ds
\end{aligned}$$

In order to prove the global relative controllability of the system (4.24) it is enough to show that $\mathcal{P}$ has a fixed point in C. To do this, we can employ the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps C into itself.

We have

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{P}(x)(t)\right\|^2 &= 5a_1\mathbb{E}\|x_0\|^2 + 5\mathbb{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds\right\|^2 \\
&+5\mathbb{E}\left\|\int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2 \\
&+5\mathbb{E}\left\|\int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right]\right\|^2 \\
&+5\mathbb{E}\left\|\int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)\right]u(s)ds\right\|^2
\end{aligned}$$

It follows from Lemma 2.5, in [84], and the above notation that:

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{P}(x)(t)\right\|^2 &\le 5a_1\mathbb{E}\|x_0\|^2 + 5a_2\frac{t^{2q-1}}{2q-1}\int_0^t \mathbb{E}\left\|f(s,x(s))\right\|^2 ds \\
&+5L_\sigma a_2\frac{t^{2q-1}}{2q-1}\int_0^t\left(\int_0^\tau \mathbb{E}\left\|\sigma(\theta,x(\theta))\right\|^2 d\theta\right)ds + 5MM_Ba_3c_1c_3 \\
&+5M\int_0^t \mathbb{E}\|u(s)\|^2 ds.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{P}(x)(t)\right\|^2 &\le 5a_1\mathbb{E}\|x_0\|^2 + 5a_2\frac{t^{2q-1}}{2q-1}\widetilde{N}\int_0^t (1+\mathbb{E}\left\|x(s)\right\|^2)ds \\
&+5L_\sigma a_2\frac{t^{2q-1}}{2q-1}\widetilde{L}\int_0^t\left(\int_0^\tau (1+\mathbb{E}\left\|x(\theta)\right\|^2)d\theta\right)ds + 5MM_Ba_3c_1c_3 \\
&+5M^2l^2\left[\mathbb{E}\|x_1\|^2 + a_1\mathbb{E}\|x_0\|^2 + a_2\frac{T^{2q-1}}{2q-1}\widetilde{N}\int_0^T (1+\mathbb{E}\|x(s)\|^2)ds.\right. \\
&\left.+\, L_\sigma a_2\frac{T^{2q-1}}{2q-1}\widetilde{L}\int_0^T\left(\int_0^\tau (1+\mathbb{E}\left\|x(\theta)\right\|^2)d\theta\right)ds + 5MM_Ba_3c_1c_3\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{P}(x)(t)\right\|^2 &\le 5M^2l^2\mathbb{E}\|x_1\|^2 + 5a_1\mathbb{E}\|x_0\|^2(1+M^2l^2) \\
&+5MM_Ba_3c_1c_3(1+M^2l^2) + 5a_2\frac{T^{2q-1}}{2q-1}\widetilde{N}(1+M^2l^2)\left(1+\|x\|_{L^2}^2\right) \\
&+5a_2L_\sigma\widetilde{L}\frac{T^{2q-1}}{2q-1}(1+M^2l^2)\left(1+T\|x\|_{L^2}^2\right).
\end{aligned}$$

It follows from from the above inequality and the condition **ii**. that there exists $\beta > 0$ such that

$$\mathbb{E}\|\mathcal{P}(x)(t)\|^2 \le \beta(1+\|x\|_{L^2}^2).$$

Therefore $\mathcal{P}$ maps C into itself. Secondly, we claim that $\mathcal{P}$ is a contraction mapping on C. For $x, y \in C$,

$$\begin{aligned}\mathbb{E}\left\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\right\|^2 &\leq 3\mathbb{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)(f(s,x(s))-f(s,y(s)))ds\right\|^2 \\ &+3\mathbb{E}\left\|\int_0^t (t-s)^{q-1}\left(\int_0^\tau (\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta)))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2 \\ &+3\mathbb{E}\left\|\int_0^t \varphi(t-s)\varphi^*(T,s)\psi^{-1}\left[\Upsilon(\phi(0),x_1;x)-\Upsilon(\phi(0),x_1;y)\right]\right\|^2.\end{aligned}$$

Using Lemma 2.5, in [84], condition **ii.**, and the above notations we get

$$\begin{aligned}\mathbb{E}\left\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\right\|^2 &\leq 3a_2\frac{T^{2q-1}}{2q-1}(1+M^2l^2T)\int_0^t \mathbb{E}\left\|f(s,x(s))-f(s,y(s))\right\|^2 ds \\ &+3a_2\frac{T^{2q-1}}{2q-1}L_\sigma(1+M^2l^2T)\int_0^t\left(\int_0^\tau \mathbb{E}\left\|(\sigma(\theta,x(\theta))-\sigma(\theta,y(\theta)))\right\|^2 d\theta\right)ds \\ &\leq 3a_2\frac{T^{2q-1}}{2q-1}(1+M^2l^2T)(N+LL_\sigma T)\int_0^t \mathbb{E}\|x(s)-y(s)\|^2 ds.\end{aligned}$$

It results that

$$\sup_{t\in[0,T]}\mathbb{E}\left\|\mathcal{P}(x)(t)-\mathcal{P}(y)(t)\right\|^2 \leq 3a_2\frac{T^{2q-1}}{2q-1}(1+M^2l^2T)(N+LL_\sigma T)\sup_{t\in[0,T]}\mathbb{E}\|x(t)-y(t)\|^2 ds.$$

Therefore we conclude that if $3a_2\frac{T^{2q-1}}{2q-1}(1+M^2l^2T)(N+LL_\sigma T)<1$, then $\mathcal{P}$ is contraction mapping on C, implies that the mapping $\mathcal{P}$ has a unique fixed point $x(.)\in C$. Hence we have

$$\begin{aligned}x(t) &= E_q(A(t)^q)x_0+\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+\int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &+\int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] \\ &+\int_0^t\left[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)\right]u(s)ds\end{aligned}$$

Thus $x(t)$ is the solution of the system (4.24), and it is easy to verify that $x(T)=x_1$. Further the control function $u(t)$ steers the system (4.24) from initial complete state $\phi(0)$ to x_1 on J. Hence the system (4.24) is globally relatively controllable on J.

□

Example

In this example, we apply the results obtained in the previous section for the following stochastic fractional dynamical systems with distributed delays in control which involves sequential Caputo derivative

$$
\begin{aligned}
&{}^cD^q x(t) = Ax(t) + \int_{-1}^{0} d_\tau B(t,\tau)u(t+\tau) + f(t,x(t)) + \sigma(t,x(t))\frac{d\omega(t)}{dt}; 0 < q < 1, t \in [0,T] \\
&x(0) = x_0
\end{aligned}
\tag{4.29}
$$

where

$$
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B(t,\tau) = \begin{pmatrix} e^\tau \cos(t) & e^\tau \sin(t) \\ -e^\tau \sin(t) & e^\tau \cos(t) \end{pmatrix}, \qquad u(t+\tau) = \begin{pmatrix} u_1(t+\tau) \\ u_2(t+\tau) \end{pmatrix},
$$

$$
f(t,x(t)) = \begin{pmatrix} x_1(t)\cos x_2(t) + 3x_2(t) \\ x_2(t)\sin x_1(t) + 2x_1(t) \end{pmatrix}, \qquad \sigma(t,x(t)) = \begin{pmatrix} (2t^2+1)x_1(t)e^{-t} & 0 \\ 0 & x_2(t)e^{-t} \end{pmatrix}.
$$

Let us introduce the variables $x_1(t) = x(t)$ and $x_2(t) = {}^cD^{\frac{q}{2}} x_1(t)$.Then

$$
{}^cD^{\frac{q}{2}} x_1(t) = {}^cD^{\frac{q}{2}} x(t) = x_2
$$

The Mittag-Leffler matrix of the given system is given by

$$
E_q(At^q) = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j t^{2jq}}{\Gamma(1+2jq)} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{(2j+1)q}}{\Gamma(1+(2j+1)q)} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j t^{(2j+1)q}}{\Gamma(1+(2j+1)q)} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{2jq}}{\Gamma(1+2jq)} \end{pmatrix}.
$$

Further

$$
E_{q,q}(A(T-(s-\tau))^q) = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{2jq}}{\Gamma[(1+2j)q]} & \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q}}{\Gamma[(1+j)2q]} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q}}{\Gamma[(1+j)2q]} & \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{2jq}}{\Gamma[(1+2j)q]} \end{pmatrix},
$$

and

$$
(T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) = \begin{pmatrix} \cos_q(t) & \sin_q(t) \\ -\sin_q(t) & \cos_q(t) \end{pmatrix},
$$

where $\cos_q(t)$ and $\sin_q(t)$ are given by

$$
\cos_q(t) = \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q-1}}{\Gamma[(1+2j)q]}
$$

$$\sin_q(t) = \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(j+1)2q-1}}{\Gamma[(1+j)2q]}$$

$$\begin{aligned} \varphi(T,s) &= \int_{-1}^{0} (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) d_\tau B_T(s-\tau,\tau) \\ &= \begin{pmatrix} \alpha(s) & \beta(s) \\ -\beta(s) & \alpha(s) \end{pmatrix} \end{aligned}$$

$$\alpha(s) = \int_{-1}^{0} \exp^{\tau} \left[\cos_q(T-(s-\tau))\cos(s-\tau) - \sin_q(T-(s-\tau))\sin(s-\tau)\right] d\tau$$

$$\beta(s) = \int_{-1}^{0} \exp^{\tau} \left[\sin_q(T-(s-\tau))\cos(s-\tau) - \cos_q(T-(s-\tau))\sin(s-\tau)\right] d\tau$$

By simple matrix calculation one can see that the controllability matrix

$$\begin{aligned} \psi_0^T &= \int_0^T \varphi(T,s)\varphi^*(T,s) ds \\ &= \int_0^T \left[\alpha^2(s) + \beta^2(s)\right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ds \end{aligned}$$

is positive definite for any $T > h$,Further the functions $f(t, x(t))$ and $\sigma(t, x(t))$ satisfies the hypothesis mentioned in Theorem 4.3.2, and so the fractional system (4.29) is globally relatively controllable on $[0, T]$.

Chapter 5

Controllability of fractional stochastic dynamical systems without delays in control

This chapter is concerned with the relative controllability for a class of dynamical control systems described by semilinear fractional stochastic differential equations with nonlocal conditions in Hilbert space. Sufficient conditions for relative controllability results are obtained using Schaefer's fixed point theorem.

5.1 Preliminaries and basic properties

In this section, we provide definitions, lemmas and notations necessary to establish our main results.Throughout this paper, we use the following notations. Let $(\Omega, \mathcal{F}, \mathbb{P})$be a complete probability space equipped with a normal filtration $\mathcal{F}_t, t \in J = [0, T]$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). We consider three real separable spaces X, E and U, and Q-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ with a linear bounded covariance operator Q such that $trQ < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}_{n\geq 1}$ on E, a bounded sequence of non-negative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda_n e_n, n = 1, 2, ...$ and a sequence $\{\beta_n\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle \beta_n(t), \qquad e \in E, t \in [0, T],$$

and $\mathcal{F}_t = \mathcal{F}_t^w$ where $\mathcal{F}_t^w$ is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the Banach space of all $\mathcal{F}_t-$ measurable square integrable random variables with values in the Hilbert space X. Let $\mathbb{E}(.)$ denote the expectation with respect to the measure $\mathbb{P}$. Let $C([0,T]; L^2(\mathcal{F}, X))$ be the Banach space of continuous maps from $[0,T]$ into $L^2(\mathcal{F}, X)$ satisfying $\sup\limits_{t\in J} \mathbb{E}\|x(t)\|^2 < \infty$. Let $H_2([0,T]; X)$ be the closed subspace of $C([0,T]; L^2(\mathcal{F}, X))$ consisting of all measurable and $\mathcal{F}_t$-adapted X-valued process $x \in C([0,T]; L^2(\mathcal{F}, X))$ endowed with the norm $\|x\|_{H_2} = (\sup\limits_{t\in J} \mathbb{E}\|x(t)\|_X^2)^{1/2}$. The purpose of this paper is to investigate the relative controllability for a class of semilinear stochastic fractional differential equation with nonlocal conditions of the form

$$\begin{aligned} {}^cD_t^\alpha x(t) + Ax(t) &= Bu(t) + f(t, x(t)) + \sigma(t, x(t))\tfrac{dw(t)}{dt}, \quad t \in J = [0,T], \\ x(0) + g(x) &= x_0, \end{aligned} \tag{5.1}$$

where $0 < \alpha < 1$; ${}^cD_t^\alpha$ denotes the Caputo fractional derivative operator of order α;$x(.)$ takes its values in the Hilbert space $X; A : D(A) \subset X \to X$ is the infinitesimal generator of an α-resolvent family $\{S_\alpha(t), t \geq 0\}$; the control function $u(.)$ is given in $L^2_{\mathcal{F}}([0,T], U)$ of admissible control functions, U is a Hilbert space. B is a bounded linear operator from U into X; $f : J \times X \to X$ and $\sigma : J \times X \to L_2^0$ are appropriate functions to be specified later; x_0 is a suitable initial random function independent of $w(t)$ and $g \in C(X, X)$ is a given function.
Let us recall the following known definitions. For more details see [35].

Definition 5.1.1. *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \qquad t > 0, \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 5.1.2. *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$${}^LD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds \qquad t > 0, n-1 < \alpha < n. \tag{5.2}$$

Definition 5.1.3. *The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$${}^cD^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^k(0) \right), \qquad t > 0, n-1 < \alpha < n \tag{5.3}$$

If $f(t) \in C^n[0,\infty)$, then

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^n(s)ds = I^{n-\alpha} f^n(s), \qquad t>0, n-1<\alpha<n$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$\mathcal{L}\{^cD^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \qquad n-1<\alpha<n.$$

Definition 5.1.4. *A two parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi i}\int_C \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \qquad \alpha,\beta \in C, Re(\alpha)>0,$$

where C is a contour which starts and ends at $-\infty$ end encircles the disc $|\mu| \leq |z|^{1/2}$ counter clockwise.

For short, $E_\alpha(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = cosh(z), E_2(-z^2) = cos(z)$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(wt^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - w}, \qquad Re(\lambda) > w^{\frac{1}{\alpha}}, w>0,$$

and for more details see [35].

Definition 5.1.5. *([91]). A closed and linear operator A is said to be sectorial if there are constants $w \in \mathbb{R}, \theta \in [\frac{\pi}{2}, \pi], M>0$ such that the following two conditions are satisfied:*

- $\rho(A) \subset \Sigma_{\theta,w} = \{\lambda \in C : \lambda \neq w, |\arg(\lambda - w)| < \theta\}$,
- $\|R(\lambda, A)\| \leq \dfrac{M}{|\lambda - w|}, \quad \lambda \in \Sigma_{\theta,w}$.

Definition 5.1.6. *Let A be a closed and linear operator with the domain D(A) defined in a Banach space X. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $w \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(X)$, where L(X) is a Banach space of all bounded linear operators from X into X and the corresponding norm is denoted by $\|.\|$, such that $\{\lambda^\alpha : Re\lambda > w\} \subset \rho(A)$ and*

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)x dt, \qquad Re\lambda > w, x \in X \tag{5.4}$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 5.1.7. *Let A be a closed and linear operator with the domain D(A) defined in a Banach space X and $\alpha > 0$ We say that A is the generator of a solution operator if there exist $w \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > w\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)xdt, \qquad Re\lambda > w, x \in X \tag{5.5}$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on α-resolvent family and solution operators, we refer the reader to [35]. Now, we give the definition of the mild solution of (5.1) based on the paper [85].

Definition 5.1.8. *([85]). A continuous stochastic process $x : J \to X$ is called a mild solution of (5.1) if the following conditions hold:*

(i) x(t) is measurable and $\mathcal{F}_t$-adapted.

(ii) $x(0) + g(x) = x_0$

(iii) x satisfies the following equation

$$x(t) = T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s)\left[Bu(s) + f(s, x(s))\right] ds + \int_0^t S_\alpha(t-s)\sigma(s, x(s))dw(s) \tag{5.6}$$

where $T_\alpha(t)E_{\alpha,1}(At^\alpha) = \dfrac{1}{2\pi i}\displaystyle\int_{\widehat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda$, $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \dfrac{1}{2\pi i}\displaystyle\int_{\widehat{B}_r} e^{\lambda t}\frac{1}{\lambda^\alpha - A}d\lambda$, *$\widehat{B}_r$ denotes the Bromwich path, $S_\alpha(t)$ is the α-resolvent family and $T_\alpha(t)$ is the solution operator generated by $-A$.*

Definition 5.1.9. *([89]). Let $x_T(x_0, u)$ be the state value of (5.1) at the terminal time T corresponding to the control u and the initial value x_0. Introduce the set*

$$R(T, x_0) = \{x(T) = x_T(x_0, u : u(.) \in L^2_{\mathcal{F}}([0, T], U)\}$$

which is called the reachable set of (5.1) at the terminal time T. Then the controlled system (1) is said to be relatively controllable at T if $R(T, x_0) = L^2(\Omega, \mathcal{F}_t, X)$.

Definition 5.1.10. *([89]). The control system (5.1) is said to be relatively approximately controllable at T if the closure set $\overline{R(T, x_0)} = L^2(\Omega, \mathcal{F}_t, X)$.*

To study the relative controllability of the fractional system (5.1), we will introduce the following equivalent conditions.

Lemma 5.1.1. *([55]). The following conditions are equivalent:*

(iv) The corresponding linear system with respect to (5.1) is relatively controllable on $[0, T]$.

(v) The corresponding linear system with respect to (5.1) is relatively approximately controllable on $[0, T]$.

(vi) The corresponding linear deterministic system with respect to (5.1) is relatively controllable on $[0, T]$.

The following lemma is required to define the control function. The reader can refer to [66] for the proof.

Lemma 5.1.2. *For any* $\tilde{x}_T \in L^2(\mathcal{F}_T, X)$ *there exists* $\tilde{g} \in L^2_{\mathcal{F}}(\Omega, L^2(0, T, L^2_0))$ *such that* $\tilde{x}_T = \mathbb{E}\tilde{x}_T + \int_0^T \tilde{g}(s)dw(s)$.

Now, we define the control function in the following form

$$\begin{aligned} u(t,x) &= B^* S^*_\alpha(T-s)\left((\psi_0^T)^{-1}\left[\mathbb{E}\tilde{x}_T - T_\alpha(T)(x_0 - g(x))\right] + \int_0^t (\psi_0^T)^{-1}\tilde{g}(s)dw(s)\right) \\ &\quad -B^* S^*_\alpha(T-t) + \int_0^t (\psi_0^T)^{-1} S_\alpha(T-s) f(s,x(s))ds \\ &\quad -B^* S^*_\alpha(T-t) + \int_0^t (\psi_0^T)^{-1} S_\alpha(T-s)\sigma(s,x(s))dw(s), \end{aligned}$$

where $\psi_0^T = \int_0^T S_\alpha(T-s)BB^*S^*_\alpha(T-s)$ is the controllability Gramian, B^* denotes the adjoint of B and $S^*_\alpha(t)$ the adjoint of $S_\alpha(t)$.

5.2 Relative controllability of semilinear fractional stochastic control systems in Hilbert spaces

In this section it will be shown that the system (5.1) is relatively (approximately) controllable under appropriate conditions.

Let us assume the following conditions:

(vii) The corresponding linear system with respect to (5.1) is relatively controllable

(viii) if $\alpha \in (0,1)$ and $A \in \mathcal{A}^{\alpha}(\theta_0, w_0)$ then for $x \in X$ and $t > 0$ we have $\|T_\alpha(t)\| \leq Me^{wt}$ and $\|S_\alpha(t)\| \leq Ce^{wt}(1+t^{\alpha-1}), w > w_0$. Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \qquad \textit{and} \qquad \|S_\alpha(t)\| \leq t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T = \sup_{0\leq t\leq T} \|T_\alpha(t)\|$, and $\widetilde{M}_S = \sup_{0\leq t\leq T} Ce^{wt}(1+t^{1-\alpha})$(for more details, see [91]).

(ix) $f \in C(J \times X, X), g \in C(X,X)$ and $\sigma \in C(J \times X, L_2^0)$. Moreover, there exists a constant $C_1 > 0$ such that for $x \in X$, $\mathbb{E}\|g(x)\|_X^2 \leq C_1$ and for $s \in J, x \in B_r$ there exist two continuous functions $\widetilde{L}_f, \widetilde{L}_\sigma : J \to (0,\infty)$ such that

$$\mathbb{E}\|f(t,x)\|_X^2 \leq \widetilde{L}_f(t)\phi(\mathbb{E}\|x\|_X^2), \qquad \mathbb{E}\|\sigma(t,x)\|_{L_2^0}^2 \leq \widetilde{L}\sigma f(t)\varphi(\mathbb{E}\|x\|_X^2),$$

where $\phi, \varphi : [0,\infty) \to (0,\infty)$ are a continuous nondecreasing functions with

$$\int_0^T \xi(s)ds \leq \int_c^\infty \frac{ds}{\phi(s)+\varphi(s)},$$

where $\xi(t) = \max\left\{\frac{5\widetilde{M}_S^2 T^\alpha}{\alpha} t^{\alpha-1}\eta\widetilde{L}_f(t), 5\widetilde{M}_S^2 t^{2(\alpha-1)}\eta\widetilde{L}_\sigma(t)\right\}$, $c = 5\widetilde{M}_T^2(\mathbb{E}\|x_0\|_X^2 + C_1)$, and $\eta = \left[1 + 3\widetilde{M}_S^4 \frac{T^{2\alpha}}{2\alpha-1}\|B\|^4 T^{2\alpha-1} l^2 \frac{T^\alpha}{\alpha}\right]$.

Our result is based on the following Schaefer's fixed point theorem.

Theorem 5.2.1. *Let K be a closed convex subset of a Banach space H such that $0 \in K$. Let $\mathcal{P} : K \to K$ be a completely continuous map. Then the set $\{x \in K; x = v\mathcal{P}x; 0 \leq v \leq 1\}$ is unbounded or $\mathcal{P}$ has a fixed point.*

Theorem 5.2.2. *The fractional stochastic system (5.1) is relatively controllable if (vii)-(ix) are satisfied.*

Proof. First, it will be show that the fractional stochastic system (5.1) has at least one mild solution on J. Let $\lambda : H_2 \to H_2$ be operator defined by

$$(\lambda x)(t) = T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s)\left[Bu(s,x) + f(s,x(s))\right]ds + \int_0^t S_\alpha(t-s)\sigma(s,x(s))dw(s)$$

In order to use the Schaefer's fixed point theorem, it will be shown that λ is a completely continuous operator. We note that the operator λ is well defined in H_2.

For the sake of convenience, we divide the proof into several steps.

Step 1. We prove that λ is continuous. Let $\{x^n\}_{n=0}^{\infty}$ be sequence in H_2 such that $x^n \to x$ in H_2. Since the function f, g, u and σ are continuous, $\lim_{n\to\infty} \mathbb{E}\|\lambda x^n(t) - \lambda x(t)\|_X^2 = 0$ in H_2 for every $t \in J$. This implies that the mapping λ is continuous on H_2.

Step 2. Next we prove that λ maps bounded sets into bounded sets in H_2. To prove that for any $r > 0$ there exists a $\gamma > 0$ such that for $x \in B_r = \{x \in H_2 : \mathbb{E}\|x\|_X^2 \leq r\}$, we have $\mathbb{E}\|\lambda x\|_X^2 \leq \gamma$. For any $x \in B_r, t \in J$, we have

$$\begin{aligned}
\mathbb{E}\|u(s,x)\|^2 \leq\ & 3\mathbb{E}\left\| B^* S_\alpha^*(T-t)\left((\psi_0^T)^{-1}\left[\mathbb{E}\widetilde{x}_T - T_\alpha(T)(x_0 - g(x))\right] + \int_0^t (\psi_0^T)^{-1}\widetilde{g}(s)dw(s)\right)\right\|^2 \\
& +3\mathbb{E}\left\| B^* S_\alpha^*(T-t) + \int_0^t (\psi_0^T)^{-1} S_\alpha(T-s) f(s,x(s))ds\right\|^2 \\
& +3\mathbb{E}\left\| B^* S_\alpha^*(T-t) + \int_0^t (\psi_0^T)^{-1} S_\alpha(T-s)\sigma(s,x(s))dw(s)\right\|^2 \\
\leq\ & 3\|B\|^2 T^{2\alpha-2}\widetilde{M}_S^2 l^2\left[\mathbb{E}\|\widetilde{x}_T\|^2 + \widetilde{M}_T^2 r + \widetilde{M}_T^2 C_1 + TL_{\widetilde{g}}\right] + 3\|B\|^2 T^{2\alpha-2}\widetilde{M}_S^4 l^2 \frac{T^\alpha}{\alpha} \\
& \times \int_0^t (T-s)^{\alpha-1}\widetilde{L}_f(s)ds + 3\|B\|^2 T^{2\alpha-2}\widetilde{M}_S^4 l^2 \varphi(r)\int_0^t (T-s)^{2\alpha-2}\widetilde{L}_\sigma(s)ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\|\lambda x(t)\|_X^2 \leq\ & 5\widetilde{M}_T^2 r + 5\widetilde{M}_T^2 C_1 + 15\widetilde{M}_S^4 \frac{T^\alpha}{\alpha}\|B\|^4 \frac{t^{2\alpha-1}}{2\alpha-1} T^{2\alpha-2} l^2\left[\mathbb{E}\|\widetilde{x}_T\|^2 + \widetilde{M}_T^2 r + \widetilde{M}_T^2 C_1 + TL_{\widetilde{g}}\right] \\
& +5\widetilde{M}_S^2 \frac{T^\alpha}{\alpha}\phi(r)\left[1 + 3\widetilde{M}_S^4 \frac{t^{2\alpha-1}}{2\alpha-1}\|B\|^4 T^{2\alpha-1} l^2 \frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{\alpha-1}\widetilde{L}_f(s)ds \\
& +5\widetilde{M}_S^2 \varphi(r)\left[1 + 3\widetilde{M}_S^4 \frac{t^{2\alpha-1}}{2\alpha-1}\|B\|^4 T^{2\alpha-1} l^2 \frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{2\alpha-2}\widetilde{L}_\sigma(s)ds \\
=\ & \gamma, \quad t \in J.
\end{aligned}$$

Step 3. We show that λ maps bounded sets into equicontinuous sets of B_r.

Let $0 < t_1 < t_2 \leq T$, for each $x \in B_r$, we have

$$\begin{aligned}
&\mathbb{E}\|\lambda x(t_2)-\lambda x(t_1)\|_X^2\\
&\leq 8\|T_\alpha(t_2)-T_\alpha(t_1)\|^2\mathbb{E}\|x_0\|_X^2+8\|T_\alpha(t_2)-T_\alpha(t_1)\|^2\mathbb{E}\|g(x)\|_X^2\\
&+8\mathbb{E}\left\|\int_0^{t_1}\left[S_\alpha(t_2-s)-S_\alpha(t_1-s)\right]f(s,x(s))ds\right\|_X^2+8\mathbb{E}\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)f(s,x(s))ds\right\|_X^2\\
&+8\mathbb{E}\left\|\int_0^{t_1}\left[S_\alpha(t_2-s)-S_\alpha(t_1-s)\right]\sigma(s,x(s))dw(s)\right\|_X^2+8\mathbb{E}\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)\sigma(s,x(s))dw(s)\right\|_X^2\\
&+8\mathbb{E}\left\|\int_0^{t_1}\left[S_\alpha(t_2-s)-S_\alpha(t_1-s)\right]Bu(s,x)ds\right\|_X^2+8\mathbb{E}\left\|\int_{t_1}^{t_2}S_\alpha(t_2-s)Bu(s,x)ds\right\|_X^2.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\mathbb{E}\|\lambda x(t_2)-\lambda x(t_1)\|_X^2\\
\leq\;& 8(r+C_1)\|T_\alpha(t_2)-T_\alpha(t_1)\|^2+\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|\,ds\\
&\times\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|\,\mathbb{E}\|f(s,x(s))\|_X^2ds\\
&+8\int_{t_1}^{t_2}\|S_\alpha(t_2-s)\|ds\int_{t_1}^{t_2}\|S_\alpha(t_2-s)\|\,\mathbb{E}\|f(s,x(s))\|_X^2ds\\
&+\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|\,ds\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|\,\|B\|^2\mathbb{E}\|u(s,x)\|_X^2ds\\
&+8\int_{t_1}^{t_2}\|S_\alpha(t_2-s)\|\,ds\int_{t_1}^{t_2}\|S_\alpha(t_2-s)\|\,\|B\|^2\mathbb{E}\|u(s,x)\|_X^2ds\\
&+8\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|^2\,\mathbb{E}\|\sigma(s,x(s))\|_{L_2^0}^2ds+8\int_{t_1}^{t_2}\|S_\alpha(t_2-s)\|^2\,\mathbb{E}\|\sigma(s,x(s))\|_{L_2^0}^2ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\|\lambda x(t_2)-\lambda x(t_1)\|_X^2\\
\leq\;& 8(r+C_1)\|T_\alpha(t_2)-T_\alpha(t_1)\|^2+8\phi(r)\widetilde{\eta}\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|\,ds\\
&\times\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|\,\widetilde{L}_f(s)ds\\
&+8\widetilde{M}_S^2\frac{(t_2-t_1)^\alpha}{\alpha}\phi(r)\left[1+3\widetilde{M}_S^4\frac{t^{2\alpha-1}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\widetilde{L}_f(s)ds\\
&+8\varphi(r)\widetilde{\eta}\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|^2\,\widetilde{L}_\sigma(s)ds\\
&+8\widetilde{M}_S^2\varphi(r)\left[1+3\widetilde{M}_S^4\frac{t^{2\alpha-1}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2}\widetilde{L}_\sigma(s)ds,
\end{aligned}$$

where $\widetilde{\eta}$ is a positive constant depending only on α, l, B, T and $\widetilde{M}_S$. Since $T_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous, $\|T_\alpha(t_2)-T_\alpha(t_1)\|\to 0$ and $\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|\to 0$ as $t_1\to t_2$. Thus, from the above inequality we have $\lim\limits_{t_1\to t_2}\mathbb{E}\|\lambda x(t_2)-\lambda x(t_1)\|_X^2=0$. Thus, the set $\{\lambda x, x\in B_r\}$

is equicontinuous. Finally, combining Step 1 to 3 with Ascoli's theorem, we conclude that the operator λ is compact.

Step 4. Next, we show that the set
$N = \{x \in H_2 \text{such that} x = q\lambda x(t) \text{for some} 0 < q < 1\}$ is bounded. Let $x \in N$ then $x(t) = q\lambda x(t)$ for some $0 < q < 1$. Then for each $t \in J$ we have

$$x(t) = q\left(T_\alpha(t)(x_0 - g(x)) + \int_0^t S_\alpha(t-s)[Bu(s,x) + f(s,x(s))]ds + \int_0^t S_\alpha(t-s)\sigma(s,x(s))dw(s)\right),$$

which implies that
$\mathbb{E}\|x(t)\|_X^2$

$$\begin{aligned}
&\leq 5\|T_\alpha\|^2\mathbb{E}\|x_0\|_X^2 + 5\|T_\alpha\|^2\mathbb{E}\|g(x)\|_X^2 + 5\int_0^t \|S_\alpha(t-s)\|\,ds \int_0^t \|S_\alpha(t-s)\|\,\mathbb{E}\|f(s,x(s))\|_X^2 ds \\
&\quad +5\int_0^t \|S_\alpha(t-s)\|\,ds\int_0^t \|S_\alpha(t-s)\|\,\mathbb{E}\|Bu(s,x)\|^2 ds + 5\int_0^t \|S_\alpha(t-s)\|^2\,\mathbb{E}\|\sigma(s,x(s))\|_{L_2^0}^2 ds \\
&\leq 5\widetilde{M}_T^2\mathbb{E}\|x_0\|_X^2 + 5\widetilde{M}_T^2 C_1 \\
&\quad +5\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\left[1 + 3\widetilde{M}_S^4\frac{t^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{\alpha-1}\widetilde{L}_f(s)\phi(\mathbb{E}\|x(s)\|_X^2)ds \\
&\quad +5\widetilde{M}_S^2\left[1 + 3\widetilde{M}_S^4\frac{t^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{2\alpha-2}\widetilde{L}_\sigma(s)\varphi(\mathbb{E}\|x(s)\|_X^2)ds
\end{aligned}$$

Consider the function $\mu(t)$ defined by
$\mu(t) = \sup\{\mathbb{E}\|x(s)\|_X^2; 0 \leq s \leq t\}$, $0 \leq t \leq T$.

$$\begin{aligned}
\mu(t) &\leq 5\widetilde{M}_T^2[\mathbb{E}\|x_0\|_X^2 + C_1] \\
&\quad +5\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\left[1 + 3\widetilde{M}_S^4\frac{t^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{\alpha-1}\widetilde{L}_f(s)\phi(\mu(s))ds \\
&\quad +5\widetilde{M}_S^2\left[1 + 3\widetilde{M}_S^4\frac{t^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]\int_0^t (t-s)^{2\alpha-2}\widetilde{L}_\sigma(s)\varphi(\mu(s))ds.
\end{aligned}$$

Denoting by $\nu(t)$ the right hand side of the last inequality, we have $\nu(0) = c = 5\widetilde{M}_T^2[\mathbb{E}\|x_0\|_X^2 + C_1]$, $\mu(t) \leq \nu(t)$, $t \in J$ Moreover,

$$\begin{aligned}
\nu'(t) &= 5\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\left[1 + 3\widetilde{M}_S^4\frac{t^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]t^{\alpha-1}\widetilde{L}_f(t)\phi(\mu(t)) \\
&\quad +5\widetilde{M}_S^2\left[1 + 3\widetilde{M}_S^4\frac{t^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]t^{2\alpha-2}\widetilde{L}_\sigma(t)\varphi(\mu(t)) \\
&\leq 5\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\left[1 + 3\widetilde{M}_S^4\frac{t^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]t^{\alpha-1}\widetilde{L}_f(t)\phi(\nu(t)) \\
&\quad +5\widetilde{M}_S^2\left[1 + 3\widetilde{M}_S^4\frac{t^{2\alpha}}{2\alpha-1}\|B\|^4T^{2\alpha-1}l^2\frac{T^\alpha}{\alpha}\right]t^{2\alpha-2}\widetilde{L}_\sigma(t)\varphi(\nu(t)),
\end{aligned}$$

or equivalently by (ix), we have

$$\int_{\nu(0)}^{\nu(t)} \frac{ds}{\phi(s)+\varphi(s)} \leq \int_0^T \xi(s)ds < \int_c^\infty \frac{ds}{\phi(s)+\varphi(s)}, \quad 0 \leq t \leq T$$

This inequality implies that there is a constant k such that $\nu(t) \leq k$, $t \in J$, and hence, $\mu(t) \leq k$. Furthermore, we get $\|x(t)\|2^{\leq}\mu(t) \leq \nu(t) \leq k$ $t \in J$. By the Schaefer's fixed point theorem, we deduce that λ has a fixed point $x(t)$ on J, with $x(T) = x_T$, which is a mild solution of (5.1). That means it is along this trajectory that the solution of (5.1) will be steered by u from x_0 to x_T. That completes the proof.

□

In order to study the approximate controllability for the fractional stochastic control system (5.1), we introduce the approximate controllability of its linear part

$$\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t) + (Bu)(t) \\ x(0) + g(x) &= x_0 \end{aligned} \tag{5.7}$$

For this purpose, we need to introduce the relevant operator

$$\begin{aligned} \psi_0^T &= \int_0^T S_\alpha(T-s)BB^*S_\alpha^*(T-s) \\ R(q, \psi_0^T) &= (qI + \psi_0^T)^{-1}, \end{aligned}$$

where $q > 0$ and ψ_0^T is a linear bounded operator.

We assume the following additional conditions

(x) $qR(q, \psi_0^T) \to 0$ as $q \to 0^+$ in the strong operator topology.

(xi) $f(t, x) : J \times X \to X$ and $\sigma(t, X) : J \times X \to L_2^0$ are bounded for $t \in J$ and $x \in X$.

Remark 5.2.1. *From [66] (Theorem 2) the condition (x) is equivalent to the fact that the linear fractional control system (5.7) is approximately controllable on* $J := [0, T]$. *Hence, by Lemma 5.1.1, (vii) is equivalent to* $qR(q, \psi_0^T) := (qI + \psi_0^T)^{-1} \to 0$ *as* $q \to 0^+$. *Moreover,(vii) can be replaced by the following more verifiable criterion:*

There exists some positive constant $\widetilde{\gamma}$ *such that* $\langle \psi_s^T z, z\rangle \leq \widetilde{\gamma}\|z\|^2$ *for all* $z \in X$.

Theorem 5.2.3. *Under the conditions (vii)-(xi), and if* $S_\alpha(t)$ *is a compact, then system (5.1) is relatively approximately controllable on* $[0, T]$.

Proof. For all $q > 0$ define the control function as

$$\begin{aligned}
u^q(t,x) &= B^* S_\alpha^*(T-s)\left((qI+\psi_0^T)^{-1}[\mathbb{E}\widetilde{x}_T - T_\alpha(T)(x_0-g(x))] + \int_0^t (qI+\psi_0^T)^{-1}\widetilde{g}(s)dw(s)\right) \\
&\quad -B^* S_\alpha^*(T-t)\int_0^t (qI+\psi_0^T)^{-1} S_\alpha(T-s) f(s,x(s))ds \\
&\quad -B^* S_\alpha^*(T-t)\int_0^t (qI+\psi_0^T)^{-1} S_\alpha(T-s)\sigma(s,x(s))dw(s),
\end{aligned} \tag{5.8}$$

and the operator $\lambda_q : H_2 \to H_2$ as follows

$$(\lambda_q x)(t) = T_\alpha(t)(x_0-g(x)) + \int_0^t S_\alpha(t-s)\left[Bu^q(s,x) + f(s,x(s))\right]ds + \int_0^t S_\alpha(t-s)\sigma(s,x(s))dw(s). \tag{5.9}$$

Replacing l^2 with $\frac{1}{q^2}$ and using the same procedure as in the proof of Theorem 5.2.2, one can prove that λ_q has a unique fixed point x_q.

By using the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned}
x_q(T) &= \widetilde{x}_T - q(qI+\psi)^{-1}[\mathbb{E}\widetilde{x}_T - T_\alpha(T)(x_0-g(x))] + q\int_0^T (qI+\psi_s^T)^{-1} S_\alpha(T-s) f(s,x_q(s))ds \\
&\quad +q\int_0^T (qI+\psi_s^T)^{-1}\left[S_\alpha(T-s)\sigma(s,x_q(s)) - \widetilde{g}(s)\right]dw(s).
\end{aligned} \tag{5.10}$$

It follows from the properties of f and σ that $\|f(s,x_q(s))\|^2 + \|\sigma(s,x_q(s))\|^2 \le L_1$. Then there is a subsequence denoted by $\{f(s,x_q(s)), \sigma(s,x_q(s))\}$ weakly converging to say $\{f(s),\sigma(s)\}$. Thus from the above equation, we have

$$\begin{aligned}
\mathbb{E}\|x_q(T)-\widetilde{x}_T\|^2 &\le 6\left\|q(qI+\psi_0^T)^{-1}[\mathbb{E}\widetilde{x}_T - T_\alpha(T)(x_0-g(x))]\right\|^2 + 6\mathbb{E}\left(\int_0^T \left\|q(qI+\psi_s^T)^{-1}\widetilde{g}(s)\right\|_{L_2^0}^2 ds\right) \\
&\quad +6\mathbb{E}\left(\int_0^T \left\|q(qI+\psi_0^T)^{-1}\right\| \left\|S_\alpha(T-s)(f(s,x_q(s))-f(s))\right\| ds\right)^2 \\
&\quad +6\mathbb{E}\left(\int_0^T \left\|q(qI+\psi_s^T)^{-1} S_\alpha(T-s)f(s)\right\| ds\right)^2 \\
&\quad +6\mathbb{E}\left(\int_0^T \left\|q(qI+\psi_s^T)^{-1}\right\| \left\|S_\alpha(T-s)(\sigma(s,x_q(s))-\sigma(s))\right\|_{L_2^0}^2 ds\right) \\
&\quad +6\mathbb{E}\left(\int_0^T \left\|q(qI+\psi_s^T)^{-1} S_\alpha(T-s)\sigma(s)\right\|_{L_2^0}^2 ds\right).
\end{aligned}$$

On the other hand, by assumption (x) for all $0 \leq s \leq T$, the operator $q(qI + \psi_s^T)^{-1} \to 0$ strongly as $q \to 0^+$, and moreover $q(qI + \psi_s^T)^{-1} \leq 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $S_\alpha(t)$ we obtain $\mathbb{E}\|x_q(T) - \widetilde{x}_T\|^2 \to 0$ as $q \to 0^+$. This gives the approximate controllability of (5.1) . Hence the proof is complete.

$\square$

Conclusion

The main goals of this thesis is to investigate the subject controllability of Fractional stochastic dynamical systems.
The third chapter contains some controllability results for stochastic systems. The first result shows that the Banach fixed point theorem can effectively be used in control problems to obtain sufficient conditions. Here it is proved that under some hypotheses together with the assumption that the linear stochastic system is completely controllable, the semilinear stochastic system iscomplete ly controllable.
Another result is obtained via the generalized implicit function theorem. It presents the result that under some natural conditions the non-linear stochastic system islocal ly null controllable provided that its linearized system is controllable.

In fourth chapter of this thesis we have study some controllability results of fractional stochastic dynamical systems with delays in control, we have study global relative controllability for the linear and nonlinear fractional stochastic dynamical systems with multiple delays in control function. The result shows that the Banach fixed point theorem can effectively be used to study the control problems for establishing sufficient conditions. Here it is proved that under some hypotheses together with the assumption that the linear stochastic system is globally relatively controllable, the nonlinear fractional stochastic system is also globally relatively controllable, and we have study also global relative controllability of linear and nonlinear stochastic fractional dynamical systems with distributed delays in control. With Lipschitz and linear growth conditions, some sufficient conditions have been presented for global relative controllability of stochastic nonlinear systems in finite dimensional space.

In the fifth chapter of this thesis we have study some controllability results of fractional stochastic dynamical systems without delays in control, we have study the relative controllability for a class of dynamical control systems described by semilinear fractional stochastic differential equations with nonlocal conditions in Hilbert space. A new set of sufficient conditions for the relative controllability of the considered system have been formulated and proved. As the differential inclusion system is considered as a generalization of the system described by differential equations, it should be pointed out that under some suitable conditions on f and σ, one can establish the relative controllability of fractional stochastic differential inclusions with nonlocal conditions by adapting the techniques and ideas established in this paper and suitably introducing the technique of single valued maps defined in [13]. This is one of our future goals.

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