Université de Saida- Docteur Moulay Tahar
Faculté des Sciences

THÈSE

Présentée pour obtenir le diplôme de

Doctorat 3ème Cycle

Spécialité: Analyse stochastique et Statistique des processus
Filière : Mathématiques

Présentée par

Meryem Chaouche

Thème:

Une Contribution à l'étude de certaines classes d'inclusions différentielles stochastiques non-linéaires

Soutenue le 22 Mars 2022 devant le jury composé de:

Président:
Mr. Fethallah Ouhbi TEBBOUNE (Professeur à l'université de Saida- Docteur Moulay Tahar)
Examinateurs:
Mr. Abdeldjebbar KANDOUCI (Professeur à l'université de Saida- Docteur Moulay Tahar)
Mr. Abdelkader GHERIBALLAH (Professeur à l'université Djillali Liabès Sidi Bel Abbes)
Mr. Mohammed Kadi ATTOUCH (Professeur à l'université Djillali Liabès Sidi Bel Abbes)
Directeur de thèse:
Mr. Toufik. GUENDOUZI (Professeur à l'université de Saida- Docteur Moulay Tahar)

Dedication

To my teachers

To my family

To my friends

Acknowledgements

To almighty Allah, who granted me patience, knowledge, and health throughout the realization of my research.

I would like to express my gratitude to my supervisor Prof. Toufik GUENDOUZI for the effective feedback, comments, and remarks throughout the realization of this Doctorat's thesis. Immense thanks to him for the great efforts did.

I would like to express my deepest gratitude to prof Fethallah Ouhbi TEBBOUNE for his guidance and insightful comments on conducting this piece of work.

I warmly thank Prof. Abdeldjebbar KANDOUCI for his help and guidance during all my study journey and I am honored to have him as a jury member.

I would like to thank prof GHERIBALLAH Abdelkader, prof Mohammed Kadi ATTOUCH who, despite their obligations, have kindly accepted to read and evaluate this piece of work.

My sincere thanks also to all the members of laboratory of stochastic models, statistic and applications, who provided me an opportunity to join their team, and who gave access to the laboratory and research facilities. Without their precious support it would not be possible to conduct this research.

Contents

General Introduction

Differential equations and inclusions with fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, mechanic, biology, ecology, aerodynamic, polymer rheology and many others. Fractional differential equations or inclusions also serve as an excellent tool for describing the memory and genetic properties of different materials and processes. As a consequence there was an intensive development of the theory of differential equations and inclusions of fractional order. One can see the monographs of Abbas et al. [52], Kilbas et al. [6], Miller and Ross [33], Podlubny [7], Zhou [64], the survey of Agarwal et al [49] [51] and the references therein. Many articles have been devoted to the existence of solutions for fractional differential equations and inclusions, for example, [12][39][44][53][70]. As for the study of the existence of mild solutions for fractional differential inclusions, please see [26][9][27].

The theory of impulsive differential equations or inclusions has also attracted increasing attention because of its wide applicability in science and engineering. Impulsive differential inclusions arising from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occurs. Such processes are naturally seen in biology, physics, medical fields, etc. Due to their significance, many authors have been established the solvability of impulsive differential inclusions. For the general theory and applications of such equations we refer the interested reader to Benchohra et al. [40], Graef et al. [30].

The deterministic systems often fluctuate due to noise, which is random or at least appears to be so. Therefor, we must move from deterministic problems to stochastic ones. As the generalization of classic impulsive differential and partial differential inclusions, impulsive stochastic differential and partial differential inclusions have attracted the researchers great interest, and some works have done on the existence results of mild solutions for these equation (see [8] [28] and references therein). Recently, attempts were made to combine fractional derivatives and stochastic differential inclusions. One can see [59][60][61][71] and references therein.

On the other hand, fractional Brownian motion has become an object of intense study, due to its interesting properties and applications in various scientific areas including telecommunication, turbulence and finance. The fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a suitable generalization of the classical Brownian motion, but exhibits lon-rang dependence, self similarity and which has stationary increments. When $H = \frac{1}{2}$ the fBm coincide with the classical Brownian motion. When $H \neq \frac{1}{2}$, the fBm is neither a semimartingale nor a Markov process. For additional details on the fractional Brownian motion, we refer the reader to [14]. A general theory for the infinite dimensional stochastic differential equations driven by a fractional Brownian motion has begun to receive attention by various researchers see e.g. [13][66]. The existence, uniqueness, stability and qualitative analysis of the mild solutions of stochastic differential equations driven by fractional Brownian motion with infinite delay have been studied by many authors (see [57] and references therein). Recently, Ren et al. [67]proved the existence and uniqueness of mild solution for a class of impulsive neutral stochastic functional integro-differential equations with infinite delay driven by standard cylindrical Wiener process and an independent cylindrical fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$ in the Hilbert space. Boudaoui et al. [2] proved the existence of mild solutions to stochastic impulsive evolution equations with time delay, driven by fractional Brownian motion and Krasnoselski Schaefer type fixed point theorem. Ren et al. [69] proved the existence and uniqueness of the integral solution for a class of non-densely defined impulsive neutral stochastic functional differential equation driven by an independent cylindrical fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$ in the Hilbert space. However, there are very few contributions regarding the existence of solutions to stochastic differential inclusions driven by fractional Brownian motion [5] [7]. An existence result of mild solutions for a first-order impulsive semilinear stochastic functional differential inclusions driven by a fractional Brownian motion with infinite delay has been proved by Boudaoui et al. [5].

This thesis is divide into four chapter. In the first one we recal some basic definitions and properties of different processes in Hilbert space, and we study the integration with respect to this processes at the end of this chapter we will present the definitions and some properties of semigroup and sectorial operator. Secondly we will generalize derivatives and integrals that have been studied in calculus to a more general setting, we start with some history of fractional calculus, we recall some definitions of how to define derivatives and integrals of arbitrary order. The third chapter is devoted to study the stochastic differential inclusion. The principal aim of this chapter is to proof the existence of mild solution for stochastic differential inclusion driven by cylindrical sub fractional Brownian motion. In the first section we give the definition of phases space, next in the second section we introduce some basic definitions and results of multivalued maps in the third section, we give the solution of the stochastic differential inclusion driven by cylindrical Wiener process at the end of this chapter we study the existence of mild solution for stochastic differential inclusion with Hilfer fractional derivative The last chapter is the heart of our present study, First we start with an introduction next in section two we give some basic definitions to establish our main results and in section three we proof the existence of mild solution of our problem at the end an example is given to illustrate our results.

Chapter 1

Stochastic Calculus In Hilbert Space

Stochastic calculus is the branch of mathematics that operates on stochastic processes. It follows a consistent theory of integration to be defined for integrals of stochastic process with respect to stochastic processes. It is used to model systems that behave randomly.

Many stochastic processes are based on functions which are continuous, but nowhere differentiable.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and $\mathscr{K}$ be a real separable Hilbert space with the norm and scalar product denoted by $\| \, . \, \|_{\mathscr{K}}$ and $\langle ., . \rangle_{\mathscr{K}}$. We will alway assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is complete, i.e., that $\mathscr{F}$ contains all subsets A of Ω with $\mathbb{P}$-outer measure zero,

$$\mathbb{P}^*(A) = \inf\{\mathbb{P}(F) : A \subset F \subset \mathscr{F}\} = 0.$$

1.1 Cylindrical Gaussian Random Variables

We introduce cylindrical standard Gaussian random variables and Hilbert-space-valued Gaussian random variables.

Definition 1.1.1. [36] *We say that $\tilde{X}$ is a cylindrical standard Gaussian random variable on $\mathscr{K}$ if $\tilde{X} : \mathscr{K} \longrightarrow L^2(\Omega, \mathscr{F}, \mathbb{P})$ satisfies the following conditions.*

1. *The mapping $\tilde{X}$ is linear.*
2. *For an arbitrary $k \in \mathscr{K}$, $\tilde{X}(k)$ is a Gaussian random variable with mean zero and variance $\| k \|^2_{\mathscr{K}}$.*
3. *If $k, k' \in \mathscr{K}$ are orthogonal, i.e., $\langle k, k' \rangle_{\mathscr{K}} = 0$, then the random variables $\tilde{X}(k)$ and $\tilde{X}(k')$ are independent.*

Note that if $\{f_j\}_{j=1}^{\infty}$ is an orthonormal basis (ONB) in $\mathscr{K}$, then $\{\tilde{X}(f_j)\}_{j=1}^{\infty}$ is a sequence of independent Gaussian random variables with mean zero and variance one. By linearity of the mapping $\tilde{X} : \mathscr{K} \longrightarrow L^2(\Omega, \mathscr{F}, \mathbb{P})$, we can represent $\tilde{X}$ as

$$\tilde{X}(k) = \sum_{j=1}^{\infty} \langle k, f_j \rangle_{\mathscr{K}} \tilde{X}(f_j),$$

with the series convergent P-a.s by Kolmogorov's three-series theorem ([45], theorem 22.3). In order to produce a $\mathscr{K}$-valued Gaussian random variable, we proceed as follows.
Let $\mathscr{L}(\mathscr{K})$ the space of linear and bounded operator. We denote by $\mathscr{L}_1(\mathscr{K})$ the space of trace- class operators on $\mathscr{K}$,

$$\mathscr{L}_1(\mathscr{K}) = \{L \in \mathscr{L}(\mathscr{K}) : \tau(L) := tr((LL^*)^{\frac{1}{2}}) < \infty\},$$

where the trace of the operator $[L] = (LL^*)^{\frac{1}{2}}$ is defined by

$$tr([L]) = \sum_{j=1}^{\infty} \langle [L] f_j, f_j \rangle_{\mathscr{K}},$$

for an ONB $\{f_j\}_{j=1}^{\infty} \subset \mathscr{K}$. It is well known [48] that $tr([L])$ is independent of the choice of the ONB and that $\mathscr{L}_1(\mathscr{K})$ equipped with the trace norm τ is a Banach space.

Let $Q : \mathscr{K} \longrightarrow \mathscr{K}$ be a symmetric nonnegative definite trace-class operator. Assume that $X : \mathscr{K} \longrightarrow L^2(\Omega, \mathscr{F}, \mathbb{P})$ satisfies the following conditions:

1. The mapping X is linear.
2. For an arbitrary $k \in \mathscr{K}$, $X(k)$ is a Gaussian random variable with mean zero.
3. For arbitrary $k, k' \in \mathscr{K}$, $E(X(k)X(k')) = \langle Qk, k' \rangle_{\mathscr{K}}$.

Let $\{f_j\}_{j=1}^{\infty}$ be an ONB in $\mathscr{K}$ diagonalizing Q, and let the eigenvalues corresponding to the eigenvectors f_j be denote λ_j, so that $Qf_j = \lambda_j f_j$. We define

$$X(\omega) = \sum_{j=1}^{\infty} X(f_j)(\omega) f_j.$$

Since $\sum_{j=1}^{\infty} \lambda_j < \infty$, the series converges in $L^2((\Omega, \mathscr{F}, \mathbb{P}), \mathscr{H})$ and hence $\mathbb{P}$-a.s.

Definition 1.1.2. [36] *We call $X : \Omega \longrightarrow \mathscr{K}$ defined above a $\mathscr{K}$-valued Gaussian random variable with covariance Q.*

Definition 1.1.3. [36] *Let $\mathscr{K}$ be a separable Hilbert space. The measure $\mathbb{P}\circ X^{-1}$ induced by a $\mathscr{K}$-valued Gaussian random variable X with the covariance Q on the measurable Hilbert space $(\mathscr{K},\mathscr{B}(\mathscr{K}))$ is called a Gaussian measure with covariance Q on $\mathscr{K}$, where $\mathscr{B}(\mathscr{K}))$ denote the Borel σ-field on $\mathscr{H}$.*

1.2 Cylindrical and Q-Wiener Process

1.2.1 Cylindrical Wiener Process

Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ be a filtered probability space, $C([0,T],\mathscr{H})$ be the Banach space of $\mathscr{H}$-valued continuous functions on $[0,T]$ and $\mathscr{K}$ be a real separable Hilbert space. We will always assume that the filtration $\mathscr{F}_t$ satisfies the usual conditions

1. $\mathscr{F}_0$ contains all $A\in\mathscr{F}$ such that $\mathbb{P}(A)=0$.

2. $\mathscr{F}_t=\bigcap_{s>t}\mathscr{F}_s$.

Definition 1.2.1. [36] *A $\mathscr{K}$-valued stochastic process $\{X_t\}_{t\geq 0}$ defined on probability space $(\Omega,\mathscr{F},\mathbb{P})$ is called Gaussian if for any positive integer n and $t_1,\ldots,t_n\geq 0$, $X_{t_1},\ldots,X_{t_n}$ is a $\mathscr{K}^n$-valued Gaussian random variable.*

A standard cylindrical Wiener process can now be introduced by using the concept of a cylindrical random variable.

Definition 1.2.2. [36] *We call a family $\{\tilde{W}_t\}_{t\geq 0}$ defined on a filtered probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ a cylindrical Wiener process in a Hilbert space $\mathscr{K}$ if:*

1. *For an arbitrary $t\geq 0$, the mapping $\tilde{W}_t:\mathscr{K}\longrightarrow L^2(\Omega,\mathscr{F},\mathbb{P})$ is linear.*

2. *For an arbitrary $k\in\mathscr{K}$, $\tilde{W}_t(k)$ is an $\mathscr{F}_t$-Brownian motion.*

3. *For arbitrary $k,k'\in\mathscr{K}$ and $t\geq 0$, $E(\tilde{W}_t(k)\tilde{W}_t(k'))=t\langle k,k'\rangle_{\mathscr{K}}$.*

For every $t>0$, $\tilde{W}_t/\sqrt{t}$ is a standard cylindrical Gaussian random variable, so that for any $k\in\mathscr{K}$, $\tilde{W}_t(k)$ can be represented as $\mathbb{P}$-a.s. convergent series

$$\tilde{W}_t(k)=\sum_{j=1}^{\infty}\langle k,f_j\rangle_K\tilde{W}_t(f_j),$$

where $\{f_j\}_{j=1}^{\infty}$ is an ONB in $\mathscr{K}$.

1.2.2 Q-Wiener Process

Let Q be a nonnegative definite symmetric trace-class operator on $\mathscr{K}$, then a $\mathscr{K}$-valued Q-Wiener process can be defined.

Definition 1.2.3. [36] *Let Q be a nonnegative definite symmetric trace-class operator on a separable Hilbert space $\mathscr{K}$, $\{f_j\}_{j=1}^{\infty}$ be an OBN in $\mathscr{K}$ diagonalizing Q, and let the corresponding eigenvalues be $\{\lambda_j\}_{j=1}^{\infty}$. Let $\{w_j(t)\}_{t>0}$, $j = 1, 2, ...,$ be a sequence of independent Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. The process*

$$W_t = \sum_{j=1}^{\infty} \lambda_j^{\frac{1}{2}} w_j(t) f_j, \tag{1.1}$$

is called a Q-Wiener process in $\mathscr{K}$.

We can assume that the Brownian motion $w_j(t)$ are continuous. Then the series 1.1 converges in $L^2(\Omega, C([0, T], \mathscr{K}))$ for every interval $[0, T]$. Therefore, the $\mathscr{K}$-valued Q-Wiener process can be assumed to be continuous. We denote

$$W_t(k) = \sum_{j=1}^{\infty} \lambda_j^{\frac{1}{2}} w_j(t) \langle f_j, k\rangle_{\mathscr{K}},$$

for any $k \in \mathscr{K}$, with the series converging in $L^2(\Omega, C([0, T], \mathscr{K}))$ on every interval $[0, T]$.

Remark 1.2.1. *A stronger convergence result can be obtained for the series* 1.1. *Since*

$$\begin{aligned} P\Big(\sup_{0\leq t\leq T} \Big\| \sum_{j=m}^{n} \lambda_j^{\frac{1}{2}} w_j(t) f_j \Big\|_{\mathscr{K}} > \epsilon\Big) &\leq \frac{1}{\epsilon^2} E\Big\| \sum_{j=m}^{n} \lambda_j^{\frac{1}{2}} w_j(T) f_j \Big\|_{\mathscr{K}}^2 \\ &= \frac{T}{\epsilon^2} \sum_{j=m}^{n} \lambda_j \longrightarrow 0, \end{aligned}$$

with $m \leq n$ and $m, n \longrightarrow \infty$, the series 1.1 *converges in probability on* $[0, T]$ *and hence, by the Lévy- Itô-Nisio theorem* ([43], *Theorem 2.4), it also converges P-a.s. uniformly on* $[0, T]$.

Some basic properties of a Q-Wiener process are summarized in the next theorem.

Theorem 1.2.1. [36] *A $\mathscr{K}$-valued Q-Wiener process $\{W_t\}_{t\geq 0}$ has the following properties:*

1. $W_0 = 0$.
2. *W_t has continuous trajectories in $\mathscr{K}$.*
3. *W_t has independent increments.*

4. *W_t is a Gaussian process with the covariance operator Q, i.e., for any $k, k' \in \mathcal{K}$ and $s, t \geq 0$,*

$$E(W_t(k)W_s(k')) = (t \wedge s)\langle Qk, k' \rangle_{\mathcal{K}}.$$

5. *For an arbitrary $k \in \mathcal{K}$, the law $\mathcal{L}((W_t - W_s)(k)) \sim \mathcal{N}(0, (t-s)\langle Qk, k \rangle_{\mathcal{K}})$.*

1.3 Cylindrical and Q-Fractional Brownian Motion

Fractional Brownian motion is a family of Gaussian processes that are indexed by the Hurst parameter $H \in (0,1)$. In a finite dimensional Euclidean space these processes were introduced by Kolomogorov [1] and some properties of these processes were given by Mandelbrot and Van Ness [14].

The fractional Brownian motion, for $H \neq \frac{1}{2}$ is not a semi martingale it is necessary to define a stochastic calculus, these processes have a self similarity in probability law and for $H \in (\frac{1}{2}, 1)$, a long range dependence property described by the covariance function.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathcal{K}$ be a real separable Hilbert space with the norm and scalar product denoted by $\| . \|_{\mathcal{K}}$.

1.3.1 Cylindrical Fractional Brownian Motion

Definition 1.3.1. [54] *A $\mathcal{K}$-valued Gaussian process $(B_t^H(k), t \geq 0, k \in \mathcal{K})$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be cylindrical fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ if:*

1. *$E(\langle k, B^H(t) \rangle_{\mathcal{K}}) = 0$ for all $t \in \mathbb{R}^+$ and $k \in \mathcal{K}$.*

2. *$E\langle k, B^H(s) \rangle_{\mathcal{K}} \langle k', B^H(t) \rangle_{\mathcal{K}} = \frac{1}{2}\langle k, k' \rangle_{\mathcal{K}}(t^{2H} + s^{2H} - |t-s|^{2H})$ for all $s, t \in \mathbb{R}^+$ and $k, k' \in \mathcal{K}$.*

Remark 1.3.1. *For $H = \frac{1}{2}$ this definition is the usual one for a standard cylindrical Wiener process.*

Definition 1.3.2. [54] *Let Q be a nonnegative, self adjoint bounded linear operator that is not nuclear, then a cylindrical fractional Brownian motion is defined by the formal series,*

$$B^H(t) = \sum_{n=1}^{\infty} e_n \beta_n^H(t) = \sum_{n=1}^{\infty} e_n \langle e_n, B^H(t) \rangle,$$

where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis in the Hilbert space $Q^{\frac{1}{2}}\mathcal{K}$ and $\{B_n^H(t)\}_{n=1}^{\infty}$ for all $t \in \mathbb{R}^+$ is a sequence of independent, real-valued standard fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$.

1.3.2 Q-Fractional Brownian Motion

If Q is a non negative, definite symmetric trace class operator on $\mathcal{K}$, then a $\mathcal{K}$-valued Q-fractional Brownian motion can be defined

Definition 1.3.3. [54] *Let$\mathcal{K}$ be a separable Hilbert space and Q be a non negative, nuclear, self adjoint operator on $\mathcal{K}$. A continuous, zero mean, $\mathcal{K}$-valued Gaussian process $(B_Q^H(t), t \in \mathbb{R}^+)$ is said to be Q-fractional Brownian motion with Hurst parameter $H \in (0,1)$ and associated with the covariance operator Q if:*

1. $E\langle k, B_Q^H(t)\rangle_{\mathcal{K}} = 0$, *for all* $k \in \mathcal{K}$ *and* $t \in \mathbb{R}^+$.
2. $E\langle k, B_Q^H(s)\rangle_{\mathcal{K}} \langle k', B_Q^H(t)\rangle_{\mathcal{K}} = \frac{1}{2}\langle Qk, k'\rangle_{\mathcal{K}}(t^{2H} + s^{2H} - |t-s|^{2H})$ *for any* $s, t \in \mathbb{R}^+$ *and* $k, k' \in \mathcal{K}$.
3. $(B_Q^H(t), t \geq 0)$ *has* $\mathcal{K}$*-valued continuous sample path* $\mathbb{P}$*.a.s.*

Definition 1.3.4. [54] *Let Q be a non negative definite symetric-class operator on a separable Hilbert space $\mathcal{K}$, $\{e_n\}_{n=1}^{\infty}$ be an ONB in $\mathcal{K}$ diagonalizing Q and the corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Let $\beta_n^H(t)$ be a sequence of real, independent standard fractional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ for $n = 1, 2, ...$ and $t \in \mathbb{R}$. The process*

$$W_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n,$$

is called a Q-fractional Brownian motion in $\mathcal{K}$.

Remark 1.3.2. *Proposition2.2* [54] *ensures the existence of fractional Brownian motion and the existence of cylindrical (i.e. $Q = Id$) fractional Brownian motion for $H > \frac{1}{2}$, but the arguments are valid for arbitrary Q.*
However B_Q^H takes values in the large Hilbert space $\mathcal{K}_1$, where $\mathcal{K} \hookrightarrow \mathcal{K}_1$ and the embedding is the Hilbert-Schmidt operator.

Remark 1.3.3. *If Q is a nuclear operator, then a cylindrical fractional Brownian motion is a Q-fractional Brownian motion.*

1.4 Cylindrical and Q-Sub-Fractional Brownian motion

As an extension of Brownian motion, recently, Bojdecki et al [55] introduced and studied a rather special class of self-similar Gaussian process. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called Sub-fractional Brownian motion.

1.4.1 Cylindrical Sub Fractional Brownian Motion

Definition 1.4.1. *Let $\mathscr{K}$ be a separable Hilbert space. A continuous, zero mean, $\mathscr{K}$-valued Gaussian process $\left(S_I^H(t), t \geq 0\right)$ is said to be cylindrical sub-fractional Brownian motion with Hurst parameter $H \in (0,1)$ if his covariance is given by*

$$E\langle k, S_I^H(s)\rangle\langle k', S_I^H(t)\rangle = \left\langle k, k^{'}\right\rangle \left[s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]\right] \text{ for all } s,t \in \mathbb{R}^+ \text{and } k, k^{'} \in \mathscr{K}.$$

Definition 1.4.2. *Let Q be a non negative, self adjoint bounded linear operator that is not nuclear, then a cylindrical sub fractional Brownian motion is defined by the formal series*

$$S_I^H(t) = \sum_{n=1}^{\infty} S_n^H(t) e_n \quad t \geq 0;$$

where $\{S_n^H(t)\}_{n=1}^{\infty}$ is a sequence of independent, real valued standard sub fractional Brownian motion with Hurst parameter $H \in (0,1)$ and $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis in the Hilbert space $\mathscr{K}$.

1.4.2 Q-Sub Fractional Brownian Motion

Let $(U, \|.\|_U, \langle.\rangle_U)$ and $(\mathscr{K} K, \|.\|_{\mathscr{K}}, \langle.\rangle_{\mathscr{K}})$ be two separable Hilbert space. Let $\mathscr{L}(\mathscr{K}, U)$ denote the space of all bounded linear operator from $\mathscr{K}$ to U and $Q \in \mathscr{L}(\mathscr{K}, U)$ be a non negative self adjoint operator.

Definition 1.4.3. *Let $\mathscr{K}$ be a separable Hilbert space and Q be a non negative self adjoint operator on $\mathscr{K}$. A continuous, zero mean $\mathscr{K}$-valued Gaussian process $\left(S_Q^H(t), t \geq 0\right)$ is said to be Q-sub fractional Brownian motion with Hurst parameter $H \in (0,1)$ associated with the covariance operator Q if:*

$$E\left\langle k, S_Q^H(s)\right\rangle\left\langle k', S_Q^H(t)\right\rangle = \left\langle Qk, k^{'}\right\rangle \left[s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]\right] \text{ for all } s,t \in \mathbb{R}^+.$$

Definition 1.4.4. *Let $Q \in \mathscr{L}(\mathscr{K}, U)$ be a non negative, self adjoint trace class operator on a separable Hilbert space $\mathscr{K}$, $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis in the Hilbert space $\mathscr{K}$ diagonalizing Q and the corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Let $\{S_n^H(t)\}_{n=1}^{\infty}$ be a sequence of real independent standard sub fractional Brownian motion, the process*

$$S_Q^H(t) = \sum_{n=1}^{\infty} S_n^H(t) Q^{\frac{1}{2}} e_n = \sum_{n=1}^{\infty} S_n^H(t)\sqrt{\lambda_n} e_n;$$

is called a $\mathscr{K}$-valued Q sub fractional Brownian motion.

1.5 Stochastic Integral

1.5.1 Stochastic integral with respect to cylindrical Wiener process

We will introduce the concept of Itô's stochastic integral with respect to a Q-Wiener process and with respect to a cylindrical Wiener process simultaneously.

Let $\mathscr{K}$ and $\mathscr{H}$ be a separable Hilbert space, and Q be either a symetric nonnegative definite trace-class operator on $\mathscr{K}$ or $Q = I_k$, the identity operator on $\mathscr{K}$.
In case Q is trace-class operator, we will always assume that its all eigenvalues λ_j, $j = 1,...$; otherwise we can start with the Hilbert space $ker(Q)^{\perp}$ instead of $\mathscr{K}$.
The associated eigenvalues forming an ONB in $\mathscr{K}$ will be denoted by f_k. Then the space $\mathscr{K}_Q = Q^{\frac{1}{2}}\mathscr{K}$ equipped with the scalar product.

$$\langle u, v\rangle_{\mathscr{K}_Q} = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle u, f_j\rangle_{\mathscr{K}} \langle v, f_j\rangle_{\mathscr{K}}$$

is a separable Hilbert space with an ONB $\{\lambda_j^{\frac{1}{2}} f_j\}_{j=1}^{\infty}$.
If $\mathscr{H}_1$, $\mathscr{H}_2$ are two real separable Hilbert spaces with $\{e_i\}_{i=1}^{\infty}$ an ONB in $\mathscr{H}_1$, then the space of Hilbert-Schmidt operators from $\mathscr{H}_1$ to $\mathscr{H}_2$ defined as

$$\mathscr{L}_2(\mathscr{H}_1, \mathscr{H}_2) = \{L \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2) : \sum_{i=1}^{\infty} \| Le_i \|_{\mathscr{H}_2}^2 < \infty\}.$$

It is well known (see [62]) that $\mathscr{L}_2(\mathscr{H}_1, \mathscr{H}_2)$ equipped with the norm

$$\| L \|_{\mathscr{L}_2(\mathscr{H}_1, \mathscr{H}_2)} = \left(\sum_{i=1}^{\infty} \| Le_i \|_{\mathscr{H}_2}^2 \right)^{\frac{1}{2}},$$

is a Hilbert space . Since the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable, the space $\mathscr{L}_2(\mathscr{H}_1, \mathscr{H}_2)$ is also separable, as Hilbert-Schmidt operators are limits of sequences of finite-dimensional linear operators.

Consider $\mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$ the space of Hilbert Schmidt operators from $\mathscr{K}_Q$ to $\mathscr{H}$. If $\{e_j\}_{j=1}^{\infty}$ is an ONB in $\mathscr{H}$, then the Hilbert-Schmidt norm of an operator $L \in \mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$

is given by

$$\begin{aligned}
\| L \|^2_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})} &= \sum_{i,j=1}^{\infty} \langle L(\lambda_j^{\frac{1}{2}} f_j), e_i \rangle^2_{\mathscr{H}} \\
&= \sum_{i,j=1}^{\infty} \langle L(Q^{\frac{1}{2}} f_j), e_i \rangle^2_{\mathscr{H}} \\
&= \| LQ^{\frac{1}{2}} \|^2_{\mathscr{L}_2(\mathscr{K},\mathscr{H})} \\
&= tr((LQ^{\frac{1}{2}})(LQ^{\frac{1}{2}})^*).
\end{aligned}$$

The scalar product between two operators $L, M \in \mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$ is defined by

$$\langle L, M \rangle_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})} = tr((LQ^{\frac{1}{2}})(MQ^{\frac{1}{2}})^*). \tag{1.2}$$

Since the Hilbert space $\mathscr{K}_Q$ and $\mathscr{H}$ are separable, the space $\mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$ is also separable.
Let $L \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, if $k \in \mathscr{K}_Q$, then

$$k = \sum_{j=1}^{\infty} \langle k, \lambda_j^{\frac{1}{2}} f_j \rangle_{\mathscr{K}_Q} \lambda_j^{\frac{1}{2}} f_j,$$

and L considered as an operator from $\mathscr{K}_Q$ to $\mathscr{H}$ defined as

$$Lk = \sum_{j=1}^{\infty} \langle k, \lambda_j^{\frac{1}{2}} f_j \rangle \lambda_j^{\frac{1}{2}} L f_j,$$

has a finite Hilbert-Schmidt norm, since

$$\begin{aligned}
\| L \|^2_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})} &= \sum_{j=1}^{\infty} \| L(\lambda_j^{\frac{1}{2}} f_j) \|^2_{\mathscr{H}} \\
&= \sum_{j=1}^{\infty} \lambda_j \| L(f_j) \|^2_{\mathscr{H}} \\
&\leq \| L \|^2_{\mathscr{L}(\mathscr{K},\mathscr{H})} \, tr(Q).
\end{aligned}$$

Thus, $\mathscr{L}(\mathscr{K}, \mathscr{H}) \subset \mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$. If $L, M \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, formula 1.2 reduce to

$$\| L \|^2_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})} = tr(LQL^*)$$

and

$$\langle L, M \rangle_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})} = tr(LQM^*),$$

allowing for separation of $Q^{\frac{1}{2}}$ and L^*. This is usually exploited in calculations where $L \in \mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$ is approximated with a sequence $L_n \in \mathscr{L}(\mathscr{K}, \mathscr{H})$.
The space $\mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$ consists of linear operators $L : \mathscr{K} \longrightarrow \mathscr{H}$ not necessarily bounded, with domain $\mathscr{D}(L) \supset Q^{\frac{1}{2}}\mathscr{K}$, and such that $tr((LQ^{\frac{1}{2}})(LQ^{\frac{1}{2}})^*)$ is finite.
If $Q = I_{\mathscr{K}}$ then $\mathscr{K}_Q = \mathscr{K}$, we denote that the space $\mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$ contains genuinely bounded linear operators from $\mathscr{K}$ to $\mathscr{H}$.

Stochastic Itô Integral for Elementary Processes

Let $\varepsilon(\mathscr{L}(\mathscr{K}, \mathscr{H}))$ denote the class of $\mathscr{L}(\mathscr{K}, \mathscr{H})$-valued elementary processes adapted to the filtration $\{\mathscr{F}_t\}_{t \leq T}$ that are of the form

$$\Phi(t, \omega) = \phi(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j(\omega)\mathbb{1}_{(t_j, t_{j+1}]}(t),$$

where $0 \leq t_1 \leq ... \leq t_n = T$, and $\phi, \phi_j,\ j = 0, 1, ..., n-1$ are respectively $\mathscr{F}_0$-measurable and $\mathscr{F}_{t_j}$-measurable $\mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$-valued random variable such that $\phi(\omega), \phi_j(\omega) \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, $j = 0, 1, ..., n-1$ (recall that $\mathscr{L}(\mathscr{K}, \mathscr{H}) \subset \mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$).
Note that if $Q = \mathbb{1}_{\mathscr{H}}$, then the random variables ϕ_j are in fact $\mathscr{L}_2(\mathscr{K}, \mathscr{H})$-valued.

We shall say that an elementary process $\Phi \in \varepsilon(\mathscr{L}(\mathscr{K}, \mathscr{H}))$ is bounded if it is bounded in $\mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$.

We define the Itô stochastic integral with respect to a Q-Wiener process W_t by

$$\int_0^t \Phi(s)dW_s = \sum_{j=0}^{n-1} \phi_j(W_{t_{j+1}\wedge t} - W_{t_j \wedge t}),$$

for $t \in [0, T]$. The term $\phi\omega_0$ is neglected since $\mathbb{P}(W_0 = 0) = 1$. This stochastic integral is an $\mathscr{H}$-valued stochastic process.

We define the Itô cylindrical stochastic integral of an elementary process $\Phi \in \varepsilon(\mathscr{L}(\mathscr{K}, \mathscr{H}))$ with respect to a cylindrical Winer process $\tilde{W}$ by

$$\Big(\int_0^t \Phi(s)d\tilde{W}_s\Big)(h) = \sum_{j=0}^{n-1} \Big(\tilde{W}_{t_{j+1}\wedge t}(\phi_j^*(h)) - \tilde{W}_{t_j\wedge t}(\phi_j^*(h))\Big),$$

for $t \in [0, T]$ and $h \in \mathscr{H}$. The following proposition states Itô's isometry, which is essential in furthering the construction of the stochastic integral.

Property 1.5.1. [36] *For a bounded elementary process* $\Phi \in \varepsilon(\mathscr{L}(\mathscr{K}, \mathscr{H}))$

$$E \parallel \int_0^t \Phi(s)dW_s \parallel^2_{\mathscr{H}} = E\int_0^t \parallel \Phi(s) \parallel_{\mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})} < \infty, \tag{1.3}$$

for $t \in [0, T]$.

We have the following counterpart of 1.3 for the Itô cylindrical stochastic integral of a bounded elementary process $\Phi \in \varepsilon(\mathscr{L}(\mathscr{K}, \mathscr{H}))$*:*

$$E\Big(\Big(\int_0^t \phi(s)d\tilde{W}_s\Big)(h)\Big)^2 = \int_0^t E \parallel \Phi^*(s)(h) \parallel_{\mathscr{K}}^2 ds < \infty.$$

Let $\Lambda_2(\mathscr{K}_Q, \mathscr{H})$ be a class of $\mathscr{L}_2(\mathscr{K}_Q, \mathscr{H})$-valued processes measurable mapping from $\Big([0,T]\times\Omega, \mathscr{B}[0,T]\bigotimes\mathscr{F}\Big)$ to $\Big(\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H}), \mathscr{B}(\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H}))\Big)$ adapted to the filtration $\{\mathscr{F}_t\}_{t\leq T}$ (thus can be replaced with $\mathscr{F}_T$) and satisfying the condition

$$E\int_0^T \parallel \Phi(t) \parallel_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})}^2 dt < \infty.$$

We note that $\Lambda_2(\mathscr{K}_Q, \mathscr{H})$ equipped with the norm

$$\parallel \Phi \parallel_{\Lambda_2(\mathscr{K}_Q,\mathscr{H})} = \Big(E\int_0^t \parallel \Phi(t) \parallel_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})}^2 dt\Big)^{\frac{1}{2}},$$

is a Hilbert space.

Property 1.5.2. [36] *If* $\Phi \in \Lambda_2(\mathscr{K}_Q, \mathscr{H})$, *then there exists a sequence of bounded elementary processes* $\Phi_n \in \varepsilon(\mathscr{L}(\mathscr{K}, \mathscr{H}))$ *approximating* Φ *in* $\Lambda_2(\mathscr{K}_Q, \mathscr{H})$, *i.e*

$$\parallel \Phi_n - \Phi \parallel_{\Lambda_2(\mathscr{K}_Q,\mathscr{H})}^2 = E\int_0^t \parallel \Phi_n(t) - \Phi(t) \parallel_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})}^2 dt \longrightarrow 0.$$

Stochastic Itô Integral with respect to a Q-Wiener process

We are ready to extend the definition of the Itô stochastic integral with respect to a Q-Wiener process to adapted stochastic processes $\Phi(s)$ satisfying the condition

$$E\int_0^T \parallel \Phi(s) \parallel_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})}^2 ds < \infty,$$

which will be further relaxed to the condition

$$\mathbb{P}\Big(\int_0^T \parallel \Phi(s) \parallel_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})}^2 ds < \infty\Big) = 1.$$

Definition 1.5.1. [36] *The stochastic integral of a process* $\Phi \in \Lambda_2(\mathscr{K}_Q, \mathscr{H})$ *with respect to a* $\mathscr{H}$*-valued Q-Wiener process* W_t *is the unique isometric linear extension of the mapping*

$$\Phi(.) \longrightarrow \int_0^T \Phi(s)dW_s,$$

from the class of bounded elementary processes to $L^2(\Omega,\mathcal{H})$, *to a mapping from* $\Lambda_2(\mathcal{K}_Q,\mathcal{H})$ *to* $L^2(\Omega,\mathcal{H})$ *such that the image of* $\Phi(t)=\phi\mathbb{1}_{\{0\}}+\sum_{j=0}^{n-1}\phi_j\mathbb{1}_{(t_j,t_{j+1}]}(t)$ *the stochastic integral process* $\int_0^t\Phi(s)dW_s$, $0\le t\le T$, *for* $\Phi\in\Lambda_2(\mathcal{K}_Q,\mathcal{H})$ *and given by*

$$\int_0^t\Phi(s)dW_s=\int_0^T\Phi(s)\mathbb{1}_{[0,t]}(s)dW_s.$$

Theorem 1.5.1. [36] *The stochastic integral* $\Phi\longrightarrow\int_0^{\cdot}\Phi(s)dW_s$ *with respect to a* $\mathcal{H}$*-valued Q-Wiener process* W_t *is an isometry between* $\Lambda_2(\mathcal{K}_Q,\mathcal{H})$ *and the space of continuous square-integrable martingales* $\mathcal{M}_T^2(\mathcal{H})$.

$$E\parallel\int_0^t\Phi(s)dW_s\parallel_{\mathcal{H}}^2=E\int_0^t\parallel\Phi(s)\parallel_{\mathcal{L}_2(\mathcal{K}_Q,\mathcal{H})}^2ds<\infty,$$

for $t\in[0,T]$.

The quadratic variation process of the stochastic integral process $\int_0^t\Phi(s)dW_s$ and the increasing process related to $\parallel\int_0^t\Phi(s)dW_s\parallel_{\mathcal{H}}^2$ are given by

$$\left\langle\left\langle\int_0^{\cdot}\Phi(s)dW_s\right\rangle\right\rangle_t=\int_0^t\left(\Phi(s)Q^{\frac{1}{2}}\right)\left(\Phi(s)Q^{\frac{1}{2}}\right)^*ds$$

and

$$\begin{aligned}\left\langle\int_0^{\cdot}\Phi(s)dW_s\right\rangle_t&=\int_0^t tr\Big((\Phi(s)Q^{\frac{1}{2}})(\Phi(s)Q^{\frac{1}{2}})^*\Big)ds\\&=\int_0^t\parallel\Phi(s)\parallel_{\mathcal{L}_2(\mathcal{K}_Q,\mathcal{H})}^2ds.\end{aligned}$$

Remark 1.5.1. *For* $\Phi\in\Lambda_2(\mathcal{K}_Q,\mathcal{H})$ *such that* $\Phi(s)\in\mathcal{L}(\mathcal{K},\mathcal{H})$, *the quadratic variation process of the stochastic integral process* $\int_0^t\Phi(s)dW_s$ *and the increasing process related to* $\parallel\int_0^t\Phi(s)dW_s\parallel_{\mathcal{H}}^2$ *simplify to*

$$\left\langle\left\langle\int_0^{\cdot}\Phi(s)dW_s\right\rangle\right\rangle_t=\int_0^t\Phi(s)Q\Phi(s)^*ds$$

and

$$\left\langle\int_0^{\cdot}\Phi(s)dW_s\right\rangle_t=\int_0^t tr\Big((\Phi(s)Q\Phi(s)^*\Big)ds.$$

Let $\mathcal{P}(\mathcal{K}_Q,\mathcal{H})$ denote the class of $\mathcal{L}_2(\mathcal{K}_Q,\mathcal{H})$-valued stochastic processes adapted to the filtration $\{\mathcal{F}_t\}_{t\le T}$, measurable as mapping from $\Big([0,T]\times\Omega,\mathcal{B}[0,T]\otimes\mathcal{F}_t\Big)$ to

$\Big(\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H}),\mathscr{B}(\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H}))\Big)$, and satisfying the condition

$$\mathbb{P}\Big\{\int_0^T \parallel \Phi(t) \parallel^2_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})} dt < \infty\Big\} = 1.$$

Obviously, $\Lambda_2(\mathscr{K}_Q,\mathscr{H}) \subset \mathscr{P}(\mathscr{K}_Q,\mathscr{H})$.

The processes from $\mathscr{P}(\mathscr{K}_Q,\mathscr{H})$ can be approximated in a suitable way by processes from $\Lambda_2(\mathscr{K}_Q,\mathscr{H})$ and, in fact, by bounded elementary processes from $\varepsilon(\mathscr{L}(\mathscr{K},\mathscr{H}))$.

Lemma 1.5.1. [36] *Let $\Phi \in \mathscr{P}(\mathscr{K}_Q,\mathscr{H})$, then there exists a sequence of bounded processes $\Phi_n \in \varepsilon(\mathscr{L}(\mathscr{K},\mathscr{H})) \subset \Lambda_2(\mathscr{K}_Q,\mathscr{H})$ such that*

$$\int_0^T \parallel \Phi(t,\omega) - \Phi_n(t,\omega) \parallel^2_{\mathscr{L}_2(\mathscr{K}_Q,\mathscr{H})} dt \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty, \tag{1.4}$$

in probability and $\mathbb{P}$-a.s.

We can define a class of $\mathscr{H}$-valued elementary processes $\epsilon(\mathscr{H})$ adapted to the filtration $\{\mathscr{F}_t\}_{t\leq T}$ as all processes of the form

$$\Psi(t,\omega) = \psi(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \psi_j(\omega)\mathbb{1}_{(t_j,t_{j+1}]}(t),$$

where $0 = t_0 \leq t_1 \leq ... \leq t_n = T$, ψ is $\mathscr{F}_0$-measurable and $\psi_j (j = 0,1,...,n-1)$ are $\mathscr{F}_{t_j}$-measurable $\mathscr{H}$-valued random variables.

Lemma 1.5.2. [36] *Let $\Psi(t)$, $t \leq T$, be an $\mathscr{H}$-valued, $\mathscr{F}_t$-adapted stochastic process satisfying the condition*

$$\mathbb{P}\Big\{\int_0^T \parallel \Psi(t) \parallel_{\mathscr{H}} dt < \infty\Big\} = 1,$$

then there exists a sequence of bounded elementary processes $\Psi_n \in \epsilon(\mathscr{H})$ such that

$$\int_0^T \parallel \Psi(t,\omega) - \Psi_n(t,\omega) \parallel_{\mathscr{H}} dt \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty,$$

in probability and almost surely.

We will need the following estimate useful estimate.

Lemma 1.5.3. [36] *Let $\Phi \subset \Lambda_2(\mathscr{K}_Q,\mathscr{H})$. Then for arbitrary $\delta > 0$ and $n > 0$*

$$\mathbb{P}\Big(\sup_{t\leq T} \parallel \int_0^t \Phi(s) dW_s \parallel_{\mathscr{H}} > \delta\Big) \leq \frac{n}{\delta^2} + \mathbb{P}\Big(\int_0^T \parallel \Phi(s) \parallel^2_{\Lambda_2(\mathscr{K}_Q,\mathscr{H})} ds > n\Big).$$

We are ready to conclude the construction of the stochastic integral now

Lemma 1.5.4. *Let Φ_n be a sequence in $\Lambda_2(\mathcal{K}_Q,\mathcal{H})$ approximating a process $\Phi \in \mathcal{P}(\mathcal{K}_Q,\mathcal{H})$ in the sens of 1.4, i.e.,*

$$\mathbb{P}\Big(\int_0^T \parallel \Phi_n(t,\omega)-\Phi(t,\omega) \parallel^2_{\mathcal{L}_2(\mathcal{K}_Q,\mathcal{H})} dt > 0\Big) \longrightarrow 0.$$

Then, there exist an $\mathcal{H}$-valued $\mathcal{F}_T$-measurable random, denote by $\int_0^T \Phi(t)dW_t$, such that

$$\int_0^T \Phi_n(t)dW_t \longrightarrow \int_0^T \Phi(t)dW_t,$$

in probability. The random variable $\int_0^T \Phi(t)dW_t$ does not depend (up to stochastic equivalence) on the choice of the approximating sequence.

Definition 1.5.2. *The $\mathcal{H}$-valued random variable $\int_0^T \Phi(t)dW_t$ defined in 1.5.4 is called the stochastic integral of a process in $\mathcal{P}(\mathcal{K}_Q,\mathcal{H})$ with respect to a Q-Wiener process. For* $0 \le t \le T$*, we define an $\mathcal{H}$-valued stochastic integral process $\int_0^t \Phi(s)dW_s$ by*

$$\int_0^t \Phi(s)dW_s = \int_0^T \Phi(s)\mathbb{1}_{[0,T]}(s)dW_s.$$

The stochastic integral process for $\Phi \in \mathcal{P}(\mathcal{K}_Q,\mathcal{H})$ may not be a martingale, but it is a local martingale.

Definition 1.5.3. *A stochastic process $\{M_t\}_{t\le T}$, adapted to a filtration $\mathcal{F}_t$, with values in a separable Hilbert space $\mathcal{H}$ is called a local martingale if there exists a sequence of increasing stopping time τ_n, with $\mathbb{P}(\lim_{n\longrightarrow\infty} \tau_n = T) = 1$, such that for every n, $M_{t\wedge\tau_n}$ is a uniformly integrable martingale.*

1.5.2 Stochastic integral with respect to cylindrical Fractional Brownian Motion

A Hilbert- valued stochastic integration is defined for an integrator that is a cylindrical fractional Brownian motion in a Hilbert space. Since the integrator is not semi martingale for the fractional Brownian motion considered, a different definition of integration is required. The approach to integration has an analogue with Skorokhod integrals for fractional Brownian motion by the basic use of derivative of some functionals of Brownian motion.

Let $K_H(t,s)$ be the kernel function, for $0 \le s \le t \le T$

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\big(\frac{1}{2}-H\big)\int_s^t (u-s)^{H-\frac{3}{2}}\big(1-\big(\frac{s}{u}\big)^{\frac{1}{2}-H}\big);$$

where $c_H = \left[\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right]^{\frac{1}{2}}$ and $H \in (0,1)$.
If $H \in (\frac{1}{2},1)$, then K_H has a simpler form as

$$K_H(t,s) = c_H(H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t \left(u-s\right)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du.$$

A definition of stochastic integral of deterministic $\mathcal{K}$-valued function with respect to a scalar fractional Brownian motion $(B(t), t \geq 0)$ is described.
Let $K_H^* : \varepsilon \longrightarrow L^2([0,T],\mathcal{K})$ be the linear operator given by

$$K_H^*\varphi(t) = \varphi(t)K_H(T,t) + \int_t^T \left(\varphi(s)-\varphi(t)\right)\frac{\partial K_H(s,t)}{\partial s}ds; \tag{1.5}$$

for $\varphi \in \varepsilon$, where ε is the linear space of $\mathcal{K}$-valued step function on $[0,T]$.
For $\varphi \in \varepsilon$,

$$\varphi(t) = \sum_{i=1}^{n-1} x_i \mathbb{1}_{[t_i,t_{i+1}]}(t),$$

where $x_i \in K$, $i \in \{1,...,n-1\}$ and $0 = t_1 < t_2 < ... < t_n = T$.
We define

$$\int_0^T \varphi dB = \sum_{i=1}^{n-1} x_i(B_{t_{i+1}} - B_{t_i}). \tag{1.6}$$

It follows directly that

$$E\left\|\int_0^T \varphi dB\right\|^2 = | K_H^*\varphi |^2_{L^2([0,T],\mathcal{K})}. \tag{1.7}$$

Let $(\mathcal{H}, \|.\|_{\mathcal{H}}, \langle .,. \rangle_{\mathcal{H}})$ be the Hilbert space obtained by the completion of the pre Hilbert space ε with the inner product $\langle \varphi, \psi \rangle_{\mathcal{H}} := \langle K_H^*\varphi, K_H^*\psi \rangle_{L^2([0,T],\mathcal{K})}$, for $\varphi, \psi \in \varepsilon$.
The stochastic integral 1.6 is extended to $\varphi \in \mathcal{H}$ by the isometry 1.7.
Thus $\mathcal{H}$ is the space of integrable functions. If $H \in (\frac{1}{2},1)$ then it is easily verified that $\widetilde{\mathcal{H}} \subset \mathcal{H}$, where $\widetilde{\mathcal{H}}$ is the Banach space of Borel measurable functions with the norm $\|.\|_{\widetilde{\mathcal{H}}}$ given by

$$\| \varphi \|^2_{\widetilde{\mathcal{H}}} = \int_0^T \int_0^T |\varphi(u)||\varphi(v)|\phi(u-v)dudv;$$

where $\phi(u) = H(2H-1)|u|^{2H-2}$ and it is elementary to verify that $L^p([0,t],\mathcal{K}) \subset \widetilde{\mathcal{H}}$ for $p > \frac{1}{H}$ then

$$E\|\int_0^T \varphi dB\|^2 = \int_0^T \int_0^T \langle \varphi(u), \varphi(v) \rangle \phi(u-v)dudv.$$

If $H \in (0,\frac{1}{2})$, then the space of integral functions is smaller than for $H \in (\frac{1}{2},1)$.
Associated with $(B(t), t \geq 0)$ is a standard cylindrical Wiener process $(W(t), t \geq 0)$ in $\mathcal{K}$

such that formally $B(t) = K_H(W(t))$.
For $x \in K \smallsetminus \{0\}$, let $B_x(t) = \langle B(t), x\rangle$, it is elementary to verify from 1.6 that there is a scalar Wiener process $(w_x(t), t \geq 0)$ such that

$$B_x(t) = \langle B(t), x\rangle = \int_0^t K_H(t,s)dw_x(s);$$

for $t \in \mathbb{R}^+$. Furthermore, $w_x(t) = B_x\left((K_H^*)^{-1}\mathbb{1}_{[0,t]}\right)$, where K_H^* is given by 1.5.
Now we define the stochastic integral $\int_0^T GdB$ for an operator-valued function $G : [0,T] \longrightarrow \mathcal{L}(\mathcal{K})$ is a $\mathcal{K}$-valued random variable.

Definition 1.5.4. [*11*] *Let* $G : [0,T] \longrightarrow \mathcal{L}(\mathcal{K})$, $(e_n, n \in \mathbb{N})$ *be a complete orthonormal basis in K,* $Ge_n(t) = G(t)e_n$, $Ge_n \in \mathcal{H}$ *for* $n \in \mathbb{N}$ *and B is a standard cylindrical fractional Brownian motion. Define*

$$\int_0^T GdB := \sum_{n=1}^{\infty} \int_0^T Ge_n dB_n; \tag{1.8}$$

provided the infinite series converges in $L^2(\Omega)$.

Property 1.5.3. [*11*] *Let* $G : [0,T] \longrightarrow \mathcal{L}(\mathcal{K})$ *and* $G(.)x \in \mathcal{H}$ *for each* $x \in V$. *Let* $\Gamma_T : \mathcal{K} \longrightarrow L^2([0,T], \mathcal{K})$ *be given as*

$$(\Gamma_T(x))(t) = (K_H^* Gx)(t);$$

for $t \in [0,T]$ *and* $x \in \mathcal{K}$. *If* $\Gamma_T \in \mathcal{L}_2(\mathcal{K}, L^2([0,T],\mathcal{K}))$ *is a Hilbert Schmidt operator then the stochastic integral (1.8) is a well-defined centered Gaussian* $\mathcal{K}$ *-valued random variable with covariance operator* $\tilde{Q}_T$ *given by*

$$\tilde{Q}_T x = \int_0^T \sum_{n=1}^{\infty} \langle (\Gamma_T e_n)(s), x\rangle (\Gamma_T e_n)(s) ds. \tag{1.9}$$

This integral does not depend on the choice of the complete orthonormal basis $(e_n, n \in \mathbb{N})$.

Remark 1.5.2. *Since* $\Gamma_T \in \mathcal{L}_2(\mathcal{K}, L^2([0,T],\mathcal{K}))$, *it follows that the map* $x \longrightarrow (\Gamma_T x)(t)$ *is the Hilbert -Schmidt on* $\mathcal{K}$ *for almost all* $t \in [0,T]$. *Let* Γ_T^* *be the adjoint of* Γ_T. *Then* Γ_T^* *is also Hilbert-Schmidt and* $\tilde{Q}_T$ *can be expressed as*

$$\tilde{Q}_T x = \int_0^T (\Gamma_T(\Gamma_T^* x))(t) dt; \tag{1.10}$$

for $x \in \mathcal{K}$.

If $H \in (\frac{1}{2}, 1)$ and G satisfies

$$\|G\|^2_{\tilde{\mathscr{H}}} = \int_0^T \int_0^T |G(u)|_{\mathscr{L}_2(\mathscr{K})} |G(v)|_{\mathscr{L}_2(\mathscr{K})} \phi(u-v)\,du\,dv < \infty;$$

then

$$\tilde{Q}_T = \int_0^T \int_0^T G(u)G^*(v)\phi(u-v)\,du\,dv;$$

where $\phi(u-v) = H(2H-1) \mid u-v \mid^{2H-2}$.

Property 1.5.4. [11] *If $\tilde{A} : Dom(\tilde{A}) \longrightarrow \mathscr{K}$ is closed linear operator, $G : [0,T] \longrightarrow \mathscr{K}$ satisfies $G([0,T]) \subset Dom(\tilde{A})$ and both G and $\tilde{A}G$ satisfy the conditions for G in property 1.5.3, then*

$$\int_0^T G dB \subset Dom(\tilde{A}) \quad \mathbb{P}.a.s;$$

and

$$\tilde{A} \int_0^T G dB = \int_0^T \tilde{A} G dB \quad \mathbb{P}.a.s.$$

1.5.3 Stochastic integral with respect to Q-cylindrical fractional Brownian motion

Let $\mathscr{K}$, U be two separable real Hilbert space. We recall that the process $B^H_Q(t)$ is given the following series:

$$B^H_Q(t) = \sum_{n=1}^{\infty} B^H_n(t) Q^{\frac{1}{2}} e_n \quad t \geq 0;$$

is said to be $\mathscr{K}$-valued Q-cylindrical fractional Brownian motion with covariance Q.

Let $\mathscr{L}^0_Q(\mathscr{K}, U)$ be the space of all $\xi \in \mathscr{L}(\mathscr{K}, U)$ such that $\xi Q^{\frac{1}{2}}$ is Hilbert Schmidt operator the norm is given by

$$\| \xi \|^2_{\mathscr{L}^0_Q(\mathscr{K},U)} = \| \xi Q^{\frac{1}{2}} \|^2_{HS} = tr(\xi Q \xi^*).$$

Let $\varphi : [0,T] \longrightarrow \mathscr{L}^0_Q(\mathscr{K}, U)$ such that:

$$\sum_{n=1}^{\infty} \|K^*_H(\varphi Q^{\frac{1}{2}} e_n)\|_{L^2([0,T],U))} < \infty. \tag{1.11}$$

Definition 1.5.5. [54] *Let $\varphi : [0,T] \longrightarrow \mathscr{L}_Q^0(\mathscr{K},U)$ satisfy 1.11, then its stochastic integral with respect to fractional Brownian motion B_Q^H is defined for $t \geq 0$, as follows*

$$\int_0^t \varphi(s)dB_Q^H(s) := \sum_{n=1}^{\infty} \varphi(s)Q^{\frac{1}{2}}e_n dB_n^H(s) = \sum_{n=1}^{\infty} \int_0^t K_H^*(\varphi Q^{\frac{1}{2}}e_n)(s)dW(s).$$

Notice that if

$$\sum_{n=1}^{\infty} \|\varphi Q^{\frac{1}{2}}e_n\|_{L^{\frac{1}{H}}([0,T],U)} < \infty, \tag{1.12}$$

then the particular 1.11 holds.

Lemma 1.5.5. *For any $\varphi : [0,T] \longrightarrow \mathscr{L}_Q^0(\mathscr{K},U)$ such that 1.12 holds, and for any $\alpha,\beta \in [0,T]$ with $\alpha > \beta$,*

$$E\|\int_\alpha^\beta \varphi(s)dB_Q^H(s)\|_U^2 \leq cH(2H-1)(\alpha-\beta)^{2H-1}\sum_{n=1}^{\infty}\int_\beta^\alpha \|\varphi(s)Q^{\frac{1}{2}}e_n\|_U^2 ds;$$

where $c = c(H)$. If in addition

$$\sum_{n=1}^{\infty} \|\varphi(t)Q^{\frac{1}{2}}e_n\|_U; \tag{1.13}$$

is uniformly convergent for $t \in [0,T]$, then

$$E\|\int_\alpha^\beta \varphi(s)dB_Q^H(s)\|_U^2 \leq cH(2H-1)\int_\beta^\alpha \|\varphi(s)\|^2_{\mathscr{L}_Q^0(\mathscr{K},U)} ds.$$

Remark 1.5.3. *If $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of nonnegative real numbers such that the nuclear operator Q satisfies $Qe_n = \sigma_n e_n$, assuming that there exists a positive constant K_φ such that*

$$\| \varphi(t) \|_{\mathscr{L}_Q^2(\mathscr{K},U)} < K_\varphi \quad \textit{uniformely in } [0,T];$$

then 1.13 holds automatically.

1.5.4 Stochastic integral with respect to Q-Sub Fractional Brownian Motion

Let ε the linear space of $\mathbb{R}$-valued step functions on $[0,T]$. For $\varphi \in \varepsilon$, we define its wiener integral with respect to one dimensional sub fractional Brownian motion

$\{S^H(t)\}_{t\geq 0}$ as follows

$$\int_0^T \varphi(s)dS^H(s) = \sum_{n=1}^{\infty} x_i(S^H_{t_i+1} - S^H_{t_i}).$$

Let $\mathscr{H}_{S^H}$ be the canonical Hilbert space associated to the sub-fBm S^H. That is $\mathscr{H}_{S^H}$ is the cloture of the linear span ε with respect to the scalar product,

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathscr{H}_{S^H}} = Cov\left(S^H(t), S^H(s)\right).$$

We know that the covariance of sub-fBm can be written as

$$\mathbb{E}\left[S^H(t)S^H(s)\right] = \int_0^t \int_0^s \phi_H(u,v)dudv = C_H(t,s), \tag{1.14}$$

where $\phi_H(u,v) = H(2H-1)\left(\mid u-v \mid^{2H-2} - (u+v)^{2H-2}\right)$.
Equation (1.14) implies that

$$\left\langle \varphi, \psi \right\rangle_{\mathscr{H}_{S^H}} = \int_0^t \int_0^t \varphi_u \psi_v \phi(u,v)dudv. \tag{1.15}$$

Now we consider the kernel

$$K_H(t,s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})} s^{3/2-H} \left(\int_s^t (x^2-s^2)^{H-3/2}dx \right) \mathbb{1}_{[0,t]}(s). \tag{1.16}$$

By Dzhaparidze and Van Zanten [31], we have

$$C_H(t,s) = c_H^2 \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du; \tag{1.17}$$

where

$$c_H^2 = \frac{\Gamma(1+2H)sin(\pi H)}{\pi}.$$

Let K_H^* be the linear operator from ε to $L^2[0,T]$ defined by

$$\left(K_H^*\varphi\right)(s) = c_H \int_s^r \varphi_r \frac{\partial K_H}{\partial r}(r,s)dr.$$

By using the equalities (1.15) (1.17), we obtain

$$
\begin{aligned}
\langle K_H^*\varphi, K_H^*\rangle_{L^2([0,T])} &= c_H^2 \int_0^T \left(\int_s^T \varphi_r \frac{\partial K_H}{\partial r}(r,s)dr\right)\left(\int_s^T \psi_u \frac{\partial K_H}{\partial u}(u,s)du\right)ds,\\
&= c_H^2 \int_0^T \int_0^T \left(\int_0^{r\wedge u} \frac{\partial K_H}{\partial r}(r,s)\frac{\partial K_H}{\partial u}(u,s)ds\right)\varphi_r\psi_u drdu,\\
&= c_H^2 \int_0^T \int_0^T \frac{\partial^2 K_H}{\partial r\partial u}(u,s)\varphi_r\psi_u drdu,\\
&= H(2H-1)\int_0^T\int_0^T \left(\mid u-r\mid^{2H-2} - (u+r)^{2H-2}\right)\varphi_r\psi_u drdu,\\
&= \langle \varphi,\psi\rangle_{\mathscr{H}_{S^H}}.
\end{aligned} \tag{1.18}
$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert space $\mathscr{H}_{S^H}$ and $L^2([0,T])$.
Hence, the process W defined by $W(t) := S^H\left((K_H^*)^{-1}\mathbb{1}_{[0,t]}\right)$ is a Wiener process, and S^H has the following Wiener integral representation:

$$
S^H(t) = c_H \int_0^t K_H(t,s)dW(s),
$$

because $(K_H^*)(1_{[0,t]})(s) = c_H K_H(t,s)$.
By Dzhapridze and Van Zanten [31], we have

$$
W(t) = \int_0^t \psi_H(t,s)dS^H(s),
$$

where

$$
\psi_H(t,s) = \frac{s^{H-1/2}}{\Gamma(3/2-H)}\left[t^{H-3/2}(t^2-s^2)^{1/2-H} - (H-3/2)\int_s^t (x^2-s^2)^{1/2-H}x^{H-3/2}dx\right]\mathbb{1}_{[0,t]}(s).
$$

In addition, for any $\varphi \in \mathscr{H}_{S^H}$,

$$
\int_0^t \varphi(s)dS^H(s) = \int_0^t (K_H^*\varphi)(t)dW(t);
$$

if and only if $K_H^*\varphi \in L^2([0,T])$.
Also, denoting $L^2_{\mathcal{H}_{S^H}}([0,T]) = \{\varphi \in \mathcal{H}_{S^H}, K_H^*\varphi \in L^2([0,T])\}$.
Since $H > \frac{1}{2}$, we have by (1.18) and lemma 2.1 of [24],

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset L^2_{\mathcal{H}_{S^H}}([0,T]). \tag{1.19}$$

Lemma 1.5.6. *(Nualart[17]) For $\varphi \in L^{\frac{1}{H}}([0,T])$,*

$$H(2H-1)\int_0^T \int_0^T |\varphi_r||\varphi_u||u-r|^{2H-2}\,dr\,du \le C_H \|\varphi\|_{L^{\frac{1}{H}}([0,T])},$$

where $C_H = \left(\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}\right)^{1/2}$, with β denoting the beta function.

To define the stochastic integral with respect to Q-sub-fractional Brownian motion we proceed as follows: Let $\mathcal{L}_Q^0(\mathcal{K},U)$ be the space of all $\xi \in \mathcal{L}(\mathcal{K},U)$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$\|\xi\|^2_{L_Q^0(\mathcal{K},U)} = \|\xi Q^{\frac{1}{2}}\|^2_{HS} = tr(\xi Q \xi^*).$$

Then ξ is called a Q-Hilbert Schmidt operator from $\mathcal{K}$ to U.
Let $\varphi : [0,T] \longrightarrow L_Q^0(\mathcal{K},U)$ such that

$$\sum_{n=1}^{\infty} \|K_H^*(\varphi Q^{\frac{1}{2}} e_n)\|_{L^2([0,T],U)} < \infty. \tag{1.20}$$

Theorem 1.5.2. *Let $\varphi : [0,T] \longrightarrow L_Q^0(\mathcal{K},U)$ satisfy 1.20. Then its stochastic integral with respect to the sub-fBm S_Q^H is defined, for $t \ge 0$, as follows*

$$\begin{aligned}\int_0^t \varphi(s)\,dS_Q^H(s) &:= \sum_{n=1}^{\infty}\int_0^t \varphi(s)Q^{\frac{1}{2}}e_n\,dS_n^H(s),\\ &= \sum_{n=1}^{\infty}\int_0^t K^*(\varphi Q^{\frac{1}{2}}e_n)\,dW(s).\end{aligned}$$

Notice that if

$$\sum_{n=1}^{\infty} \|\varphi(s)Q^{\frac{1}{2}}e_n\|_{L^{\frac{1}{H}}([0,T],U)} < \infty, \tag{1.21}$$

then in particular (1.20) holds, which follows immediately form (1.19).

The following lemma is obtained as a simple application of lemma 1.5.6.

Lemma 1.5.7. *([17]) For any $\varphi : [0,T] \longrightarrow L_Q^0(\mathcal{K}, U)$ such that 1.21 holds, and for any $u, v \in [0,T]$ with $u > v$,*

$$\mathbb{E}\Big\| \int_v^u \varphi(s) dS_Q^H(s) \Big\|_U^2 \leq C_H (u-v)^{2H-1} \sum_{n=1}^{\infty} \int_v^u \|\varphi(s) Q^{\frac{1}{2}} e_n\|_U^2 ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|\varphi(s) Q^{\frac{1}{2}} e_n\|_U^2 \quad \text{is uniformly convergent for } t \in [0,T], \tag{1.22}$$

then

$$\mathbb{E}\Big\| \int_v^u \varphi(s) dS_Q^H(s) \Big\|_U^2 \leq C_H (u-v)^{2H-1} \int_v^u \|\varphi(s)\|_{L_Q^0(\mathcal{K},U)}^2 ds.$$

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be the complete orthonormal basis of $\mathcal{K}$ introduced above. Applying lemma 1.5.6, we obtain

$$\begin{aligned} E\|\int_v^u \varphi(s) dS_Q^H(s)\|_U^2 &= E\| \sum_{n=1}^{\infty} \int_v^u \varphi(s) Q^{\frac{1}{2}} e_n dS^H(s)\|_U^2. \\ &= \sum_{n=1}^{\infty} E\|\int_v^u \varphi(s) Q^{\frac{1}{2}} dS^H(s)\|_U^2. \\ &= \sum_{n=1}^{\infty} H(2H-1) \int_v^u \int_v^u \|\varphi(t) Q^{\frac{1}{2}} e_n\|_U \|\varphi(s) Q^{\frac{1}{2}} e_n\|_U |t-s|^{2H-2} dt ds. \\ &\leq c_H \sum_{n=1}^{\infty} \left(\int_v^u \|\varphi(s) Q^{\frac{1}{2}} e_n\|_U^{\frac{1}{H}} \right)^{2H}. \\ &\leq c_H (u-v) \sum_{n=1}^{\infty} \int_v^u \|\varphi(s) Q^{\frac{1}{2}} e_n\|_U^2 ds. \end{aligned}$$

□

Remark 1.5.4. *If $\{\lambda_n\}_{n=1}^{\infty}$ is bounded sequence of non-negative real numbers such that the nuclear operator Q satisfies $Qe_n = \lambda_n e_n$, assuming that there exists a positive constant K_φ such that*

$$\|\varphi(t)\|_{\mathcal{L}_Q^2(K,U)} \leq K_\varphi \quad \text{uniformly in [0, T];}$$

then 1.22 holds automatically.

1.6 Theory of Semigroup

1.6.1 Uniformly continuous semigroups of bounded linear operators

Definition 1.6.1. [4] *Let X be a Banach space. A one parameter family* $\{T(t)\}_{t\geqslant 0}$ *of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*

1. $T(0) = I$ *(I is the identity operator on X).*
2. $T(t+s) = T(t).T(s)$ *for every* $t, s \geqslant 0$ *(the semigroup property).*

A semigroup of bounded linear operators ,$T(t)$ is uniformly continuous if

$$\lim_{t\longrightarrow 0} \| T(t) - I \| = 0.$$

Definition 1.6.2. [4] *Let* $T(t)$ *be a semigroup of bounded linear operator A with domain*

$$\mathscr{D}(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad exists \right\},$$

defined by

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

is called the infinitesimal generators of the semigroup $T(t)$.

Theorem 1.6.1. [4] *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. We have*

$$T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!},$$

the series converging in norm for every $t \geq 0$.

From the definition 1.6.1 it is clear that a semigroup $T(t)$ has a unique infinitesimal generator. If $T(t)$ is uniformly continuous its infinitesimal generator is a bounded linear operator. On the other hand, every bounded linear operator A is the infinitesimal generator of a uniformly continuous semigroup $T(t)$.
Is this semigroup is unique? the affirmative answer to this question is given next.

Theorem 1.6.2. [4] *Let* $T(t)$ *and* $S(t)$ *be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t\longrightarrow 0} \frac{T(t) - I}{t} = A = \lim_{t\longrightarrow 0} \frac{S(t) - I}{t},$$

then $T(t) = S(t)$ *for* $t \geqslant 0$.

Corollaire 1.6.1. [4] *Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*

1. *There exists a constant $w \geqslant 0$ such that $\| T(t) \| < e^{wt}$.*
2. *There exists a unique bounded linear operator A such that $T(t) = e^{tA}$.*
3. *The operator A in part (2) is the infinitesimal generator of $T(t)$.*
4. *$t \longrightarrow T(t)$ is differentiable in norm and*

$$\frac{dT(t)}{dt} = AT(t) = T(t)A.$$

1.6.2 Strongly continuous semigroups of bounded linear operator

Definition 1.6.3. [4] *A semigroup $T(t)$, $0 \leq t < \infty$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t \longrightarrow 0} T(t)x = x, \quad \textit{for every } x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0- semigroup.

Theorem 1.6.3. [4] *Let $T(t)$ be a C_0-semigroup there exist constants $w \geq 0$ and $M \geq 1$ such that*

$$\| T(t) \| \leq Me^{wt} \quad \textit{for } 0 \leq t < \infty.$$

Corollaire 1.6.2. *If $T(t)$ is a C_0-semigroup, then for every $x \in X$, $t \longrightarrow T(t)x$ is a continuous function from $\mathbb{R}^+$ into X.*

Theorem 1.6.4. [4] *Let $T(t)$ be a C_0-semigroup and let A be its infinitesimal generator. Then*

1. *For $x \in X$,*

$$\lim_{h \longrightarrow 0} \frac{1}{h} \int_t^{t+h} T(t)x\,ds = T(t)x.$$

2. *For $x \in X$, $\int_0^t T(s)x\,ds \in \mathscr{D}(A)$, and*

$$A\Big(\int_0^t T(s)x\,ds\Big) = T(t)x - x.$$

3. *For* $x \in \mathscr{D}(A)$, $T(t)x \in \mathscr{D}(A)$ *and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

4. *For* $x \in \mathscr{D}(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

Proof. Please see [4]. □

Corollaire 1.6.3. *If A is the infinitesimal generator of a* C_0*-semigroup* $T(t)$ *then* $\mathscr{D}(A)$*, the domain of A is dense in X and A is a closed linear operator.*

Theorem 1.6.5. *Let* $T(t)$ *and* $S(t)$ *be* C_0*-semigroups of bounded linear operators with infinitesimal generators A and B respectively. If* $A = B$ *then* $T(t)$=$S(t)$ *for* $t \geq 0$.

If A is the infinitesimal generator of a C_0-semigroup then by corollary 1.6.3, $\overline{\mathscr{D}(A)} = X$. Actually, a much stronger result is true. Indeed we have,

Theorem 1.6.6. *Let A be the infinitesimal generator of* C_0*-semigroup* $T(t)$*. If* $\mathscr{D}(A^n)$ *is the domain of* A^n*, then* $\bigcap_{n=1}^{\infty} \mathscr{D}(A^n)$ *is dense in X.*

The Hille Yosida Theorem

Let $T(t)$ be a C_0-semigroup. From theorem 1.6.3 it follows that there are constant $\omega \geq 0$ and $M \geq 1$ such that $\| T(t) \| \leq M \exp(\omega t)$ for $t \geq 0$.
If $\omega = 0$, $T(t)$ is called uniformly bounded if $M = 1$, it is called a C_0-semigroup of contractions.

This section is devoted to the characterization of the infinitesimal generators of C_0-semigroup of contraction. Conditions of the behavior of the resolvent of an operator A, which are necessary and sufficient for A to be the infinitesimal generator of a C_0-semigroup of contractions are given.

Recall that if A is a linear, not necessary bounded operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e. $(\lambda I - A)^{-1}$ is a bounded linear operator on X. The family $R(\Lambda; A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.

Theorem 1.6.7. *(Hille Yosida) A linear (unbounded) operator A is the infinitesimal generator of* C_0*-semigroup of contraction* $T(t)$*,* $t \geq 0$ *if and only if*

(i) *A is closed and* $\overline{\mathscr{D}(A)} = X$.

(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$,

$$\| R(A,\lambda) \| \leq \frac{1}{\lambda}.$$

Lemma 1.6.1. [4] *Let A satisfy the condition (i) and (ii) of theorem 1.6.7 and let $R(\lambda, A) = (\lambda I - A)^{-1}$. Then $\lim_{\lambda \longrightarrow \infty} \lambda R(\lambda, A)x = x$ for $x \in X$.*

Now we define, for every $\lambda > 0$, the Yosida approximation of A by $A_\lambda = \lambda A R(\lambda; A) = \lambda^2 R(\lambda; A) - \lambda I$.
A_λ is an approximation of A in the following sense:

Lemma 1.6.2. *Let A satisfy the condition (i) and (ii) of theorem 1.6.7. If A_λ is the Yosida approximation of A, then $\lim_{\lambda \longrightarrow \infty} A_\lambda x = Ax$, for $x \in \mathscr{D}(A)$.*

Lemma 1.6.3. *Let A satisfy the condition (i) and (ii) of theorem1.6.7. If A_λ is the Yosida approximation of A, then A_λ is the infinitesimal generator of uniformly continuous semigroup of contractions e^{tA_λ}.*
Furthermore, for every $x \in X$, $\lambda, \mu > 0$ we have

$$\| e^{tA_\lambda} x - e^{tA_\mu} x \| \leq t \| A_\lambda x - A_\mu x \| .$$

Corollaire 1.6.4. *Let A be the infinitesimal generator of a C_0-semigroup of contractions $T(t)$. If A_λ is the Yosida approximation of A, then*

$$T(t)x = \lim_{\lambda \longrightarrow \infty} e^{tA_\lambda} x, \;\; for \;\; x \in X.$$

Corollaire 1.6.5. *Let A be the infinitesimal generator of a C_0-semigroup of contractions $T(t)$. The resolvent set of A contains the open right half-plane, i.e., $\{\lambda : Re(\lambda) > 0\} \subset \rho(A)$ and for such λ,*

$$\| R(\lambda; A) \| \leqslant \frac{1}{Re(\lambda)}.$$

1.6.3 Sectorial operator

Let X be a Banach space and A a (single-valued linear) operator. For $0 < \omega < \pi$, let

$$S_\omega^0 = \{z \in \mathbb{C} \setminus \{0\} : |arg z| < \omega\},$$

denote the open sector symmetric about the positive real axis with opening angle ω.

Let $\overline{S}_\omega$ be its closure, that is,

$$\overline{S}_\omega = \{z \in \mathbb{C} \setminus \{0\} : |arg z| \leq \omega\} \cup \{0\}.$$

Definition 1.6.4. [3] *Let* $-1 < p < 0$ *and* $0 < \omega < \frac{\pi}{2}$. *By* $\Theta_{\omega}^{p}(X)$ *we denote the family of all linear closed operators* $A : \mathscr{D}(A) \subset X \longrightarrow X$ *which satisfy*

i) $\sigma(A) \subset \overline{S}_{\omega}$.

ii) for every $\omega < \mu < \pi$, *there exist a constant* c_{μ} *such that*

$$\| R(z,A) \| \leq c_{\mu} |z|^{p}, \ \textit{for all} \ \ z \in \mathbb{C} \smallsetminus \overline{S}_{\mu}.$$

Where the family $R(z,A) = (zI - A)^{-1}$, $z \in \rho(A)$ *of bounded linear operators is the resolvent of A.*

A linear operator A will be called an almost sectorial operator on X if $A \in \Theta_{\omega}^{p}(X)$.

Remark 1.6.1. *Let* $A \in \Theta_{\omega}^{p}(X)$. *Then the definition implies that* $0 \in \rho(A)$.

We denote the semigroup associated with A by $\{Q(t)\}_{t \geq 0}$,

$$Q(t) = e^{-tz}(A) = \frac{1}{2\pi i} \int_{\Gamma_{\theta}} e^{-tz} R(z;A)\, dz, \quad t \in S^{0}_{\frac{\pi}{2}-\omega};$$

where the integral contour $\Gamma_{\theta} = \{\mathbb{R}^{+} e^{i\theta}\} \cup \{\mathbb{R}^{+} e^{-i\theta}\}$ is oriented counter clockwise and $\omega < \theta < \mu < \frac{\pi}{2} - |arg\, t|$ forms and analytic semigroup of growth order 1+p.

Property 1.6.1. [19] *Let* $A \in \Theta_{\omega}^{p}(X)$ *with* $-1 < p < 0$ *and* $0 < \omega < \frac{\pi}{2}$, *then the following properties remain true:*

i) $Q(t)$ *is analytic in* $S^{0}_{\frac{\pi}{2}-\omega}$ *and* $\frac{d^n}{dt^n} Q(t) = (-A)^n Q(t)$, $t \in S^{0}_{\frac{\pi}{2}-\omega}$.

ii) The functional equation $Q(s+t) = Q(s)Q(t)$ *for all* $s, t \in S^{0}_{\frac{\pi}{2}-\omega}$ *holds.*

iii) There is a constant $c_0 = c_0(p) > 0$ *such that*

$$\| Q(t) \| \leq c_0 t^{-p-1} \ \ \textit{for} \, t > 0.$$

iv) if $\beta > 1 + p$, *then* $\mathscr{D}(A^{\beta}) \subset \Sigma_{Q} = \{x \in X : \lim_{t \longrightarrow 0^{+}} Q(t)x = x\}$.

v) $R(\lambda; A) = \int_{0}^{\infty} e^{-\lambda t} Q(t)\, dt$ *for every* $\lambda \in \mathbb{C}$ *with* $Re(\lambda) > 0$.

Definition 1.6.5. [26] *A closed densely defined operator A on a Banach space X is called sectorial of angle $\omega < \pi$ (in short: $A \in sect(\omega)$ if*

i) $\sigma(A) \subset \overline{S}_\omega$ *and*

ii) $M(A,\omega^{'}) := \sup\{\| \lambda R(\lambda, A) \| | \lambda \notin \overline{S}_{\omega^{'}}\} < \infty$, *for all* $\omega < \omega^{'} < \pi$.

Definition 1.6.6. *An operator A is simply called sectorial if it is sectorial of angle ω for some $\omega \in (0,\pi)$. In this case, is called the sectoriality angle of A. Analogously to the half-plane case, we say that a set A of operators is uniformly sectorial of angle $\omega < \pi$ if*

$$\sup_{A \in \mathscr{A}} M(A,\alpha) < \infty,$$

for all $\alpha \in (\omega,\pi)$.

Remark 1.6.2. *The definition of sectorial operators is not universal in the literature. Some authors require a sectorial operator to be injective and to have dense range as well. We will omit these condition from our definition and add explicitly one or both to our assumptions when necessary. Notice that for a sectorial operator A on a Banach space one always has $N(A) \cap R(A) = 0$. In particular, if A has dense range, A is injective as well.*

Remark 1.6.3. *Let A be a densely defined operator on some Banach space X. Then it is well known that $-A$ generates an analytic C_0-semigroup if and only if A is with $\omega(A) < \frac{\pi}{2}$. Moreover, if $-A$ is the generator of a C_0-semigroup, then A is sectorial with $\omega(A) < \frac{\pi}{2}$. However, there exist sectorial operators with sectorial angle equal to $\frac{\pi}{2}$ that do not generate C_0-semigroups.*

Theorem 1.6.8. *An operator A on a Banach space X is sectorial if and only if $(-\infty,0) \subseteq \rho(A)$ and $M := \sup_{t>0} \| t(t+A)^{-1} \|$.Moreover, if A is sectorial, the following assertions hold:*

i) $x \in dom(A)$ *if and only if* $\lim_{t\to\infty} t(t+A)^{-1}x = x$,
$x \in ran(A)$ *if and only if* $\lim_{t\to 0} t(t+A)^{-1}x = 0$,

ii) $ran(A) \cup ker(A) = \{0\}$.

Remark 1.6.4. *If A is sectorial, then A generates an analytic semigroup $\{T(t)\}_{t\geq 0}$.*

Chapter 2

Fractional Calculus

2.1 Birth of Fractional Calculus

In a letter dated 30th September 1695, L'Hopital wrote to Leibniz asking him particular notation he has used in his publication for the n-th derivative of a function

$$\frac{D^n f(x)}{Dx^n}$$

i.e. what would the result be if $n = \frac{1}{2}$. Leibniz's response "an apparent paradox from which one day useful consequences will be drawn." In these words fractional calculus was born. Studies over the intervening three hundred years have proven at least half right. It is clear, that with in the XX century, especially numerous applications have been found. However, these applications and mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult to grasp, the definitions are no more rigorous than integer order counterpart. Later the question became: Can n , be any number: fractional, irrational, or complex? Because the latter question was answered affirmatively, the name 'fractional calculus' has become a misnomer and might better be called 'integration and differentiation of arbitrary order' or 'arbitrary ordered differ-integrations'. In 1812, P.S. Laplace defined a fractional derivative of arbitrary order appeared in Lacroix's (1819) writings. He developed a mere mathematical exercise generalizing from a case of integer order. Starting with $y = x^m$, where m a positive integer, Lacroix easily develops n th derivative:

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \quad m \geq n.$$

Using Legengdre's symbol for the generalized factorial (the complete Gamma function), Lacroix gets:

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

He then gives example for $y = x$ and $n = \frac{1}{2}$, and obtains:

$$\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

It is interesting to note that the result of Lacroix in the manner typical of the classical formalists of the periods is same as that yielded by the formalists Riemann Liouvelli definition of fractional derivative. This expression of Lacroix is also referred to as Euler's formula (1730). Let us try and use this to evaluate fractional derivative of $f(x) = exp(t)$. The exponential function is represented as series

$$f(t) = e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!},$$

applying this term to the Euler expression (as above) we get,

$$\frac{d^\nu e^t}{dt^\nu} = \sum_{k=0}^{\infty} \frac{t^{k-\nu}}{\Gamma(k-\nu+1)},$$

where ν is a positif real number.

2.2 Special Functions of fractional calculus

We will recall in this section some results of the special functions of Fractional Calculus which are important for other parts of this work.

2.2.1 Gamma function

Definition 2.2.1. [51] *The gamma function* $\Gamma(z)$ *is defined by the integral:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Property 2.2.1. *The gamma function satisfies the following functional equation:*

$$\Gamma(z+1) = z\Gamma(z). \tag{2.1}$$

Another important property can be represented also by the following limit:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^2}{z(z+1)\dots(z+n)}, \tag{2.2}$$

where we initially suppose that $Re(z) > 0$.

2.2.2 Beta function

Definition 2.2.2. *The beta function is defined by the following integral:*

$$B(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1}\,d\tau\ ,\ \ (Re(z) > 0, Re(w) > 0)\,. \tag{2.3}$$

Property 2.2.2. *The principal property of the function Beta is:*

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{2.4}$$

from which it follows that:

$$B(z,w) = B(w,z)\,.$$

2.3 Fractional Integrals and Derivatives

In this section we give some definitions and properties of fractional calculus.

2.3.1 Riemann-Liouville fractional integrals and Derivatives

In this part we give the definitions of the Riemann Liouville fractional integrals and fractional derivatives on a finite interval, real line and present some of their properties.

Definition 2.3.1. *We consider the weighted spaces of continuous functions*

$$C_\gamma[a,b] = \{f : [a,b] \longrightarrow \mathbb{R} : (x-a)^\gamma f(x) \in C[a,b]\},$$

and

$$C_\gamma^n[a,b] = \{f \in C^{n-1}[a,b] : f^{(n)} \in C_\gamma[a,b], n \in \mathbb{N}\},$$

$$C_\gamma^0[a,b] = C_\gamma[a,b].$$

Definition 2.3.2. [51] *The Riemann Liouville fractional integrals* $I^{\alpha}_{a^+}f$ *and* $I^{\alpha}_{b^-}f$ *of order* $\alpha \geq 0$ *are defined by*

$$I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t > a$$

$$I^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{f(s)}{(s-t)^{1-\alpha}}ds, \quad t < b.$$

This integrals are called the left-sidel and the right sidel fractional integrals. Her $\Gamma(.)$ *is the gamma function.*

Property 2.3.1. [51] *If* $\alpha > 0$ *and* $\beta > 0$ *then the equations*

$$I^{\alpha}_{a^+}I^{\beta}_{a^+}f(t) = I^{\alpha+\beta}_{a^+}f(t); \quad I^{\alpha}_{b^-}I^{\beta}_{b^-}f(t) = I^{\alpha+\beta}_{b^-}f(t), \tag{2.5}$$

are satisfied at almost every point $x \in [a,b]$ *for* $f(x) \in L^p(a,b)$ $(1 \leqq p \leqq \infty)$. *If* $\alpha+\beta > 1$ *then the relations in 2.5 hold at any point of* $[a,b]$.

Lemma 2.3.1. [51] *For* $x > a$ *we have*

$$[I^{\alpha}_{a^+}(t-a)^{\beta-1}](x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \quad \alpha \geq 0, \beta > 0,$$

$$[D^{\alpha}_{a^+}(t-a)^{\alpha-1}](x) = 0, \quad 0 \leq \alpha \leq 1.$$

Lemma 2.3.2. [51] *Let* $\alpha > 0$ *and* $0 \leq \gamma \leq 1$. *Then* $I^{\alpha}_{a^+}$ *is bounded from* $C_\gamma[a,b]$ *into* $C_\gamma[a,b]$.

Lemma 2.3.3. [51] *Let* $\alpha > 0$ *and* $0 \leq \gamma \leq 1$ *if* $\gamma \leq \alpha$, *then* $I^{\alpha}_{a^+}$ *is bounded from* $C[a,b]$.

Lemma 2.3.4. [51] *Let* $0 \leq \gamma \leq 1$ *and* $f \in C_\gamma[a,b]$, *then* $I^{\alpha}_{a^+}f(a) = \lim_{x \longrightarrow a^+} I^{\alpha}_{a^+}f(x) = 0, 0 \leq \gamma \leq \alpha$.

Proof. Note that by lemma 2.3.3, $I^{\alpha}_{a^+}f \in C[a,b]$.
Since $f \in C_\gamma[a,b]$ then $(x-a)^\gamma f(x)$ is continuous on $[a,b]$ and thus $|(x-a)^\gamma f(x)| < M$, $x \in [a,b]$ for some positive constant M. Therefor

$$| I^{\alpha}_{a^+}f(x) | < M[I^{\alpha}_{a^+}(t-a)^{-\gamma}](x),$$

and by lemma 2.3.1

$$| I^{\alpha}_{a^+}f(x) | \leq M\frac{\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)}(x-a)^{\alpha-\gamma}.$$

Since $\alpha > \gamma$, the right hand side $\longrightarrow 0$ as $x \longrightarrow a^+$, This completes the proof. □

Definition 2.3.3. [68] *(**Riemann-Liouville fractional integral on the real line**) The Riemann-Liouville fractional integral on $\mathbb{R}$ are defined as*

$$\left(I_{+}^{\alpha}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}f(t)dt = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)_{+}^{\alpha-1}f(t)dt, \tag{2.6}$$

and

$$\left(I_{-}^{\alpha}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}f(t)dt = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)_{-}^{\alpha-1}f(t)dt, \tag{2.7}$$

Remark 2.3.1. *The function $f \in D\left(I_{+,-}^{\alpha}\right)$ if the corresponding integrals converge for a.a $x \in \mathbb{R}$.*

Property 2.3.2. [68]

i. Fractional integration admits the following composition formulas for fractional integrals:

$$I_{+,-}^{\alpha}I_{+,-}^{\beta}f = I_{+,-}^{\alpha+\beta}f \tag{2.8}$$

for $f \in L^{p}(\mathbb{R})$, $\alpha, \beta > 0$ and $\alpha+\beta < \frac{1}{p}$.

ii. We consider $f \in L^{p}(\mathbb{R})$, $g \in L^{q}(\mathbb{R})$, $p > 1$, $q > 1$, and $\frac{1}{p}+\frac{1}{q} = 1$, then we obtain the following integration by parts formula

$$\int_{\mathbb{R}} g(x)\left(I_{+}^{\alpha}f\right)(x)dx = \int_{\mathbb{R}} f(x)\left(I_{-}^{\alpha}g\right)(x)dx. \tag{2.9}$$

iii. (Inclusion property)
Let $C^{\lambda}(\mathbb{T})$ be the set of Hölder continuous functions $f : \mathbb{T} \to \mathbb{R}$ of order λ i.e,

$$C^{\lambda}(\mathbb{T}) = \left\{f : \mathbb{T} \to \mathbb{R} \mid \| f \|_{\lambda} := sup_{t\in\mathbb{T}} \mid f(t) \mid + sup_{s,t\in\mathbb{T}} \mid f(s)-f(t) \mid (t-s)^{-\lambda} < \infty\right\}.$$

If $\alpha > 0$, and $\alpha p > 1$, then

$$I_{+,-}^{\alpha}\left(L^{p}(\mathbb{R})\right) \subset C^{\lambda}[a,b]$$

for any $-\infty < a < b < \infty$ and $0 < \lambda < \alpha - \frac{1}{p}$.

Definition 2.3.4. [51] *The Riemann-Liouville fractional derivative $I_{a^{+}}^{\alpha}f$ and $I_{b^{-}}^{\alpha}f$ of order $\alpha \geq 0$ are defined by*

$${}^{(R-L)}D_{a^{+}}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-s)^{n-1-\alpha}f(s)ds, \quad t > a$$

$${}^{(R-L)}D_{b^{-}}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{-d}{dt}\right)^{n}\int_{t}^{b}(s-t)^{n-1-\alpha}f(s)ds, \quad t < b$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Remark 2.3.2. *When $\alpha = n$ be a natural number, then we have*

$$(D^0_{a^+} f)(t) = (D^0_{b^-} f)(t) = f(t).$$

$$(D^n_{a^+} f)(t) = f^{(n)}(t).$$

$$(D^n_{b^-} f)(t) = (-1)^n f^{(n)}(t).$$

Where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n.

Lemma 2.3.5. [68] *If f be a continuous function and $f \in L^p[a,b]$ with $p > 0$ and $t > a$ then the following equalities*

$$\left(D^p_{a^+} I^p_{a^+} f\right)(t) = f(t),$$

and

$$\left(D^p_{b^-} I^p_{b^-} f\right)(t) = f(t),$$

hold almost everywhere on $[a,b]$.

Property 2.3.3. [68] *If $0 < q < p$ then for $f(x) \in L^p([a,b])$, the relations:*

$$\left(D^q_{a^+} I^p_{a^+} f\right)(t) = \left(I^{p-q}_{a^+} f\right)(t),$$

$$\left(D^q_{b^-} I^p_{b^-} f\right)(t) = \left(I^{p-q}_{b^-} f\right)(t),$$

hold almost everywhere on $[a,b]$.

2.3.2 Caputo Fractional Derivatives

In this section we present the definitions and some properties of the Caputo fractional derivatives. Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$.

Definition 2.3.5. [68] *The fractional derivatives $({}^cD^\alpha_{a^+} f)(t)$ and $({}^cD^\alpha_{b^-} f)(t)$ of order $\alpha >$ 0 on $[a,b]$ are defined via the above Riemann-Liouville fractional derivatives by*

$$\left({}^cD^\alpha_{a^+} f\right)(t) = \left(D^\alpha_{a^+}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k}(x-a)^k\right]\right). \tag{2.10}$$

$$({}^cD^\alpha_{b^-} f)(t) = \left(D^\alpha_{b^-}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k}(b-x)^k\right]\right). \tag{2.11}$$

These derivatives are called left-sided and right sided Caputo fractional derivatives of order α.

Remark 2.3.3. *If α be a real number then the Caputo fractional derivative 2.10 and 2.11 coincide with the Riemann Liouville fractional derivatives,*

$$\left({}^cD^{\alpha}_{a^+}f\right)(t) = (D^{\alpha}_{a^+}f)(t).$$

Remark 2.3.4. *If $\alpha = n \in \mathbb{N}$ and the derivative $f^{(n)}(t)$ of order n exists, then $\left({}^cD^{\alpha}_{a^+}f\right)(t)$ concides with $f^{(n)}(t)$.*

$$\left({}^cD^{n}_{a^+}f\right)(t) = f^{(n)}(t) \quad and \quad \left({}^cD^{n}_{b^-}f\right)(t) = (-1)^n f^{(n)}(t). \tag{2.12}$$

The Caputo fractional derivatives $\left({}^cD^{n}_{a^+}f\right)$ and $\left({}^cD^{n}_{b^-}f\right)(t)$ are defined for functions $f(t)$ for which the Riemann-Liouville fractional derivatives of the right hand sides of 2.10 and 2.11 exist.
In particular, they are defined for $f(t)$ belonging to the space $AC^n[a,b]$ of absolutely continuous functions.

Theorem 2.3.1. [68] *Let $a > 0$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. If $f(t) \in AC^n[a,b]$, then the Caputo fractional derivatives $\left({}^cD^{\alpha}_{a^+}f\right)(t)$ and $\left({}^cD^{\alpha}_{b^-}f\right)(t)$ exist almost every where on $[a,b]$.*

a) If $a \notin \mathbb{N}$, $\left({}^cD^{\alpha}_{a^+}f\right)(t)$ and $\left({}^cD^{\alpha}_{b^-}f\right)(t)$ are represented by

$$\left({}^cD^{\alpha}_{a^+}f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^n(s)}{(t-s)^{\alpha-n+1}}ds = (I^{n-\alpha}_{a^+}D^n f)(t).$$

and

$$\left({}^cD^{\alpha}_{b^-}f\right)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b \frac{f^n(s)}{(s-t)^{\alpha-n+1}}ds = (-1)^n(I^{n-\alpha}_{b^-}D^n f)(t),$$

where $D = d/dx$.

b) If $\alpha = n \in \mathbb{N}$, then $\left({}^cD^{n}_{a^+}f\right)(t)$ and $\left({}^cD^{n}_{b^-}f\right)(t)$ are represented by 2.12. In particular

$$\left({}^cD^{0}_{a^+}f\right)(t) = \left({}^cD^{0}_{b^-}f\right)(t) = f(t).$$

Theorem 2.3.2. [68] *Let $\alpha > 0$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. Also let $f(t) \in C^n[a,b]$. Then the Caputo fractional derivatives $\left({}^cD^{\alpha}_{a^+}f\right)(t)$ and $\left({}^cD^{\alpha}_{b^-}f\right)(t)$ are continuous on $[a,b]$.*

Remark 2.3.5. *The Caputo derivatives have similar properties to those of the Riemann-Liouville fractional derivatives.*

Lemma 2.3.6. *Let $\alpha > 0$ and let $f(t) \in C[a,b]$*

- *If $\alpha = n \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$ then*

$$({}^cD^{\alpha}_{a^+}I^{\alpha}_{a^+}f)(t) = f(t); \quad ({}^cD^{\alpha}_{b^-}I^{\alpha}_{b^-}f)(t) = f(t).$$

2.3.3 Hifler fractional derivative

Hilfer [50] proposed a general operator for fractional derivative, called "Hilfer fractional derivative," which combines Caputo and Riemann-Liouville fractional derivatives. Hilfer fractional derivative is performed, for example, in the theoretical simulation of dielectric relaxation in glass forming materials.

Definition 2.3.6. [*50*] *The Hilfer fractional derivative of order* $0 \le \alpha \le 1$ *and* $0 < \beta < 1$ *for a function f is defined by*

$$D_{0^+}^{\alpha,\beta} f(t) = I_{0^+}^{\alpha(1-\beta)} \frac{d}{dt} I_{0^+}^{(1-\alpha)(1-\beta)} f(t).$$

Remark 2.3.6. *When* $\alpha = 0,\ 0 < \beta < 1,$ *the Hilfer fractional derivative coincides with classical Riemann-Liouville farctional derivative*

$$D_{0^+}^{0,\beta} f(t) = \frac{d}{dt} I_{0^+}^{1-\beta} f(t) =^L D_{0^+}^{\beta} f(t).$$

When $\alpha = 1,\ 0 < \beta < 1,$ *the Hilfer fractional derivative coincides with classical Caputo fractional derivative*

$$D_{0^+}^{1,\beta} f(t) = I_{0^+}^{1-\beta} \frac{d}{dt} f(t) =^c D_{0^+}^{\beta} f(t).$$

Now, we introduce the space

$$C_{1-\gamma}^{\alpha,\beta}[a,b] = \{f \in C_{1-\gamma}[a,b] : D_{a^+}^{\alpha,\beta} f \in C_{1-\gamma}[a,b]\},$$

and

$$C_{1-\gamma}^{\gamma}[a,b] = \{f \in C_{1-\gamma}[a,b] : D_{a^+}^{\gamma} f \in C_{1-\gamma}[a,b]\}.$$

Since $C_{1-\gamma}^{\gamma}[a,b] \subset C_{1-\gamma}^{\alpha,\beta}[a,b]$.

The following lemma follows directly from the semigroup property in property 2.3.1

Lemma 2.3.7. [*34*] *Let* $0 \le \alpha \le 1, 0 \le \beta \le 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$*. If* $f \in C_{1-\gamma}^{\gamma}[a,b]$ *then*

$$I_{a^+}^{\gamma} D_{a^+}^{\gamma} f = I_{a^+}^{\alpha} D_{a^+}^{\alpha,\beta} f,$$

and

$$D_{a^+}^{\gamma} I_{a^+}^{\alpha} f = D_{a^+}^{\beta(1-\alpha)} f.$$

Lemma 2.3.8. [*34*] *Let* $f \in L^1[a,b]$*. If* $D_{a^+}^{\beta(1-\alpha)} f$ *exists and in* $L^1[a,b]$ *then*

$$D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} f = I_{a^+}^{\beta(1-\alpha)} D_{a^+}^{\beta(1-\alpha)} f.$$

Proof. $D^{\alpha,\beta}_{a^+} I^{\alpha}_{a^+} = I^{\beta(1-\alpha)}_{a^+} D I^{(1-\beta)(1-\alpha)}_{a^+} I^{\alpha}_{a^+} = I^{\beta(1-\alpha)}_{a^+} D I^{1-\beta(1-\alpha)}_{a^+} = I^{\beta(1-\alpha)}_{a^+} D^{\beta(1-\alpha)}_{a^+}$ □

Lemma 2.3.9. [34] *Let* $0 \le \alpha \le 1$, $0 \le \beta \le 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $f \in C_{1-\gamma}[a,b]$ *and* $I^{1-\beta(1-\alpha)}_{a^+} f \in C^1_{1-\gamma}[a,b]$ *then* $D^{\alpha,\beta}_{a^+} I^{\alpha}_{a^+} f$ *exists in* $(a,b]$ *and*

$$D^{\alpha,\beta}_{a^+} I^{\alpha}_{a^+} f(x) = f(x) \quad x \in (a,b].$$

Chapter 3

Stochastic Differential Inclusions

Differential inclusion is a generalization of the notion of an ordinary differential equation, therefore all problems considered for differential equation, that is, existence of solutions, continuation of solution dependence on initial conditions and parameters, are present in the theory of differential inclusions.

3.1 Phase Space

The notation of the space $\mathscr{B}$ play an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato [29].

1. If $x : (-\infty, b) \to \mathscr{H}$, $b > 0$, is continuous on $(0, b]$ and x_0 in $\mathscr{B}$, then for every $t \in [0, a)$ the following conditions hold:
   (a) x_t is in $\mathscr{B}$.
   (b) $\| x(t) \|_{\beta} \leq \tilde{H} \| x_t \|_{\mathscr{B}}$.
   (c)$\| x_t \|_{\mathscr{B}} \leq K(t) \sup\{\| x(s) \|_{\beta} : 0 \leq s \leq t\} + M(t) \| x_0 \|_{\mathscr{B}}$, where $\tilde{H} \geq 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded, and $\tilde{H}$, K, M are independent of $x(.)$.

2. For the function $x(.)$ in i., x_t is a $\mathscr{B}$-valued function $[0, a)$.

3. The space $\mathscr{B}$ is complete.

3.2 Multi-valued mapps

A multivalued map F of a set X into a set Y is a correspondence which associates to every $x \in X$ a nonempty subset $F(x) \subset Y$ called the value of x. Denoting by $\mathscr{P}(Y)$ the collection of all-nonempty subsets of Y we write this correspondence as

$$F : X \rightrightarrows \mathscr{P}(Y).$$

The notion of multivalued arises naturally in various branches of modern mathematics, such as mathematical economics, theory of games, convex analysis, ect. Now we give some basic definitions and properties of multivalued function.

Let $(X, \|\,.\,\|)$ be a Banach space and Y be a subset of X. We use the notations

$$\mathscr{P}(X) = \{Y \in X : Y \neq \emptyset\},$$

$$\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : Y \text{closed}\},$$

$$\mathscr{P}_{b}(X) = \{Y \in \mathscr{P}(X) : Y \text{bounded}\},$$

$$\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y \text{compact}\},$$

$$\mathscr{P}_{cv}(X) = \{Y \in \mathscr{P}(X) : Y \text{convex}\},$$

$$\mathscr{P}_{cp,cv}(X) = \{Y \in \mathscr{P}(X) : Y \text{compact and convex}\}.$$

Let $A, B \in \mathscr{P}(X)$. Consider $H_d : \mathscr{P}(X) \times \mathscr{P}(X) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ the Hausdorff distance between A and B given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$$

where $d(A, B) = \inf_{a\in A} d(a, b)$ and $d(a, B) \inf_{b\in B} d(a, b)$. As usual, $d(x, \emptyset) = +\infty$.
Then $(\mathscr{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathscr{P}_{cl}(X), H_d)$ is a generalized (complete) metric space.

Definition 3.2.1. [8] *A multivalued operator* $N : X \longrightarrow \mathscr{P}_{cl}(X)$ *is called*

1. γ*-Lipschitz if there exists* $\gamma > 0$ *such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \quad \textit{for all } x, y \in X.$$

2. *A contraction if it is* γ*-Lipschitz with* $\gamma < 1$.

Definition 3.2.2. [8] *A multivalued* $F : J \longrightarrow \mathscr{P}_{cl}(X)$ *is said to be measurable if, for each* $y \in X$*, the function*

$$t \longrightarrow d(y, F(x)) = \inf\{d(x, z) : z \in F(t)\}$$

is measurable.

Definition 3.2.3. [8] *The selection set of a multivalued map* $G : J \longrightarrow \mathscr{P}(X)$ *is defined by*

$$S_G = \{u \in L^1 : u(t) \in G(t), a.e t \in J\},$$

The set $S_{F \circ u}$ *known as the set of selectors from F is defined by*

$$S_{F\circ u} = \{v \in L^1(J) : v(t) \in F(t, u(t)), a.e. t \in J\}.$$

Definition 3.2.4. [8] *Let X and Y be metric space. A set-valued F from X to Y is characterized by its graph*

$$Gr(F) := \{(x, y) \in X \times Y : \quad y \in F(x)\}.$$

Definition 3.2.5. [8] *Let* $(X, |.|)$ *be a Banach space. A multivalued map* $F : X \longrightarrow \mathscr{P}(X)$ *is convex closed if* $F(X)$ *is convex (closed) for all* $x \in X$.
The map F is bounded on bounded sets if $F(B) = \bigcup_{x \in B} F(x)$ *is bounded in X for all* $B \in \mathscr{P}_b(X)$ *i.e*

$$\sup_{x \in B}\{\sup\{|y| : \quad y \in F(x)\}\} < \infty.$$

Definition 3.2.6. [8] *A multivalued map F is called upper semi continuous (u.s.c) on X if for each* $x_0 \in X$, *the set* $F(x_0)$ *is a nonempty, closed subset of X, and for each open set U of X containing* $F(x_0)$, *there exists an open neighborhood V of* x_0 *such that* $F(V) \subseteq U$.

A set-valued map F is said to be u.s.c if it is so at every point $x_0 \in X$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathscr{P}_b(X)$.
If the multivalued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has closed graph

$$(i.e. x_n \longrightarrow x_*, y_n \longrightarrow y_*, y_n \in G(x_n) \text{ imply } y_* \in F(x_*)).$$

The map F has a fixed point is there exists $x \in X$ such that $x \in Gx$. The set of fixed point of the multivalued operator G will be denoted by FixG.

Definition 3.2.7. [8] *A measurable multivalued function* $F : J \longrightarrow \mathscr{P}_{b,cl}(X)$ *is said to be integrably bounded if there exists a function* $g \in L^1(\mathbb{R}^+)$ *such that* $|f| \leq g(t)$ *for almost all* $t \in J$ *for all* $f \in F(t)$.

Lemma 3.2.1. [8] *Let G be a completely continuous multivalued map with nonempty compact values. Then G is u.s.c if and only if G has a closed graph*

$$(i.e u_n \longrightarrow u, w_n \longrightarrow w, w_n \in G(u_n) \textit{ imply } w \in G(u)).$$

Definition 3.2.8. [8] *A multivalued map $F : J\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathscr{P}(\mathbb{R})$ is said to be L^1-caratheodory if*

1. $t \longrightarrow F(t,x,y)$ *is measurable for each* $x, y \in \mathbb{R}$.
2. $x \longrightarrow F(t,x,y)$ *is u.s.c for almost all* $t \in J$.
3. *For each* $q > 0$ *there exists* $\varphi_q \in L^1(J,\mathbb{R}^+)$ *such that*

   $\| F(t,x,y) \|_{\mathscr{P}} = \sup\{|f| : f \in F(t,x,y)\} \leq \varphi_q(t)$ *for all* $|x| \leq q, |y| \leq q$ *and for a.e.* $t \in J$.

The multivalued map F is said to be caratheodory if it satisfies (1) and (2).

Lemma 3.2.2. *Let X be a Banach space. Let* $F : J\times X \longrightarrow \mathscr{P}_{cp,cv}(X)$ *be an* L^1*-caratheodory multivalued map, and let* Λ *be a linear continuous mapping from* $L^1(J,X)$ *to* $C(J,X)$. *Then the operator*

$$\begin{array}{cccc} \Lambda\circ S_{F\circ u} & : & C(J,X) & \longrightarrow & \mathscr{P}_{cp,cv}(C(J,X)) \\ & & w & \longrightarrow & (\Lambda\circ S_{F\circ u})(w) := (\Lambda S_{F\circ u})(w) \end{array}$$

is a closed graph operator in $C(J,X)\times C(J,X)$.

Property 3.2.1. *Let* $F : X \longrightarrow Y$ *be an u.s.c map with closed values. Then Gr(F) is closed.*

Lemma 3.2.3. *Let X be a seperable metric space. Then every measurable multivalued map* $F : X \longrightarrow \mathscr{P}_{cl}(X)$ *has a measurable selection.*

Definition 3.2.9. [8] *The multivalued map* $F : J\times\mathscr{H} \longrightarrow \mathscr{P}(\mathscr{H})$ *is said to be* L^2*-Carathéodory if*

i) $t \longrightarrow F(t,v)$ *is measurable for each* $v \in \mathscr{H}$.

ii) $t \longrightarrow F(t,v)$ *is u.s.c for almost all* $t \in J$.

iii) *For each* $q > 0$, *there exists* $h_q \in L^1(J,\mathbb{R}^+)$ *such that* $\|F(t,v)\|^2 = \sup_{f\in F(t,v)} E\|f\|^2 \leq h_q(t)$, *for all* $\|v\|^2_{\mathscr{H}} \leq q$ *and for a.e.* $t \in J$.

3.3 Semilinear stochastic inclusions in a Hilbert space

Let us consider the semilinear stochastic evolution inclusion with delays in a Hilbert space, defined by

$$\begin{cases} dx(t) \in [Ax(t) + F(x(\varphi(t)))]dt + \sigma(x\tau(t))dw_t \quad t \in J = [0,T] \\ x(t) = \phi(t), t \in J_0 = [-r,0], \end{cases} \tag{3.1}$$

where ϕ is $\mathscr{F}_0$-measurable and A is the infinitesimal generator of strongly continuous semigroup of closed linear operator $S(t), t \geqslant 0$ on the separable Hilbert space $\mathscr{H}$ with inner product $\langle .,. \rangle$ and norm $\| \,.\, \|$. $\rho, \tau : [0, +\infty[\longrightarrow [-r, +\infty], \ r \geqslant 0$ are suitable delay functions. $x : [-r, T] \longrightarrow \mathscr{H}$ and $\phi : J_0 \longrightarrow \mathscr{H}$ is the initial datum such that $\phi(t)$ is $\mathscr{F}_0$-measurable for all $t \in J_0$, $E \| \phi(0) \|^p < \infty$ and $\int_{-r}^0 E \| \phi(0) \|^p \, ds < \infty$, $p \geq 2$.

Let $\mathscr{K}$ be another separable Hilbert space with inner product $\langle .,. \rangle_{\mathscr{K}}$ and norm $\| \,.\, \|_{\mathscr{K}}$. Suppose $w(t)$ is a given $\mathscr{K}$-valued Brownian motion or wiener process with a finite trace nuclear covariance operator $Q \geq 0$.

Let $\mathscr{L}(\mathscr{K}, \mathscr{H})$ denotes the Banach space of all bounded linear operators from $\mathscr{K}$ into $\mathscr{H}$. Assume $F : \mathscr{H} \longrightarrow 2^H \setminus \emptyset$, the space of nonempty subsets of the space $\mathscr{H}$ and $\sigma : \mathscr{H} \longrightarrow \mathscr{L}(\mathscr{K}, \mathscr{H})$, are two measurable mappings in $\mathscr{H}$-norm and $L^2(\mathscr{K}, \mathscr{H})$-norm, respectively.

Definition 3.3.1. [46] *Let A be the infinitesimal generator of strongly continuous semigroup of closed linear operators $S(t)$, $t \geq 0$. Let ϕ be $\mathscr{F}_0$-measurable $\mathscr{H}$-valued stochastic process satisfying $E \mid \phi \mid^p < \infty$, and $f \in L^p(\mathscr{H})$ is a selection of $F(x(\varphi(t)))$. The function $x(t)$ is given by*

$$\begin{cases} x(t) = S(t)\phi(0) + \int_0^t S(t-s) f(s) ds + \int_0^t S(t-s)\sigma(x(\tau(s))) dw(s) \quad t \in J \\ x(t) = \phi(t), t \in J_0 = [-r, 0]. \end{cases} \tag{3.2}$$

is the mild solution of the problem 3.1.

We denotes by $BCC(\mathscr{H})$ the set of all nonempty bounded, closed and convex subsets of $\mathscr{H}$.

Lemma 3.3.1. [46] *Let $\mathscr{H}$ be a Hilbert space and $\Phi : \mathscr{H} \longrightarrow BCC(\mathscr{H})$ a u.s.c and condensing map. If the set*

$$U = \{x \in \mathscr{H} : \lambda x \in \Phi x \ \ \textit{for some} \ \ \lambda > 1\} \ \ \textit{is bonded}, \textit{then } \Phi \textit{ has a fixed point.}$$

The following lemma is crucial in the prof.

Lemma 3.3.2. [46] *Let I be a compact interval and Y be a Hilbert space. Let F be a multivalued map which is measurable for each $u \in \mathscr{H}$ upper semi continuous with respact to u and for each fixed $u \in \mathscr{H}$ the set,*

$$N_{F,u} = \{f \in L^p(\mathscr{H}) : f(t) \in F(u) \ \ \textit{for a.e. } t \in J\} \ \ \textit{is nonempty.}$$

Also let Γ be a linear continuous mapping from $L^p(I, Y)$ to $C(I, Y)$. Then the operator

$$\begin{array}{rcccl}\Gamma\circ N_F & : & C(I,Y) & \longrightarrow & BCC(C(I,Y)) \\ & & x & \longrightarrow & (\Gamma\circ N_F)(x)=\Gamma(N_{F,x})\end{array}$$

is a closed graph operator in $C(I,Y)\times C(I,Y)$.

Let us introduce the following hypothesis

(H1) $A:D(A)\subset\mathcal{H}\longrightarrow\mathcal{H}$ is the infinitesimal generator of a strongly continuous semi group $S(t)$ in $\mathcal{H}$, wich is a compact for $t\geq 0$ such that

$$\| S(t)\|\leq Me^{-\gamma t} \quad \text{for all } \; t\geq 0 \; \text{ where } \; M\geq 1 \; \text{ and } \; \gamma>0.$$

(H2) $\rho,\tau:[0,\infty)\longrightarrow[-r,+\infty)$ $r\geqslant 0$, are continuous functions such that

$$-r\leqslant\rho(t)\leqslant t \quad \text{and} \quad -r\leqslant\tau(t)\leqslant t \quad \text{for all } \; t\geqslant 0.$$

(H3) There exists constants $c_1,c_2\geq 0$ such that

$$E\|\sigma(u)\|^p\leq c_1E\|u\|^p+c_2 \quad u\in\mathcal{H}, \quad p\geq 2.$$

(H4) $F:\mathcal{H}\longrightarrow BCC(\mathcal{H})$; $u\longrightarrow F(u)$ is a measurable for each $u\in\mathcal{H}$, upper semi continuous with respect to u and for each fixed $u\in\mathcal{H}$ the set

$$N_{F,u}=\{f\in L^p(\mathcal{H}):f(t)\in F(u) \; \text{ for a.e. } \; t\in J\} \; \text{ is nonempty.}$$

(H5) $E\mid F(u)\mid^p=\sup\{E\mid v\mid^p: v\in F(u)\}\leqslant\eta(t)\Psi(E\mid u\mid^p)$ for almost all $t\in J$ and $u\in\mathcal{H}$, where $\eta\in L^p(J,\mathbb{R}^+)$ and $\Psi:\mathbb{R}^+\longrightarrow(0,\infty)$ is continuous and increasing function with

$$\int_0^T\overline{m}(s)ds<\int_c^\infty\frac{du}{1+u+\Psi(u)}$$

where $c=3^{p-1}M^p\|\phi\|^p$ and

$$\overline{m(t)}=\max[3^{p-1}T^{p-1}M^pe^{p\gamma t}\eta(t),3^{p-1}T^{\frac{p}{2}-1}M^pc_1e^{p\gamma t},3^{p-1}T^{\frac{p}{2}-1}M^pc_2e^{p\gamma t}].$$

(H6) The function σ is completely continuous and for any bounded set $V\subseteq\mathcal{H}$ the set $\{t\longrightarrow\sigma(x(\tau(t))):x\in V\}$ is equicontinuous in $\mathcal{H}$.

Remark 3.3.1. $N_{F,u}$ *is nonempty if and only if the function* $X:J\longrightarrow\mathbb{R}$ *defined by* $X(t)=\inf\{E|v|^p: \; v\in F(u)\}$ *belongs to* $L^p(J,\mathbb{R})$.

Theorem 3.3.1. [46] *Assume that hypotheses (H1)–(H6) hold. Then the stochastic inclusion 3.1 has at least one mild solution on J.*

Proof. We transform the problem 3.1 into a fixed point problem. We consider the multivalued map $\Phi : C \longrightarrow 2^C$ defined by

$$(\Phi x)(t) = \begin{cases} \phi(t), t \in J_0 \\ S(t)\phi(0) + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)\sigma((x(\tau(s))))dW_s, t \in J. \end{cases} \tag{3.3}$$

Where $f \in N_{F,x} = \{f \in L^p(\mathscr{H} : f(t) \in F(x(\rho(s))))$ for a.e. $t \in J\}$.
It is clearly that the fixed point of Φ are mild solution to 3.1.
First we prove that Φ is a completely continuous multivalued map, u.s.c, with convex closed values.
The proof is given by the following steps. In the step one we shall that Φx is convex for every $x \in C$. Next in step two we prove that Φ is a completely continuous operator.
As a consequence of step two and the hypotheses (H6) together with the Arzela-Ascoli theorem it is concluded that $\Phi : C \longrightarrow 2^C$ is a compact multivalued map, after we show that Φ has a closed graph.
By this steps, and by lemma 3.3.1 we deduce that Φ has a fixed point which is a solution of the stochastic inclusion 3.1. □

Proof. See [46] for more details. □

Example 3.3.1. *As an application of the above result, consider a one-dimensional rod of length π whose ends are maintained at 0^0 and whose sides are insulated. Suppose there is an exothermic reaction taking place inside the rod with heat being produced proportionally to the temperature at a previous time $t-r$ (for the sake of simplicity, we assume the delay $r \geq 0$ is constant). Consequently, the temperature in the rod may be modeled to satisfy*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + hu(t-r,x) \;\; 0 < x < \pi, \;\; t > 0 \\ u(t,0) = u(t,\pi) = 0 \qquad\qquad t > 0 \\ u(t,x) = \phi(t,x) \qquad\qquad t \in [-r,0], x \in [0,\pi] \end{cases} \tag{3.4}$$

where h depends on the rate of reaction and $\phi : [-r,0] \times [0,\pi] \longrightarrow \mathbb{R}$ is a given function. We observe that, when there is no heat production (i.e., $h = 0$), the problem 3.4 has solutions given by

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx),$$

where $r = 0$ *and* $\phi(0, x) = \sum_{n=1}^{\infty} a_n \sin(nx)$.
However, it often occurs that the exothermic reaction can be random. In some cases, this can be modeled by writing the term $hu(t-r, x)$ *in the form* $(h_0 + h_1 w'(t))u(t-r, x)$ *where* $w(t)$ *is real standard Brownian motion,* $h_0 : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ *is measurable with respect to first argument and for each second argument, it is u.s.c. satisfying Lipschitz continuity and* $h_1 : \mathbb{R} \longrightarrow \mathbb{R}$ *is completely continuous. Thus, 3.4 can be written as*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} \in \frac{\partial^2 u(t,x)}{\partial x^2} + h_0 u(t-r, x) + h_1 u(t-r, x) w'(t) & 0 < x < \pi, \quad t > 0 \\ u(t, 0) = u(t, \pi) = 0 & t > 0 \\ u(t, x) = \phi(t, x) & t \in [-r, 0], x \in [0, \pi] \end{cases} \tag{3.5}$$

and setting $H = L^p(0, \pi)$ *and* $K = \mathbb{R}$ *and* A *the operator* $A = \frac{d^2}{dx^2}$ *with domain*

$$D(A) = \{y \in H : \frac{dy}{dx}, \frac{d^2 y}{dx^2} \in H, y(0) = y(\pi) = 0\},$$

$F(u) = h_0 u$, $\sigma(u) = h_1 u$ *and* $\rho(t), \tau(t) = t - r$, *the problem 3.4 can be reformulated as follows (see Caraballo and Liu [58]):*

$$\begin{cases} du(t) \in [Au(t) + F(u(\sigma(t)))]dt + \sigma(u(\sigma(t)))dw(t), & t > 0, \\ u(t) = \phi(t), & t \in [-r, 0]. \end{cases} \tag{3.6}$$

One can compute immediately that A *generates a strongly continuous analytic semi group* $S(t)$ *and* $|S(t)| \leq Me^{\gamma t}$, *for all* $t \geq 0$ *where* $\gamma = M = 1$.
Under these assumptions, Theorem 3.3.1 applies and hence the problem 3.6 has a mild solution.

3.4 Stochastic differential inclusion with Hilfer fractional derivative

Let us consider the stochastic differential inclusion driven by sub-fractional Brownian motion with Hilfer fractional derivative of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta} x(t) \in Ax(t) + F(t, x_t) + g(t)\frac{dS_Q^H}{dt}, t \in J = [0, b], \\ (I_0^{1-\gamma} x)(t)|_{t=0} = \varphi \in \mathscr{B}. \end{cases} \tag{3.7}$$

Where $D_{0^+}^{\alpha,\beta}$ is the generalized Hilfer fractional derivative of orders $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$. A is the infinitesimal generator of strongly continuous semigroup of bounded

linear operator $\{T(t)\}_{t\geq 0}$.
Assume that $F : J \times \mathscr{H} \to \mathscr{P}(\mathscr{H})$ is a bounded, closed and convex multivalued map, $g : J \to \mathscr{L}_Q^0(\mathscr{K}, \mathscr{H})$, $\mathscr{K}$ is a real separable Hilbert space with product $\langle .,. \rangle_{\mathscr{K}}$. Here $\mathscr{L}_Q^0(\mathscr{K}, \mathscr{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathscr{K}$ into $\mathscr{H}$ and S_Q^H is an Q-sub-fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$. $I_0^{1-\gamma}$ is the fractional integral of orders $1-\gamma$ ($\gamma = \alpha + \beta - \alpha\beta$).

Lemma 3.4.1. *Let I be a compact interval and $\mathscr{H}$ be a Hilbert space. Let F be an L^2-Carathéodory multivalued map with $S_{F,x} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^2(J, \mathscr{H})$ to $C(J, \mathscr{H})$. Then the operator $F \circ S_F : C(J, \mathscr{H}) \longrightarrow \mathscr{P}_{cp,cv}(\mathscr{H}), x \longrightarrow (\Gamma \circ S_F)(x) = \Gamma(S_{F,x})$ is a closed graph operator in $C(J, \mathscr{H}) \times C(J, \mathscr{H})$, where $S_{F,x}$ is known as the selectors set from F and given by*

$$f \in S_{F,x} = \{f \in L^2([0,t], \mathscr{H}) : f(t) \in F(t,x) \ \textit{for a.e.} \ t \in [0,T]\}.$$

Now we introduce the space $\mathscr{PC}$ formed by all $\mathscr{F}_t$-adapted measurable square integrable $\mathscr{H}$-valued stochastic process $\{x(t) : t \in [0,b]\}$ with norm $\|x\|^2_{\mathscr{PC}} = \sup_{t\in[0,b]} E\|x(t)\|^2$, then $(\mathscr{PC}, \|.\|_{\mathscr{PC}})$ is a Banach space.
We define $\mathscr{PC}_\gamma = \{x : (-\infty, b] \longrightarrow \mathscr{H} : t^{1-\gamma}x(t) \in \mathscr{PC}\}$ with norm $\|.\|_{\mathscr{PC}_\gamma}$ defined by

$$\|.\|^2_{\mathscr{PC}_\gamma} = \sup_{t\in[0,b]} E\|t^{1-\gamma}x(t)\|^2.$$

Obviously, $\mathscr{PC}_\gamma$ is a Banach space.

Let us define the operators $\{S_{\alpha,\beta}(t) : t \geq 0\}$ and $\{P_\beta(t) : t \geq 0\}$ by

$$S_{\alpha,\beta}(t) = I_{0^+}^{\alpha(1-\beta)} P_\beta(t),$$

$$P_\beta(t) = t^{\beta-1} T_\beta(t),$$

$$T_\beta(t) = \int_0^\infty \beta\theta\Psi_\beta(\theta) T(t^\beta\theta) d\theta;$$

where

$$\Psi_\beta(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)\Gamma(1-n\beta)}, 0 < \beta < 1, \theta \in (0,\infty)$$

is a function of wright type which satisfies

$$\int_0^\infty \theta^\xi \Psi_\beta(\theta) d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+\beta\xi)}, \ \ \xi \in (-1,\infty).$$

Lemma 3.4.2. [48] *The operator $S_{\alpha,\beta}$ and P_β have the following properties*

i) For any fixed $t \geq 0$, $S_{\alpha,\beta}(t)$ and P_β are bounded linear operators, and

$$\|P_\beta(t)x\|^2 \leq M\frac{t^{2(\beta-1)}}{(\Gamma(\beta))^2}\|x\|^2 \text{ and}$$

$$\|S_{\alpha,\beta}(t)x\|^2 \leq M\frac{t^{2(\alpha-1)(1-\beta)}}{(\Gamma(\alpha(1-\beta)+\beta))^2}\|x\|^2.$$

ii) $\{P_\beta(t) : t \geq 0\}$ is compact if $\{T(t) : t \geq 0\}$ is compact.

Definition 3.4.1. *An $\mathcal{H}$-valued stochastic process $\{x(t)\}$ is said to be mild solution of system 3.7 if the process x satisfies the following equation:*

$$x(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s,x(s))ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s), \ \ t \in J.$$

3.4.1 Existence of Mild Solution

The convex case.

In this section, we will show the existence results of mild solutions for convex case of system 3.7. So we impose the following assumptions to show the main results:
(H1) The operator A is the infinitesimal generator of a strongly continuous of bounded linear operators $\{S(t)\}_{t\geq 0}$ which is compact for $t > 0$ in $\mathcal{H}$ such that $\| S(t) \|^2 \leq M$ for each $t \in [0, b]$.
(H2)The maps $F : J \times \mathcal{H} \longrightarrow \mathcal{P}_{cp,cv}(\mathcal{H})$ is an L^2-Caratheodory function and for any $t \in [0, b]$ the multifunction $t \longrightarrow F(t, x(t))$ is measurable.
(H3) There exists a function $h_q \in L^2(J,\mathcal{H})$ such that

$$\| F(t, x) \|^2 \leq h_q(t).$$

(H4) There exist a constant $k \geq 0$ such that

$$\| F(t, x_2(t)) - F(t, x_1(t)) \|^2 \leq K \| x_2 - x_1 \|^2 .$$

(H5) There exist a constant $p > \frac{1}{2\beta-1}$ such that $g : J \longrightarrow L_2^0(J,\mathcal{H})$ satisfies $\int_0^b \|g(s)\|_{L_2^0}^{2p} ds < \infty$.

Theorem 3.4.1. *If the assumptions (H1)-(H4) are satisfied then system 3.7 has a unique mild solution on $\mathscr{P}\mathscr{C}_\gamma$ provided that*

$$\frac{\tilde{M}b^{2(\beta-\gamma)+1}}{(\Gamma(\beta))^2(2\beta-1)} < 1.$$

Proof. For an arbitrary x, we define the operator Φ on $\mathscr{P}\mathscr{C}_\gamma$ as follows

$$(\Phi x)(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s,x(s))ds + \int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

We will prove that Φ has a fixed point on $\mathscr{P}\mathscr{C}_\gamma$, the proof will be given in serval steps.
Step1: We show that Φ maps $\mathscr{P}\mathscr{C}_\gamma$ into itself.
We divide the proof into two claims
Claim1: from lemma 3.4.2, Holder's inequality and hypotheses (H1)-(H4), we have

$$\begin{aligned} E\left\|t^{1-\gamma}x(t)\right\|^2 &= E\left\|t^{1-\gamma}S_{\alpha,\beta}(t)\varphi + t^{1-\gamma}\int_0^t P_\beta(t-s)F(s,x(s))ds + t^{1-\gamma}\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\right\|^2 \\ &\leq 3E\left\|t^{1-\gamma}S_{\alpha,\beta}(t)\varphi\right\|^2 + 3E\left\|t^{1-\gamma}\int_0^t P_\beta(t-s)F(s,x(s))ds\right\|^2 \\ &+ 3E\left\|t^{1-\gamma}\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\right\|^2 \\ &\leq I_1 + I_2 + I_3. \end{aligned}$$

$$\begin{aligned} I_1 :&= 3E\left\|t^{1-\gamma}S_{\alpha,\beta}(t)\varphi\right\|^2 \\ &\leq 3t^{2(1-\gamma)}M\frac{t^{2(\gamma-1)}}{(\Gamma(\gamma))^2}E\|\varphi\|^2 \\ &\leq 3\frac{M}{(\Gamma(\gamma))^2}E\|\varphi\|^2. \end{aligned}$$

$$
\begin{aligned}
I_2 :&= 3E\left\| t^{1-\gamma}\int_0^t P_\beta(t-s)F(s,x(s))ds\right\|^2\\
&\le 3b^{2(1-\gamma)}E\left(\int_0^t \|P_\beta(t-s)F(s,x(s))\|ds\right)^2\\
&\le 3b^{2(1-\gamma)}\frac{M}{(\Gamma(\beta))^2}E\left(\int_0^t (t-s)^{(\beta-1)}\|F(s,x(s))\|ds\right)^2\\
&\le \frac{3Mb^{2\alpha(\beta-1)}}{(\Gamma(\beta))^2(2\beta-1)}E\int_0^t \|F(s,x(s))\|^2 ds\\
&\le \frac{3Mb^{2\alpha(\beta-1)}}{(\Gamma(\beta))^2(2\beta-1)}E\int_0^t h_q(s)ds.
\end{aligned}
$$

$$
\begin{aligned}
I_3 :&= 3E\left\| t^{1-\gamma}\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\right\|^2\\
&\le 3t^{2(1-\gamma)}c_H(-t)^{2H-1}\int_0^t \|P_\beta(t-s)g(s)\|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})}ds\\
&\le 3b^{2(1-\gamma)}c_H(-b)^{2H-1}\frac{M}{(\Gamma(\beta))^2}\int_0^t (t-s)^{2(\beta-1)}\|g(s)\|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})}ds\\
&\le 3b^{2(1-\gamma)}c_H(-b)^{2H-1}\frac{M}{(\Gamma(\beta))^2}\left(\int_0^t (t-s)^{\frac{2p(\beta-1)}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^t \|g(s)\|^{2p}_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})}ds\right)^{\frac{1}{p}}\\
&\le 3b^{1-2\gamma+2H}c_H\frac{M}{(\Gamma(\beta))^2}\left(\int_0^t (t-s)^{\frac{2p(\beta-1)}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^t \|g(s)\|^{2p}_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})}ds\right)^{\frac{1}{p}}.
\end{aligned}
$$

Therefore Φ maps $\mathscr{P}\mathscr{C}_\gamma$ into itself.

Claim2: We prove that $(\Phi x)(t)$ is continuous on J for any $x\in\mathscr{P}\mathscr{C}_\gamma$.

Let $\varepsilon>0$ and $t\in J$, then

$$\begin{aligned}
\| (\Phi x)(t+\varepsilon)-(\Phi x)(t) \|^2_{\mathscr{P}\mathscr{C}_\gamma} &= \sup_{0\le t\le b} E \| t^{(1-\gamma)}((\Phi x)(t+\varepsilon)-(\Phi x)(t)) \|^2 \\
&= \sup_{0\le t\le b} t^{2(1-\gamma)} E \| (\Phi x)(t+\varepsilon)-(\Phi x)(t) \|^2 \\
&\le \sup_{0\le t\le b} t^{2(1-\gamma)} E \| S_{\alpha,\beta}(t+\varepsilon)\varphi + \int_0^{t+\varepsilon} P_\beta(t+\varepsilon-s)F(s,x(s))ds \\
&+ \int_0^{t+\varepsilon} P_\beta(t+\varepsilon-s)g(s)dS^H_Q(s) - S_{\alpha,\beta}(t)\varphi - \int_0^t P_\beta(t-s)F(s,x(s))ds \\
&- \int_0^t P_\beta(t-s)g(s)dS^H_Q(s) \|^2 \\
&\le 3 \sup_{0\le t\le b} t^{2(1-\gamma)} E \| S_{\alpha,\beta}(t+\varepsilon)\varphi - S_{\alpha,\beta}(t)\varphi \|^2 + 3 \sup_{0\le t\le b} t^{2(1-\gamma)} \\
&\quad E \| \int_0^{t+\varepsilon} P_\beta(t+\varepsilon-s)F(s,x(s))ds - \int_0^t P_\beta(t-s)F(s,x(s))ds \|^2 \\
&+ 3 \sup_{0\le t\le b} t^{2(1-\gamma)} E \| \int_0^{t+\varepsilon} P_\beta(t+\varepsilon-s)g(s)dS^H_Q(s) - \int_0^t P_\beta(t-s)g(s)dS^H_Q(s) \|^2 .
\end{aligned}$$

By lemma3.4.2 and hypothesis (H1)-(H4), we deduce that the right hand side of the above inequality tends to zero as $\varepsilon \longrightarrow 0$, then $(\Phi x)(t)$ is continuous.

Step2: (Φx) is convex for each $x \in \mathscr{P}\mathscr{C}_\gamma$.

If $\rho_1, \rho_2 \in \Phi(x)$, then we have

$$\rho_i = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s,x_i(s))ds + \int_0^t P_\beta(t-s)g(s)dS^H_Q(s).$$

Let $0 \le \delta \le 1$, then for each $t \in [0,b]$ we have

$$\begin{aligned}
(\delta\rho_1+(1-\delta)\rho_2)(t) =& \delta S_{\alpha,\beta}(t)\varphi + \delta \int_0^t P_\beta(t-s)F(s,x_1(s))ds + \delta \int_0^t P_\beta(t-s)g(s)dS^H_Q(s) + (1-\delta)S_{\alpha,\beta}(t)\varphi \\
&+ (1-\delta)\int_0^t P_\beta(t-s)F(s,x_2(s))ds + (1-\delta)\int_0^t P_\beta(t-s)g(s)dS^H_Q(s) \\
=& S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)(\delta F(s,x_1(s)) + (1-\delta)F(s,x_2(s))ds + \int_0^t P_\beta(t-s)g(s)dS^H_Q(s)
\end{aligned}$$

$F(t,x)$ has a convex values, then $\delta\rho_1 + (1-\delta)\rho_2 \in \Phi(x)$.

Step3: Φ is a contraction.

For any x_1 and $x_2 \in \mathscr{P}\mathscr{C}_\gamma$, we have

$$(\Phi x_1)(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t-s)F(s,x_1(s))ds - \int_0^t P_\beta(t-s)g(s)dS^H_Q(s).$$

$$\begin{aligned}
\|(\Phi x_2)(t)-(\Phi x_1)(t)\|^2_{\mathscr{PC}_\gamma} &= \sup_{0\le t\le b} E\left\| t^{1-\gamma}\left((\Phi x_2)(t)-(\Phi x_1)(t)\right)\right\|^2\\
&\le \sup_{0\le t\le b} t^{2(1-\gamma)} E\|((\Phi x_2)(t)-(\Phi x_1)(t))\|^2\\
&\le \sup_{0\le t\le b} t^{2(1-\gamma)} E\|\int_0^t P_\beta(t-s)(F(s,x_2(s))-F(s,x_1(s))ds\|^2\\
&\le \sup_{0\le t\le b} t^{2(1-\gamma)} E\int_0^t \|P_\beta(t-s)(F(s,x_2(s))-F(s,x_1(s))\|^2 ds\\
&\le b^{2(1-\gamma)}\frac{M}{(\Gamma(\beta))^2}\|F(s,x_2(s))-F(s,x_1(s))\|^2\int_0^t (t-s)^{2(\beta-1)}ds\\
&\le \frac{\tilde{M}}{(\Gamma(\beta))^2(2\beta-1)} b^{2(\beta-\gamma)+1}\|x_2-x_1\|^2.
\end{aligned}$$

Step4: $\Phi(x)$ is closed for each $x\in\mathscr{PC}_\gamma$.
Let $\{h_n\}_{n\ge 0}\in\Phi(x)$ such that $h_n\longrightarrow h$ in $\mathscr{PC}_\gamma$. Then $h\in\mathscr{PC}_\gamma$ and there exist $\{v_n\}\in S_{F,x}$ such that for each $t\in J$

$$h_n(t)=S_{\alpha,\beta}(t)\varphi+\int_0^t P_\beta(t-s)v_n(s)ds+\int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

Due to the fact that F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^2(J,\mathscr{H})$ and hence $v\in S_{F,x}$. Then for each $t\in J$

$$h_n(t)\longrightarrow h(t)=S_{\alpha,\beta}(t)\varphi+\int_0^t P_\beta(t-s)v(s)ds+\int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

Thus, $h\in\Phi(x)$. □

The non convex case.

In this section, we give a non convex version of system 3.7.
Let $\mathscr{A}$ be a subset of $J\times\mathscr{B}$. $\mathscr{A}$ is $\mathscr{L}\otimes D$ measurable if $\mathscr{A}$ belongs to the σ-algebra generated by all sets of the form $\mathscr{J}\times\mathscr{B}$, where $\mathscr{J}$ is Lebesgue measurable in J and $\mathscr{B}$ is Borel measurable in $\mathscr{B}$. A subset $\mathscr{A}$ of $L^2(J,\mathscr{H})$ is decomposable if for all $w,v\in\mathscr{A}$ and $\mathscr{J}\in J$ measurable, $w\mathscr{X}_\mathscr{J}+v\mathscr{X}_{J-\mathscr{J}}\in A$, where $\mathscr{X}$ denotes the characteristic function.
Let $F:J\times\mathscr{H}\longrightarrow\mathscr{P}_{cp}(\mathscr{H})$. Assign to F the multivalued operator

$$\mathscr{F}:C(J,\mathscr{H})\longrightarrow\mathscr{P}(L^2(J,\mathscr{H})),$$

Let $\mathscr{F}(x)=S_{F,x}=\{f\in L^2(J,\mathscr{H}):f(t)\in F(t,x(t))\ \text{for a.e}\ \ t\in J\}$. The operator $\mathscr{F}$ is called the Niemytzki operator associated to F.

Definition 3.4.2. [41] *Let Y be a separable metric space and let $N:Y\longrightarrow\mathscr{P}(L^2(J,\mathscr{H}))$ be a multivalued operator. We say that N has property (BC) if*

1) N is lower semi continuous.

2) N has nonempty closed and decomposable values .

Definition 3.4.3. [41] $F : J \times \mathcal{H} \longrightarrow \mathcal{P}_{cp}(\mathcal{H})$ *be a multivalued function with nonempty compact values. We say that F is lower semi continuous type (l.s.c type) if its associated Niemytski operator* $\mathcal{F}$ *is l.s.c and has nonempty closed and decomposable values.*

Consider $H_d : \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \longrightarrow \mathbb{R} \cup \{\infty\}$ given by $H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$, where $d(A, b) = \inf_{a\in A} d(a, b)$.

Now, we give a selection theorem due to Bressan and Colombo [10].

Theorem 3.4.2. [41] *Let Y be a separable metric space and let* $N : Y \longrightarrow \mathcal{P}(L^2(J, \mathcal{H}))$ *be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued)* $\tilde{g} : Y \longrightarrow L^2(J, \mathcal{H})$ *such that* $\tilde{g}(y) \in N(y)$ *for every* $y \in Y$.

Lemma 3.4.3. *Let* (X, d) *be a complete metric space. If the multivalued operator* $G : X \longrightarrow \mathcal{P}_{cl}(X)$ *is a contraction then G has at least one fixed point.*

Now, we introduce the following hypothesis

(H6) $F : J \times \mathcal{H} \longrightarrow \mathcal{P}(\mathcal{H})$ is nonempty compact valued multifunction map such that
a) $(t, y) \longrightarrow F(t, y)$ is $\mathcal{L} \times \mathcal{D}$ measurable and for every $t \in J$, the multifunction $t \longrightarrow F(t, y_t)$ is measurable.
b) $(t, y) \longrightarrow F(t, y)$ is lower semi continuous for a.e. $t \in J$.

Theorem 3.4.3. *Under assumption (H1)-(H6), the problem 3.7 has at least one* $\mathcal{PC}_\gamma$*-mild solution.*

Proof. the proof is given in serval steps.
Consider the problem 3.7 on $[0, b]$

$$\begin{cases} D^{\alpha,\beta}_{0^+} x(t) \in Ax(t) + F(t, x_t) + g(t)\dfrac{dS^H_Q}{dt}, t \in J = [0, b], \\ (I^{1-\gamma}_0 x)(t)|_{t=0} = \varphi \in \mathcal{B}. \end{cases} \tag{3.8}$$

Let $\mathcal{PC}_\gamma = \{x : (-\infty, b] \longrightarrow \mathcal{H} : t^{1-\gamma} x(t) \in \mathcal{PC}\}$, with $\| x \|_{\mathcal{PC}_\gamma} = (\sup_{t\in J} E \| t^{1-\gamma} x(t) \|^2)^{\frac{1}{2}}$.
Thus $(\mathcal{PC}_\gamma, \| . \|_{\mathcal{PC}_\gamma})$ is a Banach space.
Let $\mathcal{D} = \mathcal{B} \cap \mathcal{PC}_\gamma$.

We transform the problem into fixed point theorem. Consider the multivalued operator $\Phi : \mathcal{D} \longrightarrow \mathcal{P}(\mathcal{D})$ defined by

$$\Phi(x) = \{\rho \in \mathcal{D} : \rho(t) = S_{\alpha,\beta}(t)\varphi + \int_0^t P_\beta(t - s)F(s, x(s))ds + \int_0^t P_\beta(t - s)g(s)dS^H_Q(s)\}.$$

Let $\hat{\phi}:[0,b]\longrightarrow \mathscr{H}$ be a function defined by $\hat{\phi}(t)=S_{\alpha,\beta}(t)\varphi$. Then $\hat{\phi}(t)$ is an element of $\mathscr{D}$. Let $x(t)=z(t)+\hat{\phi}(t)$ for $t\in[0,b]$, with
$z(t)=\int_0^t P_\beta(t-s)f(s)ds+\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)$, where $f(s)\in F(t,z_t+\hat{\phi}_t)$ for a.e. $t\in[0,b]$.

Let consider the operator $\hat{\Phi}:\mathscr{PC}_\gamma\longrightarrow\mathscr{P}(\mathscr{PC}_\gamma)$ defined by

$$\hat{\Phi}(z)=\{\hat{\rho}\in\mathscr{PC}_\gamma:\hat{\rho}(t)=\int_0^t P_\beta(t-s)f(s)ds+\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\}.$$

Now we show that $\hat{\Phi}$ satisfies the assumption of lemma 3.4.3.

Step1: $\hat{\Phi}(t)\in\mathscr{PC}_\gamma$ for each $z\in\mathscr{PC}_\gamma$.
Let $z_n\in\hat{\Phi}(z)$ and $\| z_n-z\|^2_{\mathscr{PC}_\gamma}\longrightarrow 0$ for $z\in\mathscr{PC}_\gamma$ and there exist $f_n\in S_{F,z+\hat{\phi}}$ such that

$$z_n(t)=\int_0^t P_\beta(t-s)f_n(s)ds+\int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

Since $F(t,z(t)+\hat{\phi}(t))$ is compact values and from hypothesis (H6), we pass to a subsequence if necessary to get that f_n converges to f in $L^2(J,\mathscr{H})$.
Then for each $t\in[0,b]$,

$$E\| z_n(t)-\int_0^t P_\beta(t-s)f(s)ds-\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)\|\longrightarrow 0 \text{ as } n\longrightarrow 0,$$

so there exist a $f(.)\in S_{F,z_t+\hat{\phi}}$ such that $z(t)=\int_0^t P_\beta(t-s)f(s)ds+\int_0^t P_\beta(t-s)g(s)dS_Q^H(s)$.
Step2: There exist $\delta<1$ such that $EH_d^2(\hat{\Phi}(z_1),\hat{\Phi}(z_2))\le\delta\| z_1-z_2\|_{\mathscr{PC}_\gamma}$ for any $z_1,z_2\in\mathscr{PC}_\gamma$.
Since for all $h_1\in\hat{\Phi}(z_1)$, there exist $f_1(.)\in S_{F,z_1+\hat{\phi}}$ such that

$$h_1(t)=\int_0^t P_\beta(t-s)f_1(s)ds+\int_0^t P_\beta(t-s)g(s)dS_Q^H(s).$$

We have $H_d(F(t,z_1(t))+\hat{\phi}(t),F(t,z_2(t))+\hat{\phi}(t))\le l(t)\| z_1-z_2\|$, so there exist
$h_2(t)=\int_0^t P_\beta(t-s)f_2(s)ds+\int_0^t P_\beta(t-s)g(s)dS_Q^H$.

We have

$$\begin{aligned}\| h_2(t)-h_1(t)\|^2_{\mathscr{P}\mathscr{C}_\gamma} &= \| \int_0^t P_\beta(t-s)(f_2(s)-f_1(s))ds \|^2_{\mathscr{P}\mathscr{C}_\gamma} \\ &\leq \sup_{0\leq t\leq b} t^{2(1-\gamma)} E\int_0^t \| P_\beta(t-s)(f_2(s)-f_1(s)) \|^2 ds \\ &\leq b^{2(1-\gamma)}\frac{M}{(\Gamma(\beta))^2} l_f(t) \| z_2(t)-z_1(t) \|^2 \int_0^t (t-s)^{2(\beta-1)}ds \\ &\leq \frac{Ml_f(t)}{(\Gamma(\beta))^2(2\beta-1)} b^{2(\beta-\gamma)+1} \| z_2-z_1 \|^2 \\ &\leq \tilde{l}(t) \| z_1-z_2 \|^2 .\end{aligned}$$

with $\tilde{l}(t)=\frac{b^{2\beta-2\gamma+1}}{(2\beta-1)(\Gamma(\beta))^2}Ml_f(t)$.

$EH_d^2(\hat{\Phi}(z_1)-\hat{\Phi}(z_2)) \leq \tilde{l}(t) \| z_2-z_1 \|^2$. So we conclude that $\hat{\Phi}$ is a contraction, and thus by lemma 3.4.3, $\hat{\Phi}$ has a fixed point so the problem admit at least one mild solution. □

3.4.2 An example

Consider the following stochastic differential inclusion

$$\begin{cases} D_{0^+}^{\frac{1}{2},\frac{1}{4}} y(t,\xi) \in \frac{\partial^2 y(t,\xi)}{\partial \xi^2} + F(t,x_t) + g(t)\frac{dS_Q^H}{dt}, t\in J=[0,b], \xi\in[0,\pi], \\ (I_0^{1-\gamma}y)(0)=y_0, \\ y(t,0)=y(t,\pi)=0. \end{cases}$$

Where $D_{0^+}^{\frac{1}{2},\frac{1}{4}}$ denotes the Hilfer fractional derivative.
Let $\mathscr{H}=L^2([0,\pi],\mathbb{R})$, $F:[0,b]\times\mathscr{H}\longrightarrow\mathscr{P}(\mathscr{H})$ is bounded, closed and convex multivalued map and satisfies the condition (H1)-(H3).
The operator $A: D(A)\subset\mathscr{H}\longrightarrow\mathscr{H}$ is defined by

$$D(A)=\{y\in\mathscr{H}/ y, y^{'} \text{are obsolutely continuous}, x^{''}\in\mathscr{H} | y(0)=y(\pi)=0\}.$$

S_Q^H is Q-sub fractional Brownian motion with Hurst parameter $H\in(\frac{1}{2},1)$.
$I_0^{1-\gamma}$ is the fractional integral of orders $1-\gamma$.
$Ay=y^{''}$ then $Ay=\sum_{n=1}^{\infty} n^2 < y, y_n > y_n$. where $y_n(t)=\sqrt{\frac{2}{n}}\sin(nt)$ $n=1,2,...$
We see that A generates a compact analytic semi group $\{T(t)\}_{t>0}$ in $\mathscr{H}$.
We assume that $f_i:[0,b]\times\mathscr{H}\longrightarrow\mathscr{H}, i=1,2$ such that

i) f_1 and f_2 are u.s.c.

ii) $f_1 < f_2$.

iii) For every $s > 0$ there exists a function $h_q \in L^2([0,b] \times \mathscr{H})$ such that $f_i(t,x) \leq h_q(t)$.

Let $g : J \longrightarrow L_2^0([0,4],\mathscr{H})$ such that $\int\limits_0^4 \frac{\sin(t)}{t^{\frac{1}{3}}} ds < \infty, \;\; p > -\frac{1}{2}$.

We take $F(t,x) = [f_1(t,x), f_2(t,x)]$.

All the assumptions in theorem 3.4.1 are verified thus this inclusion has a mild solution.

Chapter 4

Impulsive fractional stochastic differential inclusions

4.1 Introduction

In this section, we aim to study this interesting problem. We prove the existence of $\mathscr{PC}$ - mild solutions for impulsive fractional stochastic differential inclusions driven by sub-fractional Brownian motion with infinite delay and non-instantaneous impulses of the form

$$\begin{cases} {}^cD_t^{\alpha}x(t) \in Ax(t) + F(t,x_t) + g(t)\frac{dS_Q^H}{dt}, t \in (s_i, t_{i+1}], i = 0,1,...,N \\ x(0) = \varphi \in \mathscr{B}, \\ x(t) = I_i(t,x_t), t \in (t_i, s_i], i = 1,...,N \end{cases} \tag{4.1}$$

Where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative operator of order $\alpha \in (0,1)$ with the lower limit zero; $x(.)$ takes its values in the separable Hilbert spaces $\mathscr{H}$ with inner product $\langle.,.\rangle_{\mathscr{H}}$ and norm $\|\,.\,\|_{\mathscr{H}}$; A is a fractional sectorial operator defined on $\mathscr{H}$; $F : J \times \mathscr{H} \to 2^H - \{\emptyset\}$ is a multifunction, $J := [0,b]$, $0 = t_0 = s_0 < t_1 \le s_1 \le s_2 \le t_2 < ... < t_{N-1} \le s_N \le t_N \le t_{N+1} = b$ be prefixed numbers; $g : J \to \mathscr{L}_Q^0(\mathscr{K},\mathscr{H})$, $\mathscr{K}$ is another real separable Hilbert space with inner product $\langle.,.\rangle_{\mathscr{K}}$ and norm $\|\,.\,\|_{\mathscr{K}}$. Here $\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathscr{K}$ into $\mathscr{H}$ and S_Q^H is an Q-sub-fBm with Hurst parameter $H \in (\frac{1}{2},1)$. The history $x_t : (-\infty,0] \to \mathscr{H}$, $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase $\mathscr{B}$, $I_i \in C((t_i,s_i] \times \mathscr{B},\mathscr{H})$, for all $i = 1,...,N$. The initial data $\left\{\varphi(t) : -\infty < t \le 0\right\}$ is an $\mathscr{F}_0$-adapted $\mathscr{B}$-valued random variable independent of the sub-fBm with infinite second moment.

4.2 Preliminaries

In this part, we discuss some basic definitions, notations, theorems, lemmas and some basic facts about sub-fractional Brownian motion, the fractional calculus and sectorial operators.
Let $(\mathscr{H}, \| \, . \, \|_{\mathscr{H}}, \langle ., . \rangle_{\mathscr{H}})$ and $(\mathscr{K}, \| \, . \, \|_{\mathscr{K}}, \langle ., . \rangle_{\mathscr{K}})$ the separable Hilbert spaces. The notation $C(J, \mathscr{H})$ stand for the Banach space of continuous functions from J to $\mathscr{H}$ with supermum norm i.e., $\| x \|_J = \sup_{t \in J} \| x(t) \|$ and $L^1(J, \mathscr{H})$ denotes the Banach space of function $x : J \to \mathscr{H}$ which are Bochner integrable normed by $\| x \|_{L^1} = \int_0^b \| x(t) \| \, dt$, for all $x \in L^1(J, \mathscr{H})$. A measurable function $x : J \to \mathscr{H}$ is Bochner integrable if and only if $\| x \|$ is Lebesgue integrable. $B(\mathscr{H})$ is a Banach space of all linear bounded operator from $\mathscr{H}$ into itself with norm $\| F \|_{B(\mathscr{H})} = \sup \{ \| F(x) \| : \| x \| \leq 1 \}$.
Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathscr{F}_0$ containing all $\mathbb{P}$-null sets).

Lemma 4.2.1. *([72]) Let $x : (-\infty, b] \to \mathscr{H}$ be an $\mathscr{F}_t$-adapted measurable process such that the $\mathscr{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^0(\Omega, \mathscr{B})$ and the restriction $x : J \to L_2^{\mathscr{F}}(\Omega, \mathscr{B})$ is continuous, then*

$$\| x_s \|_{\mathscr{B}} \leq M_b \mathbb{E} \| \varphi \|_{\mathscr{B}} + K_b \sup_{0 \leq s \leq b} \mathbb{E} \| x(s) \|_{\mathscr{B}},$$

where $K_b = \sup\{K(t) : t \in J\}$ and $M_b = \sup\{M(t) : t \in J\}$.

We introduce the space $\mathscr{PC}$ formed by all $\mathscr{F}_t$-adapted measurable square integrable $\mathscr{H}$-valued stochastic processes $\{x(t) : t \in [0, b]\}$ such that x is continuous at $t \neq t_i$, $x(t_t) = x(t_t^-)$ and $x(t_t^+)$ exist for all $i = 1, \ldots, N$. We always assume that $\mathscr{PC}$ is endowed with the norm

$$\| x \|_{\mathscr{PC}} = \left(\sup_{0 \leq t \leq b} \mathbb{E} \| x(t) \|^2 \right)^{\frac{1}{2}}.$$

Then $(\mathscr{PC}, \| \, . \, \|_{\mathscr{PC}})$ is a Banach space.

Definition 4.2.1. *([25]) Let $\{\mathscr{Y}_n\}_{n \in \mathbb{N} \geq 1}$ be a sequence of subsets of $\mathscr{H}$. Suppose there is a compact and convex subset $\mathscr{Y} \subset \mathscr{H}$ such that for any neighborhood N of $\mathscr{Y}$ there is an n so that for any $m \geq n : \mathscr{Y}_m \subset N$. Then $\bigcap_{N > 0} \overline{conv}(\bigcup_{n \geq N} \mathscr{Y}_n) \subset \mathscr{Y}$.*

Lemma 4.2.2. *([25]) Every semicompact sequence in $L^1([0, b], \mathscr{H})$ is weakly compact in $L^1([0, b], \mathscr{H})$.*

Now, we introduce the Hausdorff measure of noncompactness $\chi_Z(.)$ defined by

$$\chi_Z(B) = \inf\{\varepsilon > 0 : B \textit{ admits a finite cover by balls of radius} \leq \varepsilon \quad in \quad \mathcal{Z}\}$$

for any Hilbert space $\mathcal{Z}$.
Some basic properties of $\chi_{\mathcal{Z}}(.)$ are given in the following lemma.

Lemma 4.2.3. *Let Z be a real Hilbert space and B be a bounded set in Z. Then, the following properties are satisfied:*

i. *B is pre-compact if and only if* $\chi_Z(B) = 0$.

ii. $\chi_Z(B) = \chi_Z(\overline{B}) = \chi_Z(convB)$, *where* $\overline{B}$ *and* $convB$ *are the closure and the convex hull of B, respectively.*

iii. $\chi_Z(B) \leq \chi_Z(C)$ *when* $B \subseteq C$.

iv. $\chi_Z(B+C) \leq \chi_Z(B) + \chi_Z(C)$ *where* $B + C = \{x + y : x \in B, y \in C\}$.

v. $\chi_Z(B \cup C) = max\{\chi_Z(B), \chi_Z(C)\}$.

vi. $\chi_Z(\lambda B) \leq |\lambda| \chi_Z(B)$ *for any* $\lambda \in \mathbb{R}$.

vii. *If the map* $\phi : D(\phi) \subseteq \mathcal{Z} \to \mathcal{Z}'$ *is Lipschitz continuous with constant k then* $\chi_Z(\phi B) \leq k\chi_Z(B)$ *for any bounded subset* $B \subseteq D(\phi)$, *where* $\mathcal{Z}'$ *is another real Hilbert space.*

viii. *If* $\{V_n\}_{n=1}^{\infty}$ *is a decreasing sequence of bounded closed nonempty subset of* $\mathbb{Z}$ *and* $\lim_{n\to\infty} \chi_Z(V_n) = 0$, *then* $\bigcap_{n=1}^{\infty} V_n$ *is nonempty and compact in* $\mathcal{Z}$.

Lemma 4.2.4. *Let W be a closed convex subset of a Banach space X and* $R : W \to P_{cl}(X)$ *be a closed multifunction which is* $\mathscr{X}$*-condensing where* $\mathscr{X}$ *is a non singular measure of noncompactness defined on subsets of W, then R has a fixed point.*

Lemma 4.2.5. *Let W be a closed subset of a Banach space X and* $R : W \to P_k(X)$ *be a closed multifunction which is* $\mathscr{X}$*-condensing on every bounded subset of W, where* $\mathscr{X}$ *is a monotone measure of noncompactness defined on X. if the set of fixed points for R is a bounded subset of X then it is compact.*

Lemma 4.2.6. *Let* (X, d) *be a complete metric space. If* $R : X \to P_{clb}(X)$ *is contraction, then R has a fixed point.*

Lemma 4.2.7. *Let B be a bounded set in Z. Then for every $\varepsilon > 0$ there is a sequence $(x_n)_{n\geq 1}$ in B such that*

$$\chi(B) \leq 2\chi\{x_n : n \geq 1\} + \varepsilon.$$

Lemma 4.2.8. *Let $\chi_{C(J,\mathscr{H})}$ be the Hausdorff measure of noncompactness on $C(J,\mathscr{H})$. If $W \subseteq C(J,\mathscr{H})$ is bounded, then for every $t \in J$,*

$$\chi(W(t)) \leq \chi_{C(J,\mathscr{H})}(W)$$

where $W(t) = \{x(t) : x \in W\}$. Furthermore, if W is equicontinuous on J, Then the map $t \to \chi\{x(t) : x \in W\}$ is continuous on J and

$$\chi_{C(J,\mathscr{H})}(W) = \sup_{t\in J} \chi\{x(t) : x \in W\}.$$

Lemma 4.2.9. *Let $\{f_n : n \in \mathbb{N}\} \subset L^p(J,\mathscr{H})$, $p \geq 1$ be an integrable bounded sequence such that $\chi\{f_n : n \geq 1\} \leq \gamma(t)$, a.e.$t \in J$, where $\gamma \in L^1(J,\mathbb{R}^+)$. Then for each $\varepsilon > 0$ there exists a compact $K_\varepsilon \subseteq E$, a measurable set $J_\varepsilon \subset J$, with measure less than ε, and a sequence of functions $\{g_n^\varepsilon\} \subseteq L^p(J,\mathscr{H})$, $t \in J$ and*
$\| f_n(t) - g_n^\varepsilon(t) \| < 2\gamma(t) + \varepsilon$, for every $n \geq 1$ and every $t \in J - J_\varepsilon$.

Next, we are ready to recall some facts of fractional Cauchy problem. Bajlekova [18] studied the following linear fractional Cauchy problem

$$\begin{cases} D_\alpha^c x(t) = Ax(t) \\ x(0) = x_0 \in \mathscr{H} \end{cases} \tag{4.2}$$

where A is linear closed and $D(A)$ is dense.

Definition 4.2.2. *A family $\{S_\alpha(t) : t \geq 0\} \subset \mathscr{L}(\mathscr{H})$ is called a solution operator for (4.2) if the following conditions are satisfied:*
(a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(t) = I$.
(b) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for $x \in D(a)$ and $t \geq 0$.
(c) $S_\alpha(t)x$ is a solution of (4.2) for all $x \in D(A)$ and $t \geq 0$.

Definition 4.2.3. *An operator A is said to be belong to $e^\alpha(M,\omega)$ if the solution operator $S_\alpha(.)$ of 4.2 satisfies*

$$\| S_\alpha(t) \|_{\mathscr{L}(\mathscr{H})} \leq Me^{\omega t}, t \geq 0$$

for some constants $M \geq 1$ and $\omega \geq 0$.

Definition 4.2.4. *A solution operator $S_\alpha(t)$ of 4.2 is called analytic if it admits an analytic extension to a sector $\Sigma_{\theta_0} = \{\lambda \in \mathbb{C} - \{0\} : \|arg\lambda\| < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type (θ_0, ω_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is an $M = M(\theta, \omega)$ such that*
$\| S_\alpha(t) \|_{\mathcal{L}(\mathcal{H})} \leq Me^{\omega \Re t}$, $t \in \Sigma_\theta$
Set
$e^\alpha(\omega) := \bigcup\{e^\alpha(M, \omega) : M \geq 1\}$ *and* $e^\alpha := \bigcup\{e^\alpha(\omega) : \omega \geq 0\}$,
$A^\alpha(\theta_0, \omega_0) = \{A \in e^\alpha :$ *A generates an analytic solution operator* S_α *of type* $(\theta_0, \omega_0)\}$

Lemma 4.2.10. *If $A \in A^\alpha(\theta_0, \omega_0)$ then*
$\| S_\alpha(t) \|_{\mathcal{L}(\mathcal{H})} \leq Me^{\omega t}$ *and* $\| T_\alpha(t) \|_{\mathcal{L}(\mathcal{H})} \leq Ce^{\omega t}(1 + t^{\alpha-1})$
for every $t > 0$, $\omega > \omega_0$. So putting
$\overline{M_s} := \sup_{0 \leq t \leq b} \| S_\alpha(t) \|_{\mathcal{L}(\mathcal{H})}$, $\overline{M_T} := \sup_{0 \leq t \leq b} Ce^{\omega t}(1 + t^{1-\alpha})$.
We get

$$\| S_\alpha(t) \|_{\mathcal{L}(\mathcal{H})} \leq \overline{M_s}, \| T_\alpha(t) \|_{\mathcal{L}(\mathcal{H})} \leq t^{\alpha-1}\overline{M_T}. \tag{4.3}$$

Definition 4.2.5. *Let $A \in A^\alpha(\theta_0, \omega_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in \mathbb{R}$. A function x is called a mild solution of (4.1) if*

$$x(t) = \begin{cases} S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_0 \\ S_\alpha(t)x_0 + S_\alpha(t-t_1)I_1(t_1^-) + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_1 \\ . \\ . \\ . \\ S_\alpha(t)x_0 + \sum_{i=1}^{N} S_\alpha(t-t_i)I_i(x(t_i^-)) + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_N \end{cases} \tag{4.4}$$

where $f \in S_{F(.,x(.))}$.
$S_{F(.,x(.)}$ is the set of the measurable selections of the multivalued map such that $S_{F(.,x(.))} = \{f \in L^2(J, \mathcal{H}) : f(t) \in F(t, x(t))\}$.

4.3 Existence of mild solution

Theorem 4.3.1. *Let $A \in A^\alpha(\theta_0, \omega_0)$ with $\theta \in (0, \frac{\pi}{2}]$ and $w_0 \in \mathbb{R}$, $F : J \times \mathcal{H} \to \mathcal{P}_{cv,cp}(\mathcal{H})$ a multifunction, $g : J \to \mathcal{L}_Q^0(\mathcal{K}, \mathcal{H})$ and $I_i \in C([t_i, s_i] \times \mathcal{B}, \mathcal{H})$.*
We assume the following conditions:
(H1) For any $x \in \mathcal{H}$, the multifunction $t \to F(t, x)$ is measurable and for all $t \in J$,

$x \to F(t,x)$ is upper semicontinuous.
(H2) There exists a function $\varphi \in L^{\frac{1}{q}}(J,\mathbb{R}^+)$, $q \in (0,\alpha)$ and a nondecreasing continuous function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $x \in \mathscr{H}$
$\| F(t,x) \| \leq \varphi(t)\Theta \| x \|$
(H3) i) There exist a function $\beta \in L^{\frac{1}{q}}(J,\mathbb{R}^+)$, $q \in (0,\alpha)$ satisfying

$$4\eta\overline{M_T} \| \beta \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} < 1, \tag{4.5}$$

where $\eta = \frac{b^{\alpha-q}}{\overline{w}^{1-q}}$ and $\overline{w} = \frac{\alpha-q}{1-q}$.
ii) For every bounded subset $Z \subseteq \mathscr{H}$
$\mathscr{X}(F(t,Z)) \leq \beta(t)\mathscr{X}(Z)$, for a.e. $t \in J$, where $\mathscr{X}$ is the Hausdorff measure of noncompactness in $\mathscr{H}$.

(H4) For $g : [0,b] \to \mathscr{L}_Q^0(\mathscr{K},\mathscr{H})$ we assume the following conditions: for the complete orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in K, we have:

$$\sum_{n=1}^{\infty} \| gQ^{\frac{1}{2}}e_n \|_{L^2([0,b],\mathscr{H})} < \infty$$

$$\sum_{n=1}^{\infty} \| g(t)Q^{\frac{1}{2}}e_n \|_{\mathscr{H}} \text{ converges uniformly for } t \in [0,b].$$

(H5) The function $g : J \to \mathscr{L}_Q^0(\mathscr{K},\mathscr{H})$ satisfies

$$\int_0^b \| g(s) \|^2_{\mathscr{L}_Q^0} ds = \Lambda < \infty.$$

(H6) For any $i = 1,2,...,N$, I_i is continuous and there exists a positive constant h_i such that

$$\| I_i(t,x) \|^2 \leq h_i \| x \|^2, x \in \mathscr{H}$$

Then the problem (4.1) has a mild solution provided that there is $r > 0$ such that

$$\begin{aligned} 3\overline{M_s}^2 e^{2\omega Rb} E \| x_0 \|^2 + \frac{3}{\alpha}\overline{M_T}^2 b^\alpha \int_0^b (b-s)^{\frac{\alpha-1}{2}} E \| f(s) \|^2 ds \\ + 3c_H b^{2H-1} \sum_{n=1}^{\infty} \int_0^t \| T_\alpha(b-s)Q^{\frac{1}{2}e_n} \|^2_{\mathscr{H}} ds \leq r \end{aligned} \tag{4.6}$$

Proof. We transform the problem (4.1) into a fixed point problem, we define a multifunction $R : PC(J,\mathscr{H}) \to 2^{PC(J,\mathscr{H})}$ as follows:

For $x \in PC(J,\mathscr{H})$, $R(x)$ is the set of all functions $y \in R(x)$ such that

$$y(t) = \begin{cases} S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_0 \\ S_\alpha(t)x_0 + S_\alpha(t-t_1)I_1(t_1^-) + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_1 \\ . \\ . \\ . \\ S_\alpha(t)x_0 + \sum_{i=1}^{N} S_\alpha(t-t_i)I_i(x(t_i^-)) + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_N \end{cases} \tag{4.7}$$

where $f \in S^1_{F(.,x(.))}$. By the hypothesis (H1) the values of R are nonempty. It is easy to see that any fixed point for R is a mild solution for 4.1. so our aim is to show, by using lemma 4.2.5, that R has a fixed point. The proof will be given in the following steps.
Step 1. We proof that the values of R are closed.
Let $x \in PC(J,\mathscr{H})$ and $\{y_n : n \geq 1\}$ be a sequence in $R(x)$ which is convergent to y in $PC(J,\mathscr{H})$. Then according to the definition of R, there is a sequence $\{f_n : n \geq 1\}$ in $S^1_{F(.,x(.))}$ such that for any $t \in J_i$, $i = 0,1,...,N$, we have

$$y_n(t) = \begin{cases} S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f_n(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_0 \\ S_\alpha(t)x_0 + S_\alpha(t-t_1)I_1(t_1^-) + \int_0^t T_\alpha(t-s)f_n(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_1 \\ . \\ . \\ . \\ S_\alpha(t)x_0 + \sum_{i=1}^{N} S_\alpha(t-t_i)I_i(x(t_i^-)) + \int_0^t T_\alpha(t-s)f_n(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_N \end{cases} \tag{4.8}$$

By the assumption (H2) for every $n \geq 1$, and for a.e. $t \in J$

$$\| f_n(t) \| \leq \varphi(t)\Theta(\| x \|) \leq \varphi(t)\Theta(\| x \|_{PC(J,\mathscr{H})})$$

This show that the set $\{f_n : n \geq 1\}$ is integrally bounded. Therefore for a.e. $t \in J$ $\{f_n : n \geq 1\} \subset F(t,x(t))$, the set $\{f_n : n \geq 1\}$ is relatively compact in $\mathscr{H}$ for a.e.$t \in J$. Moreover, the set $\{f_n : n \geq 1\}$ is semicompact and then by lemma 4.2.2 it is weakly compact in $L^1(J,\mathscr{H})$. So, without loss of generality we can assume that f_n converges weakly to a function $f \in L^1(J,\mathscr{H})$. From Mazur's lemma, for any $j \in \mathbb{N}$ there exist a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}$, $k = j,...,k_0(j)$ such that $\sum_{k=j}^{k_0(j)} \lambda_{j,k} = 1$ and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0(j)} \lambda_{j,k} f_k$, $j \geq 1$ converges strongly to f in $L^1(J,\mathcal{H})$ as $j\to\infty$. so we can suppose that $z_j(t)\to f(t)$ for a.e. $t\in J$. Since F takes convex and closed values, we obtain for a.e. $t\in J$

$$f(t)\in\bigcap_{j\geq 1}\{z_k(t):k\geq j\}\subseteq\bigcap_{j\geq 1}\overline{conv}\{f_k:k\geqslant j\}\subset F(t,x(t)).$$

Noting that, by (4.3) for every $t,s\in J$, $s\in[0,t]$ and $n\geq 1$

$$\| T_\alpha(t-s)z_n(s)\|\leq (t-s)^{\alpha-1}\overline{M_T}\varphi(s)\Theta\| x\|_{PC(J,\mathcal{H})}.$$

Next taking $\tilde{y}_n(t)=\sum_{k=j}^{k(j)}\lambda_{jk}y_k$,(4.8) implies

$$\tilde{y}_n(t)=\begin{cases} S_\alpha(t)x_0+\int_0^t T_\alpha(t-s)z_n(s)ds+\int_0^t T_\alpha(t-s)g(s)dS_Q^H(s),\ t\in J_0\\ S_\alpha(t)x_0+S_\alpha(t-t_1)I_1(t_1^-)+\int_0^t T_\alpha(t-s)z_n(s)ds+\int_0^t T_\alpha(t-s)g(s)dS_Q^H(s),\ t\in J_1\\ .\\ .\\ .\\ S_\alpha(t)x_0+\sum_{i=1}^{N}S_\alpha(t-t_i)I_i(x(t_i^-))+\int_0^t T_\alpha(t-s)z_n(s)ds+\int_0^t T_\alpha(t-s)g(s)dS_Q^H(s),\ t\in J_N\end{cases}\tag{4.9}$$

But $\tilde{y}_n(t)\to y(t)$ and $\tilde{z}_n(t)\to f(t)$ for a.e. $t\in J$, therefore, by tending n to ∞ in (4.9), we get from the Lebesgue dominated convergence theorem that for every $i=0,1,...,N$.

$$y(t)=\begin{cases} S_\alpha(t)x_0+\int_0^t T_\alpha(t-s)f(s)ds+\int_0^t T_\alpha(t-s)g(s)dS_Q^H(s),\ t\in J_0\\ S_\alpha(t)x_0+S_\alpha(t-t_1)I_1(t_1^-)+\int_0^t T_\alpha(t-s)f(s)ds+\int_0^t T_\alpha(t-s)g(s)dS_Q^H(s),\ t\in J_1\\ .\\ .\\ .\\ S_\alpha(t)x_0+\sum_{i=1}^{N}S_\alpha(t-t_i)I_i(x(t_i^-))+\int_0^t T_\alpha(t-s)f(s)ds+\int_0^t T_\alpha(t-s)g(s)dS_Q^H(s),\ t\in J_N\end{cases}\tag{4.10}$$

This proves that $R(x)$ is closed.

Step 2. Set $B_0=\{x\in PC(J,\mathcal{H}):\| x\|_{PC}\leq r\}$. Obviously, B_0 is a bounded, closed and convex subset of $PC(J,\mathcal{H})$. We want to prove that $R(B_0)\subseteq B_0$. to show that, let $x\in B_0$

and $y \in R(x)$. By using (4.3), (4.6), (4.9); (H2) and Holder's inequality, we get for $t \in J_0$.

$$\begin{aligned}
E\left\|y(t)\right\|^2 &= E\left\|S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s)\right\|^2 \\
&\leq 3E\|S_\alpha(t)x_0\|^2 + 3E\left\|\int_0^t T_\alpha(t-s)f(s)ds\right\|^2 + 3E\left\|\int_0^t T_\alpha(t-s)g(s)dS_Q^H(s)\right\|^2 \\
&\leq 3\overline{M}_s^2 e^{2\omega R t} E\parallel x_0\parallel^2 + 3\overline{M}_T^2 \frac{t^\alpha}{\alpha}\int_0^t (t-s)^{\frac{\alpha-1}{2}} E\parallel f(s)\parallel^2 ds \\
&+ 3c_H t^{2H-1}\sum_{n=1}^{\infty}\int_0^t \parallel T_\alpha(t-s)Q^{\frac{1}{2}}e_n\parallel^2 ds.
\end{aligned}$$

We get for every $t \in J_i, i = 1,2,...,N$

$$\parallel y(t)\parallel_{PC}^2 \leq r < \infty.$$

Therefore $R(B_0) \subseteq B_0$.

Step 3. Let $Z = R(B_0)$. In this step we will show that the set defined as follows

$$Z_{|\overline{J}_i} = \{y^* \in C(\overline{J}_i, \mathcal{H}) : y^*(t) = y(t), t \in J_i, y^*(t_i) = y(t_i^+) y \in Z\}$$

is equicontinuous for every $i = 1,2,...,N$.

Let $y \in Z$. Then there is $x \in B_0$ with $y \in R(x)$. According to the definition of R, there is $f \in S^1_{F(.,x(.))}$ such that

$$y(t) = \begin{cases}
S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_0 \\
S_\alpha(t)x_0 + S_\alpha(t-t_1)I_1(t_1^-) + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_1 \\
. \\
. \\
. \\
S_\alpha(t)x_0 + \sum_{i=1}^{N} S_\alpha(t-t_i)I_i(x(t_i^-)) + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_N.
\end{cases} \quad (4.11)$$

We consider the following cases:

Case1. When $i = 0$, we consider two points t and $t+\delta$ be two points in $\overline{J}_0$, then:

$$\begin{aligned}
\left\| y^*(t+\delta) - y^*(t) \right\| &= \| S_\alpha(t+\delta)x_0 + \int_0^{t+\delta} T_\alpha(t+\delta-s)f(s)ds + \int_0^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \\
&\quad - S_\alpha(t)x_0 - \int_0^t T_\alpha(t-s)f(s)ds - \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \| \\
&= \| (S_\alpha(t+\delta) - S_\alpha(t))x_0 + \int_0^{t+\delta} T_\alpha(t+\delta-s)f(s)ds - \int_0^t T_\alpha(t-s)f(s)ds \\
&\quad + \int_0^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) - \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \| \\
&= \| (S_\alpha(t+\delta) - S_\alpha(t))x_0 + \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))f(s)ds \\
&\quad + \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds + \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))g(s)dS_Q^H(s) \\
&\quad + \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \| \\
&\le \| (S_\alpha(t+\delta) - S_\alpha(t))x_0 \| + \| \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))f(s)ds \| \\
&\quad + \| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \| + \| \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))g(s)dS_Q^H(s) \| \\
&\quad + \| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \|,
\end{aligned}$$

$$\begin{aligned}
E \| y^*(t+\delta) - y^*(t) \|^2 \le & 3E \| (S_\alpha(t+\delta) - S_\alpha(t))x_0 \|^2 \\
&+ 3E \| \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))f(s)ds \|^2 \\
&+ 3E \| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \|^2 \\
&+ 3E \| \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))g(s)dS_Q^H(s) \|^2 \\
&+ 3E \| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \|^2 := 3(G_1 + G_2 + G_3 + G_4 + G_5).
\end{aligned} \tag{4.12}$$

Where

$$G_1 = E \parallel (S_\alpha(t+\delta) - S_\alpha(t))x_0 \parallel^2,$$
$$G_2 = E \parallel \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))f(s)ds \parallel^2,$$
$$G_3 = E \parallel \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))g(s)dS_Q^H(s) \parallel^2,$$
$$G_4 = E \parallel \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \parallel^2,$$
$$G_5 = E \parallel \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \parallel^2 .$$

We only need to check $G_i \to 0$ as $\delta \to 0$ for every $i = 1,2,3,4,5$.
for G_1 we have

$$\begin{aligned} G_1 &= E \parallel S_\alpha(t+\delta) - S_\alpha(t))x_0 \parallel^2 \\ &\leq \parallel S_\alpha(t+\delta) - S_\alpha(t) \parallel^2 E \parallel x_0 \parallel^2 \\ &\leq \parallel S_\alpha(t+\delta) - S_\alpha(t) \parallel^2 r^{\frac{1}{2}}. \end{aligned}$$

$$\sup_{0\leq t\leq b} E \parallel S_\alpha(t+\delta) - S_\alpha(t))x_0 \parallel^2 \leq \sup_{0\leq t\leq b} \parallel S_\alpha(t+\delta) - S_\alpha(t) \parallel r^{\frac{1}{2}}$$
$$\parallel (S_\alpha(t+\delta) - S_\alpha(t))x_0 \parallel_{PC}^2 \leq \sup_{0\leq t\leq b} \parallel S_\alpha(t+\delta) - S_\alpha(t) \parallel r^{\frac{1}{2}}$$
$$\lim_{\delta\to 0} \parallel (S_\alpha(t+\delta) - S_\alpha(t))x_0 \parallel_{PC}^2 \leq \lim_{\delta\to 0} \sup_{0\leq t\leq b} \parallel S_\alpha(t+\delta) - S_\alpha(t) \parallel r^{\frac{1}{2}} = 0.$$

uniformly for $x \in B_0$.
For G_2, we apply the Lebesgue dominated convergence theorem to get

$$\begin{aligned} G_2 &= E \parallel \int_0^t (T_\alpha(t+\delta-s) - T_\alpha(t-s))f(s)ds \parallel^2 \\ &\leq E\left(\int_0^t \parallel (T_\alpha(t+\delta-s) - T_\alpha(t-s)) \parallel \parallel f(s) \parallel ds\right)^2 \end{aligned}$$
$$\sup_{0\leq t\leq b} G_2 \leq \sup_{0\leq t\leq b} E\left(\int_0^t \parallel (T_\alpha(t+\delta-s) - T_\alpha(t-s)) \parallel \parallel f(s) \parallel ds\right)^2$$
$$\lim_{\delta\to 0} \sup_{0\leq t\leq b} G_2 \leq \lim_{\delta\to 0} \sup_{0\leq t\leq b} E\left(\int_0^t \parallel (T_\alpha(t+\delta-s) - T_\alpha(t-s)) \parallel \parallel f(s) \parallel ds\right)^2 = 0.$$

For G_3 we use holder's inequality we obtain

$$\begin{aligned}\| \int_0^t (T_\alpha(t+\delta-s)-T_\alpha(t-s))g(s)dS_Q^H(s) \| &= \| \int_0^t T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \\ &\quad - \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \| \\ &\leq \| \int_0^t T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \| + \| \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \| \\ &\leq \sup_{0\leq t\leq b} T_\alpha(t+\delta-s) \| \int_0^t g(s)dS_Q^H(s) \| + \sup_{0\leq t\leq b} T_\alpha(t-s) \| \int_0^t g(s)dS_Q^H(s) \| \\ &\leq (t+\delta-s)^{\alpha-1}\overline{M}_T \| \int_0^t g(s)dS_Q^H(s) \| + \overline{M}^T (t-s)^{\alpha-1} \| \int_0^t g(s)dS_Q^H(s) \|\end{aligned}$$

$$\begin{aligned}\| \int_0^t (T_\alpha(t+\delta-s)-T_\alpha(t-s))g(s)dS_Q^H(s) \|^2 &\leq 2(t+\delta-s)^{2(\alpha-1)}\overline{M}_T^2 \| \int_0^t g(s)dS_Q^H(s) \|^2 \\ &\quad + 2\overline{M}_T^2 (t-s)^{2(\alpha-1)} \| \int_0^t g(s)dS_Q^H(s) \|^2\end{aligned}$$

$$\begin{aligned}E \| \int_0^t (T_\alpha(t+\delta-s)-T_\alpha(t-s))g(s)dS_Q^H(s) \|^2 &\leq 2(t+\delta-s)^{2(\alpha-1)}\overline{M}_T^2 E \| \int_0^t g(s)dS_Q^H(s) \|^2 \\ &\quad + 2\overline{M}_T^2 (t-s)^{2(\alpha-1)} E \| \int_0^t g(s)dS_Q^H(s) \|^2 \\ &\leq 2(t+\delta-s)^{2(\alpha-1)}\overline{M}_T^2 c_H t^{2H-1} \int_0^t \| g(s) \|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})} ds \\ &\quad + 2\overline{M}_T^2 (t-s)^{2(\alpha-1)} c_H t^{2H-1} \int_0^t \| g(s) \|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})} ds\end{aligned}$$

$$\begin{aligned}\sup_{0\leq t\leq b} E \| \int_0^t (T_\alpha(t+\delta-s)-T_\alpha(t-s))g(s)dS_Q^H(s) \|^2 &\leq \sup_{0\leq t\leq b} \Big[2(t+\delta-s)^{2(\alpha-1)}\overline{M}_T^2 c_H t^{2H-1} \\ &\quad \int_0^t \| g(s) \|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})} ds + 2\overline{M}_T^2 (t-s)^{2(\alpha-1)} c_H \int_0^t \| g(s) \|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})} ds \Big]\end{aligned}$$

$$\begin{aligned}\Big(\sup_{0\leq t\leq b} E \| \int_0^t (T_\alpha(t+\delta-s)-T_\alpha(t-s))g(s)dS_Q^H(s) \|^2 \Big)^{\frac{1}{2}} &\leq \Big(\sup_{0\leq t\leq b} [2(t+\delta-s)^{2(\alpha-1)}\overline{M}_T^2 c_H t^{2H-1} \\ &\quad \int_0^t \| g(s) \|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})} ds + 2\overline{M}_T^2 (t-s)^{2(\alpha-1)} c_H t^{2H-1} \int_0^t \| g(s) \|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})} ds] \Big)^{\frac{1}{2}}.\end{aligned}$$

$$\begin{aligned}\lim_{\delta\to 0} \Big(\sup_{0\leq t\leq b} E \| \int_0^t (T_\alpha(t+\delta-s)-T_\alpha(t-s))g(s)dS_Q^H(s) \|^2 \Big)^{\frac{1}{2}} &\leq 2(t-s)^{\alpha-1} c_H^{\frac{1}{2}} t^{\frac{2H-1}{2}} \\ &\quad \Big(\int_0^t \| g(s) \|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})} ds \Big)^{\frac{1}{2}}.\end{aligned}$$

$$\| \int_0^t [T_\alpha(t+\delta-s)-T_\alpha(t-s)]g(s)dS_Q^H(s) \|_{PC} \le 2(t-s)^{\alpha-1} c_H^{\frac{1}{2}} t^{\frac{2H-1}{2}} \left(\int_0^t \| g(s) \|^2_{\mathscr{L}_Q^0(\mathscr{K},\mathscr{H})} ds \right)^{\frac{1}{2}}.$$

For G_4, by the Holder's inequality we have

$$\begin{aligned}
\| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \| &\le \int_t^{t+\delta} \| T_\alpha(t+\delta-s) \| \| f(s) \| ds \\
&\le \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \overline{M}_T \| f(s) \| ds \\
&\le \overline{M}_T \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \| f(s) \| ds \\
&\le \overline{M}_T \left(\int_t^{t+\delta} (t+\delta-s)^{(\alpha-1)p} ds \right)^{\frac{1}{p}} \left(\int_t^{t+\delta} \| f(s) \|^q ds \right)^{\frac{1}{q}} \\
&\le \overline{M}_T \left(\int_t^{t+\delta} (t+\delta-s)^{\frac{(\alpha-1)q}{q-1}} ds \right)^{\frac{q-1}{q}} \left(\int_t^{t+\delta} \| \varphi^q(s) \| \Theta^p \| x \| ds \right)^{\frac{1}{p}} \\
&\le \overline{M}_T \left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{q-1}} ds \right)^{q-1} \Theta \| x \| \left(\int_t^{t+\delta} \| \varphi^q(s) \| ds \right)^{\frac{1}{q}} \\
&\le \overline{M}_T \left(\frac{\delta^\omega}{\omega} \right)^{q-1} \Theta \| x \| \| \varphi \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}.
\end{aligned}$$

Where $\omega = \left(\frac{\alpha-1}{q-1}\right) q + 1$

$$\begin{aligned}
\| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \| &\le \overline{M}_T \left(\frac{\delta^\omega}{\omega} \right)^{q-1} \Theta \| x \| \| \varphi \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \\
\| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \|^2 &\le \overline{M}_T^2 \left(\frac{\delta^\omega}{\omega} \right)^{2(q-1)} \Theta^2 \| x \| \| \varphi \|^2_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \\
E \| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \|^2 &\le \overline{M}_T^2 \left(\frac{\delta^\omega}{\omega} \right)^{2(q-1)} E\left(\Theta^2 \| x \| \| \varphi \|^2_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \right) \\
\sup_{0\le t\le b} E \| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \|^2 &\le \sup_{0\le t\le b} \overline{M}_T^2 \left(\frac{\delta^\omega}{\omega} \right)^{2(q-1)} E\left(\Theta^2 \| x \| \| \varphi \|^2_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \right) \\
\left(\sup_{0\le t\le b} E \| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \|^2 \right)^{\frac{1}{2}} &\le \left(\sup_{0\le t\le b} \overline{M}_T^2 \right)^{\frac{1}{2}} \left(\frac{\delta^\omega}{\omega} \right)^{q-1} E^{\frac{1}{2}} \left(\Theta^2 \| x \| \| \varphi \|^2_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \right)
\end{aligned}$$

$$\lim_{\delta\to 0} \left(\sup_{0\le t\le b} E \| \int_t^{t+\delta} T_\alpha(t+\delta-s)f(s)ds \|^2 \right)^{\frac{1}{2}} \le 0.$$

For G_5 we have

$$\left\| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \right\| \leq \sup_{0\leq t\leq b} T_\alpha(t+\delta-s) \left\| \int_t^{t+\delta} g(s)dS_Q^H(s) \right\|$$
$$\left\| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \right\| \leq \sup_{0\leq t\leq b} T_\alpha(t+\delta-s) \left\| \int_t^{t+\delta} g(s)dS_Q^H(s) \right\|$$

$$\left\| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \right\|^2 \leq \left(\sup_{0\leq t\leq b} T_\alpha(t+\delta-s) \right)^2 \left\| \int_t^{t+\delta} g(s)dS_Q^H(s) \right\|^2$$
$$E\left\| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \right\|^2 \leq \left(\sup_{0\leq t\leq b} T_\alpha(t+\delta-s) \right)^2 E\left\| \int_t^{t+\delta} g(s)dS_Q^H(s) \right\|^2$$
$$\leq M^2 2H(t+\delta-s)^{2H-1} \int_t^{t+\delta} \| g(s) \|^2_{L_Q^0(K,\mathscr{H})} ds.$$

$$\sup_{0\leq t\leq b} E\left\| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \right\|^2 \leq M^2 2H\delta^{2H-1} \int_t^{t+\delta} \|g(s)\|^2_{L_Q^0(K,\mathscr{H})} ds$$
$$\left(\sup_{0\leq t\leq b} E\left\| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \right\|^2 \right)^{\frac{1}{2}}$$
$$\leq M\sqrt{2H}\delta^{\frac{2H-1}{2}} \left(\int_t^{t+\delta} \|g(s)\|^2_{L_Q^0(K,\mathscr{H})} ds \right)^{\frac{1}{2}}.$$

$$\lim_{\delta\to 0} \left(\sup_{0\leq t\leq b} E\left\| \int_t^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) \right\|^2 \right)^{\frac{1}{2}} \leq 0.$$

Case2. For $i \in \{1,2,...,N\}$, let $t, t+\delta$ be two points in J_i. According to the definition of R, we have

$$\| y^*(t+\delta) - y^*(t) \| = \| y(t+\delta) - y(t) \|.$$

$$\| y(t+\delta) - y(t) \| = \| S_\alpha(t+\delta)x_0 + \sum_{i=1}^{N} S_\alpha(t+\delta-t_i)I_i(x(t_i^-)) + \int_0^{t+\delta} T_\alpha(t+\delta-s)f(s)ds$$
$$+ \int_0^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) - S_\alpha(t)x_0 - \sum_{i=1}^{N} S_\alpha(t-t_i)I_i(x(t_i^-))$$
$$- \int_0^t T_\alpha(t-s)f(s)ds - \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \|.$$

$$\begin{aligned}\| y(t+\delta)-y(t) \| \leq& \| (S_\alpha(t+\delta)-S_\alpha(t))\,x_0 \| + \sum_{k=1}^{i} \left\| S_\alpha(t+\delta-t_k)I_k(x(t_k^-)) - S_\alpha(t-t_k)I_k(x(t_k^-)) \right\| \\ &+ \left\| \int_0^{t+\delta} T_\alpha(t+\delta-s)f(s)ds - \int_0^t T_\alpha(t-s)f(s)ds \right\| \\ &+ \left\| \int_0^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) - \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \right\|\end{aligned}$$

$$\begin{aligned}\| y(t+\delta)-y(t) \|^2 \leq& 3\,\| (S_\alpha(t+\delta)-S_\alpha(t))\,x_0 \|^2 + 3\sum_{k=1}^{i} \left\| S_\alpha(t+\delta-t_k)I_k(x(t_k^-)) - S_\alpha(t-t_k)I_k(x(t_k^-)) \right\|^2 \\ &+ 3\left\| \int_0^{t+\delta} T_\alpha(t+\delta-s)f(s)ds - \int_0^t T_\alpha(t-s)f(s)ds \right\|^2 \\ &+ 3\left\| \int_0^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) - \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \right\|^2.\end{aligned}$$

$$\begin{aligned}E\| y(t+\delta)-y(t) \|^2 \leq& 3E\| (S_\alpha(t+\delta)-S_\alpha(t))\,x_0 \|^2 \\ &+ 3E\sum_{k=1}^{i} \left\| S_\alpha(t+\delta-t_k)J_k(x(t_k^-)) - S_\alpha(t-t_k)I_k(x(t_k^-)) \right\|^2 \\ &+ 3E\left\| \int_0^{t+\delta} T_\alpha(t+\delta-s)f(s)ds - \int_0^t T_\alpha(t-s)f(s)ds \right\|^2 \\ &+ 3E\left\| \int_0^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) - \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \right\|^2.\end{aligned}$$

$$\begin{aligned}\sup_{0\leq t\leq b} E\| y(t+\delta)-y(t) \|^2 \leq& 3\sup_{0\leq t\leq b} E\| (S_\alpha(t+\delta)-S_\alpha(t))\,x_0 \|^2 \\ &+ 3\sup_{0\leq t\leq b} E\sum_{k=1}^{i} \left\| S_\alpha(t+\delta-t_k)J_k(x(t_k^-)) - S_\alpha(t-t_k)I_k(x(t_k^-)) \right\|^2 \\ &+ 3\sup_{0\leq t\leq b} E\left\| \int_0^{t+\delta} T_\alpha(t+\delta-s)f(s)ds - \int_0^t T_\alpha(t-s)f(s)ds \right\|^2 \\ &+ 3\sup_{0\leq t\leq b} E\left\| \int_0^{t+\delta} T_\alpha(t+\delta-s)g(s)dS_Q^H(s) - \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s) \right\|^2.\end{aligned}$$

As in the first case we get

$$\lim_{\delta\to 0} \| y(t+\delta)-y(t) \|_{PC} = 0.$$

Case3. When $t = t_i$, $i = 1,2,\ldots,N$, let $\lambda > 0$ be such that $t_i + \lambda \in J_i$ and $\sigma > 0$ such that

$t_i < \sigma < t_i + \delta \leq t_{i+1}$, then we have

$$\| y^*(t_i+\delta) - y^*(t_i) \|_{PC} = \lim_{\sigma \to t_i^+} \| y(t_i+\delta) - y(\sigma) \|_{PC}.$$

According to the definition of R we get

$$\begin{aligned}\| y(t_i+\delta) - y(\sigma) \| = &\| S_\alpha(t_i+\delta)x_0 + \sum_{k=1}^{N} S_\alpha(t_i+\delta-t_k)I_k(x(t_k^-)) + \int_0^{t_i+\delta} T_\alpha(t_i+\delta-s)f(s)ds \\ &+ \int_0^{t_i+\delta} T_\alpha(t_i+\delta-s)g(s)dS_Q^H(s) - S_\alpha(\sigma)x_0 - \sum_{k=1}^{N} S_\alpha(\sigma-t_k)I_k(x(t_k^-)) \\ &- \int_0^{\sigma} T_\alpha(\sigma-s)f(s)ds - \int_0^{t} T_\alpha(\sigma-s)g(s)dS_Q^H(s) \|.\end{aligned}$$

$$\begin{aligned}\| y(t_i+\delta) - y(\sigma) \| \leq &\| (S_\alpha(t_i+\delta) - S_\alpha(\sigma))\, x_0 \| + \left\| \sum_{k=1}^{N} (S_\alpha(t_i+\delta-t_k) - S_\alpha(\sigma-t_k))\, I_k(x(t_k^-)) \right\| \\ &+ \left\| \int_0^{t_i+\delta} T_\alpha(t_i+\delta-s)f(s)ds - \int_0^{\sigma} T_\alpha(\sigma-s)f(s)ds \right\| \\ &+ \left\| \int_0^{t_i+\delta} T_\alpha(t_i+\delta-s)g(s)dS_Q^H(s) - \int_0^{\sigma} T_\alpha(\sigma-s)g(s)dS_Q^H(s) \right\|.\end{aligned}$$

Arguing as in the first case we can see that

$$\lim_{\delta \to 0, \sigma \to t_i^+} \| y(t_i+\delta) - y(\sigma) \| = 0 \tag{4.13}$$

From the inequalities (4.12)-(4.13) we conclude that $Z_{|\bar{J}_i}$ is equicontinuous for every $i = 1,2,..,m$.

Now for every $n \geq 1$, the set $B_n = \overline{convR}(B_{n-1})$. From step 1, B_n is a nonempty, closed and convex subset of $\mathscr{PC}(J,\mathscr{H})$. Moreover $B_1 = \overline{convR}(B_0) \subseteq B_0$. Also $B_2 = \overline{convR}(B_1) \subseteq \overline{convR}(B_0) \subseteq B_1$ by induction the sequence (B_n), $n \geq 1$ is decreasing sequence of nonempty, closed and bounded subsets of $\mathscr{PC}(J,\mathscr{H})$.

We need only to show that the subset $B = \bigcap_{n=1}^{\infty} B_n$ is nonempty and compact in $\mathscr{PC}(J,\mathscr{H})$. by lemma4.2.3, it is enough to show that

$$\lim_{n \to \infty} \chi_{PC}(B_n) = 0. \tag{4.14}$$

where χ_{PC} is the Hausdorff measure of noncompactness on $PC(J,\mathscr{H})$ defined in section 2. In the next step we prove the equation (4.14).

Step 4. Let $n \geq 1$ be a fixed natural number and $\varepsilon > 0$. In view of lemma4.2.7, there exists a sequence $(y_k)_{k\geq 1}$ in $R(B_{n-1})$ such that

$$\chi_{PC}(B_n) = \chi_{PC}R(B_{n-1}) \leq 2\chi_k\left\{y_k : k \geq 1\right\} + \varepsilon.$$

From the definition of χ_{PC}, the above inequality becomes

$$\chi_{PC}(B_n) \leq 2 \max_{i=0,1,\ldots,N} \chi_i(S_{|\bar{J}_i}) + \varepsilon \tag{4.15}$$

Where $S = \left\{y_k : k \geq 1\right\}$ and χ_i is the Hausdorff measure of noncompactness on $C(\bar{J}_i, \mathcal{H})$. As we have done in the previous step, we can show that $B_{n|\bar{J}_i}$, $i = 0, 1, \ldots, N$ is equicontinuous. Then, by applying lemma4.2.9 we get:

$$\chi_i(S_{|\bar{J}_i}) = \sup_{t\in\bar{J}_i} \chi(S(t)),$$

where χ is the Hausdorff measure of noncompactness on Z. Therefore, by using the nonsingularity of χ, the inequality (4.15) becomes

$$\chi_{PC}(B_n) \leq 2 \max_{i=0,1,\ldots,N} \left[\sup_{t\in\bar{J}_i} \chi(S(t))\right] + \varepsilon = 2\sup_{t\in J} \chi(S(t)) = 2\sup_{t\in J} \chi\left\{y_k(t) : k \geq 1\right\} + \varepsilon \tag{4.16}$$

Now, since $y_k \in R(B_{n-1})$, $k \geq 1$ there exists $x_k \in B_{n-1}$ such that $y_k \in R(x_k)$, $k \geq 1$. By recalling the definition of R for every $k \geq 1$ there is $f_k \in S^1_{F(.,x_k(.))}$ such that for every $t \in J$

$$\chi\left\{y_k(t) : k \geq 1\right\} \leq \begin{cases} \chi\left\{S_\alpha(t)x_0\right\} + \chi\left\{\int_0^t T_\alpha(t-s)f_k(s)ds : k \geq 1\right\} \\ +\chi\left\{\int_0^t T_\alpha(t-s)g_k(s)dS_Q^H(s) : k \geq 1\right\}, t \in J_0 \\ . \\ . \\ . \\ \chi\left\{S_\alpha(t)x_0\right\} + \sum_{p=1}^{N} \chi\left\{S_\alpha(t-t_p)I_p(x(t_p^-)) : k \geqslant 1\right\} \\ +\chi\left\{\int_0^t T_\alpha(t-s)f_k(s)ds : k \geq 1\right\} + \chi\left\{\int_0^t T_\alpha(t-s)g_k(s)dS_Q^H(s) : k \geq 1\right\}, t \in J_N \end{cases} \tag{4.17}$$

Hence, for every $t \in J$ we have

$$\chi\{S_\alpha(t)x_0 : k \geq 1\} = 0. \tag{4.18}$$

Moreover for every $p = 1, 2, \ldots, N$ and every $t \in J$

$$\chi\{S_\alpha(t-t_p)(I_p(x_k(t_p^-))) : k \geq 1\} = 0. \tag{4.19}$$

In order to be able to estimate
$\chi\{\int_0^t T_\alpha(t-s)f_k(s)ds : k \geq 1\}$
We can see that from (H3) it holds that for a.e. $t \in J$

$$\begin{aligned}\chi\{f_k(t) : k \geq 1\} &\leq \chi\{F(t, x_k(t)) : k \geq 1\}\\ &\leq \beta(t)\chi\{x_k(t) : k \geq 1\}\\ &\leq \beta(t)\chi(B_{n-1}(t))\\ &\leq \beta(t)\chi_{PC}(B_{n-1}(t)) = \gamma(t).\end{aligned}$$

Furthermore, for any $k \geq 1$, by (H2), for almost $t \in J$, we have $\| f_k(t) \| \leq \varphi(t)\Theta(r)$. Consequently, $f_k \in L^{\frac{1}{q}}(J, \mathscr{H})$, $k \geq 1$. Note that $\gamma \in L^{\frac{1}{q}}(J, \mathbb{R}^+)$. Then from lemma4.2.9, there exists a compact set $K_\varepsilon \subseteq H$ and a measurable set $J_\varepsilon \subset J$. With a measure less than ε, and a sequence of functions $\{g_k^\varepsilon\} \subset L^{\frac{1}{q}}(J, \mathscr{H})$ such that for every $s \in J$, $\{g_k^\varepsilon(s) : k \geq 1\} \subseteq K_\varepsilon$, and $\| f_k(s) - g_k^\varepsilon(s) \| \leq 2\gamma(s) + \varepsilon$, for every $k \geq 1$ and every $s \in J_\varepsilon^{'} = J - J_\varepsilon$, then using Minkowski's inequality, we get

$$\begin{aligned}\| \int_{J_\varepsilon^{'}} T_\alpha(t-s)(f_k(s) - g_k^\varepsilon(s))ds \| &\leq \overline{M}_T\eta \left[\int_{J_\varepsilon^{'}} (2\gamma(s) + \varepsilon)^{\frac{1}{q}} ds\right]^q\\ &\leq \overline{M}_T\eta \| 2\gamma(s) + \varepsilon \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\\ &\leq \overline{M}_T\eta \left[\| 2\gamma(s) \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} + \| \varepsilon \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\right]\\ &\leq \overline{M}_T\eta \left[\| 2\gamma(s) \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} + 2 \| \varepsilon \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\right]\\ &\leq 2\overline{M}_T\eta \left[\| \gamma(s) \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} + \left(\int_J \varepsilon^{\frac{1}{q}} ds\right)^q\right]\\ &\leq 2\overline{M}_T\eta \left[\| \gamma(s) \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} + \varepsilon b^q\right]\\ &\leq 2\overline{M}_T\eta \left[\| \beta\chi_{PC}(B_{n-1}) \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} + \varepsilon b^q\right]\\ &\leq 2\overline{M}_T\eta \left[\chi_{PC}(B_{n-1}) \| \beta \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} + \varepsilon b^q\right].\end{aligned}$$

Finally we get:

$$\left\| \int_{J_\varepsilon^{'}} T_\alpha(t-s)(f_k(s) - g_k^\varepsilon(s))ds \right\| \leq 2\overline{M}_T\eta \left[\chi_{PC}(B_{n-1}) \| \beta \|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} + \varepsilon b^q\right] \tag{4.20}$$

By Holder's inequality, we have:

$$\begin{aligned}
\| \int_{J_\varepsilon} T_\alpha(t-s) f_k(s) ds \| &\leq \int_{J_\varepsilon} \| T_\alpha(t-s) \| \| f_k(s) \| ds \\
&\leq \left(\int_{J_\varepsilon} \| T_\alpha(t-s) \|^{\frac{1}{p}} ds \right)^p \left(\int_{J_\varepsilon} \| f_k(s) \|^{\frac{1}{q}} ds \right)^q \\
&\leq \left((t-s)^{\frac{\alpha}{p}} \right)^p \left(\overline{M}_T^{\frac{1}{p}} \right)^p \left(\int_{J_\varepsilon} ds \right)^p \left(\int_{J_\varepsilon} \| f_k(s) \|^{\frac{1}{q}} ds \right)^q \\
&\leq \eta \overline{M}_T \left(\int_{J_\varepsilon} \left(\Theta(r) \varphi(s) \right)^{\frac{1}{q}} ds \right)^q \\
&\leq \eta \overline{M}_T \Theta(r) \left(\int_{J_\varepsilon} \varphi^{\frac{1}{q}}(s) ds \right)^q .
\end{aligned}$$

Consequently we get

$$\| \int_{J_\varepsilon} T_\alpha(t-s) f_k(s) ds \| \leq \eta \overline{M}_T \Theta(r) \left(\int_{J_\varepsilon} \varphi^{\frac{1}{q}}(s) ds \right)^q \tag{4.21}$$

So by (4.20) (4.21), we derive

$$\begin{aligned}
\chi \left\{ \int_0^t T_\alpha(t-s) f_k(s) ds : k \geq 1 \right\} &\leq \chi \left\{ \int_{J'_\varepsilon} T_\alpha(t-s) f_k(s) ds : k \geq 1 \right\} \\
&+ \chi \left\{ \int_{J_\varepsilon} T_\alpha(t-s) f_k(s) ds : k \geq 1 \right\} \\
&\leq \chi \left\{ \int_{J'_\varepsilon} T_\alpha(t-s) \left(f_k(s) - g_k^\varepsilon(s) \right) ds : k \geq 1 \right\} \\
&+ \chi \left\{ \int_{J'_\varepsilon} T_\alpha(t-s) g_k^\varepsilon(s) ds : k \geq 1 \right\} \\
&+ \chi \left\{ \int_{J_\varepsilon} T_\alpha(t-s) f_k(s) ds : k \geq 1 \right\} \\
&\leq 2 \overline{M}_T \eta \left[\chi_{PC}(B_{n-1}) \| \beta \|_{L^{\frac{1}{q}}(J, \mathbb{R}^+)} + \varepsilon b^q \right] + \\
& \eta \overline{M}_T \Theta(r) \left(\int_{J_\varepsilon} \varphi^{\frac{1}{q}}(s) ds \right)^q .
\end{aligned}$$

By taking into account that ε is arbitrary, we get for all $t \in J$

$$\chi \left\{ \int_0^t T_\alpha(t-s) f_k(s) ds : k \geq 1 \right\} \leq 2 \overline{M}_T \eta \chi_{PC}(B_{n-1}) \| \beta \|_{L^{\frac{1}{q}}(J, \mathbb{R}^+)} .$$

In order to estimate

$$\chi \left\{ \int_0^t T_\alpha(t-s) g_k(s) dS_Q^H : k \geq 1 \right\} .$$

We use lemma 1.5.7 in order to calculate the following majoration:

$$E \parallel \int_0^t T_\alpha(t-s)g_k(s)dS_Q^H(s) \parallel_H^2 \leq c_H t^{2H-1} \sum_{n=1}^{\infty} \int_0^t \parallel T_\alpha(t-s)g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2 ds$$

$$\sup_{0\leq t\leq b} E \parallel \int_0^t T_\alpha(t-s)g_k(s)dS_Q^H(s) \parallel_H^2 \leq c_H t^{2H-1} \sum_{n=1}^{\infty} \int_0^b \parallel T_\alpha(t-s)g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2 ds$$

$$\parallel \int_0^t T_\alpha(t-s)g_k(s)dS_Q^H(s) \parallel_{PC}^2 \leq c_H t^{2H-1} \sum_{n=1}^{\infty} \int_0^b \parallel T_\alpha(t-s)g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2 ds.$$

In an other hand we have:

$$\begin{aligned}\sum_{n=1}^{\infty} \int_0^b \parallel T_\alpha(t-s)g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2 ds &\leq \sum_{n=1}^{\infty} \int_0^b \parallel T_\alpha(t-s) \parallel^2 \parallel g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2 ds \\ &\leq \sum_{n=1}^{\infty} \parallel g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2 \int_0^b \parallel T_\alpha(t-s) \parallel^2 ds \\ &\leq \overline{M}_T^2 \frac{b^{2\alpha-1}}{2\alpha-1} \sum_{n=1}^{\infty} \parallel g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2,\end{aligned}$$

and we know that

$$\sum_{n=1}^{\infty} \parallel g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2 < \infty$$

So we have

$$\parallel \int_0^t T_\alpha(t-s)g_k(s)dS_Q^H(s) \parallel_{PC} \leq c_H \frac{b^{2H+2\alpha-2}}{2\alpha-1} \overline{M}_T^2 K.$$

where

$$K = \sum_{n=1}^{\infty} \parallel g_k(s)Q^{\frac{1}{2}}e_n \parallel_H^2 < \infty$$

Then for every $t \in J$

$$\chi\left\{\int_0^b T_\alpha(t-s)g(s)dS_Q^H(s) : k \geq 1\right\} \leq 0,$$

$$\chi\left\{y_k(t) : k \geq 1\right\} \leq 2\overline{M}_T \eta \chi_{PC}(B_{n-1}) \parallel \beta \parallel_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}.$$

The inequality (4.16) and the fact that ε is arbitrary, imply

$$\chi_{PC}(B_n) \leq 2\left[2\overline{M}_T \eta \chi_{PC}(B_{n-1}) \parallel \beta \parallel_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\right].$$

By the previous steps (1,2,3,4) we find that:

$$0 \le \chi_{PC}(B_n) \le \left(4\overline{M}_T\eta \parallel \beta \parallel_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\right)^{n-1} \chi_{PC}(B_1)$$

Since this inequality is true for every $n \in \mathbb{N}$, by (4.5) and by tending $n \to \infty$, we obtain

$$\lim_{n\to\infty} \chi_{PC}(B_n) = 0.$$

Step 5. In this step, we will apply lemma4.2.3. The goal is to prove that the set $B = \bigcap_{n=1}^{\infty} B_n$ is a nonempty and compact subset of $PC(J,\mathcal{H})$. Moreover for every B_n being bounded, closed and convex, B is also bounded closed and convex. Let us check that $R(B) \subseteq B$. Indeed, $R(B) \subseteq R(B_n) \subseteq \overline{conv}R(B_n) = B_{n+1}$.
For every $n \ge 1$, therefore $R(B) \subset \bigcap_{n=2}^{\infty} B_n$. On the other hand $B_n \subset B_1$ for every $n \ge 1$. So,

$$R(B) \subset \bigcap_{n=2}^{\infty} B_n = \bigcap_{n=1}^{\infty} B_n = B$$

Step 6. In this step we show that the graph of the multi-valued function $R_{|B} : B \to 2^B$ is closed. We consider a sequence $\{x_n\}_{n\ge1}$ in$\mathcal{H}$ with $x_n \to x$ in $\mathcal{H}$ and let $y_n \in R(x_n)$ with $y_n \to y$ in $PC(J,\mathcal{H})$. we will show that $y \in R(x)$. By recalling the definition of R, there is $f_n \in S^1_{F(.,x_n(.))}$ for any $n \ge 1$, such that

$$y_n(t) = \begin{cases} S_\alpha(t)x_0 + \int_0^t T_\alpha(t-s)f_n(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_0 \\ S_\alpha(t)x_0 + S_\alpha(t-t_1)I_1(t_1^-) + \int_0^t T_\alpha(t-s)f(s)ds + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_1 \\ . \\ . \\ . \\ S_\alpha(t)x_0 + \sum_{k=0}^{i} S_\alpha(t-t_k)I_k(x(t_k^-)) + \int_0^t T_\alpha(t-s)f_n(s)ds \\ + \int_0^t T_\alpha(t-s)g(s)dS_Q^H(s), t \in J_i, 1 \le i \le N \end{cases} \tag{4.22}$$

Observe that for every $n \ge 1$ and for a.e. $t \in J$

$$\parallel f_n(t) \parallel \le \varphi(t)\Theta(\parallel x_n(t) \parallel) \le \varphi(t)\Theta(r)$$

This show that the set $\{\{f_n : n \ge 1\}$ is integrably bounded. In addition, the set $\{\{f_n(t) : n \ge 1\}$ is relatively compact for a.e. $t \in J$ by the assumption (H3) and the convergence of $\{x_n\}_{n\geq 1}$, imply that

$$\chi\{f_n(t) : n \geq 1\} \leq \chi\{F(t, x_n) : n \geq 1\} \leq \beta(t)\chi\{x_n(t) : n \geq 1\},$$

then $\chi\{f_n(t) : n \geq 1\} = 0$.
So the sequence $\{f_n\}_{n\geq 1}\}$ is semi-compact, hence by lemma4.2.2 it is weakly compact in $L^1(J, \mathscr{H})$. So without loss of generality we can assume that f_n converges weakly to a function $L^1(J, \mathscr{H})$. From Mazur's lemma, for every $j \in \mathbb{N}$ there exist a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,k}, k = j, ..., k_0(j)$ such that $\sum_{k=j}^{k_0(j)} \lambda_{j,k} = 1$ and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0(j)} \lambda_{j,k} f_k$, $j \geq 1$ converges strongly to f in $L^1(J, \mathscr{H})$ as $j \to \infty$. So we may suppose that $z_j(t) \to f(t)$ for a.e.$t \in J$.
Let t be such that $F(t, .)$ is upper semicontinuous. Then, for any neighborhood U of $F(t, .)$, there is a natural number $n_0 \in \mathbb{N}$ so that for any $n \geq n_0$ we have $F(t, x_n(t)) \subseteq U$.
Because the values of F are convex and compact, definition4.2.1 tells us that
$\bigcap_{j\geq 1} \overline{conv}\left(\bigcup_{n\geq j} F(t, x_n(t))\right) \subseteq F(t, x(t))$.
As in step 1, from Mazur's theorem, there is a sequence $\{z_n : n \geq 1\}$ of convex combinations of f_n such that for a.e.$t \in J$
$f(t) \in \bigcap_{j\geq 1} \overline{\{z_n(t) : n \geq j\}} \subseteq \bigcap_{j\geq 1} \overline{conv}\{\{f_n(t) : n \geq j\}$
and z_n converges strongly to $f \in L^1(J, \mathscr{H})$. then, for a.e. $t \in J$
$f(t) \in \bigcap_{j\geq 1} \overline{\{z_n(t) : n \geq j\}} \subseteq \bigcap_{j\geq 1} \overline{conv}\{\{f_n(t) : n \geq j\} \subseteq \bigcap_{j\geq 1} \overline{conv}\left(\bigcup_{n\geq j} F(t, x_n(t))\right) \subseteq F(t, x(t))$.
Then, by the continuity of g, $S_\alpha, T_\alpha, I_k (k = 1, 2, ..., N)$ and by the same arguments used in step 1, we get from relation 4.22 that

$$y(t) = \begin{cases} S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)ds + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) g(s) dS_Q^H(s), t \in J_0 \\ . \\ . \\ . \\ S_\alpha(t)x_0 + \sum_{k=0}^{i} S_\alpha(t-t_k) I_k(x(t_k^-)) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) g(s) dS_Q^H(s), t \in J_i, 1 \leq i \leq N \end{cases} \tag{4.23}$$

Therefore, $y \in R(x)$. This show that the graph of R is closed.
As a result of the step 1-5 the multivalued $R_{|B} : B \to 2^B$ is closed and χ_{PC}-condensing, with nonempty convex compact values. By applying the fixed point theorem and lemma4.2.4 there exist $x \in B$ such that $x \in R(x)$. Then x is a PC-mild solution for the problem 4.1. □

4.4 Example

We consider the differential stochastic inclusion of the form

$$\begin{cases} {}^cD_t^{\frac{1}{2}} y(t,z) \in \Delta y(t,z) + F(t,y_t) + g(t)\frac{dS_Q^H}{dt} \quad t \in [0,1], \;\; z \in [0,\pi] \\ y(t,0) = y(t,\pi) = 0 \\ y(\tau,z) = \varphi(\tau,z)(\tau,z) \in [0,1] \times [0,\pi] \\ y(t,z) = \int_0^t \eta_i(t-s)y(s,z)ds \end{cases}$$

Where $\eta_i : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.
We take $\mathscr{H} = L^2[0,1]$ Hilbert spaces endowed with the norm $\| \, . \, \|$ and
$g : J \longrightarrow \mathscr{L}_Q^0(\mathscr{H},\mathscr{H})$, where $\mathscr{L}_Q^0(\mathscr{H},\mathscr{H})$ be the space of all operators Q Hilbert Schmidt.
Now we define the operator $A = \Delta$.
$D(A) = \{u \in C^{2+\lambda}[0,\pi] : u(0) = \pi \;\; and \;\; u(\pi) = 0\}$,
it is easy to see that the operator A is sectorial.
Now we suppose that $f_i : [0,1] \times \mathscr{H} \longrightarrow \mathscr{H}$

i f_1, f_2 are measurable and upper semi continuous.

ii f_1, f_2 are increasing functions.

iii $f_i(t) < \varphi(t)\Theta \| x \|, \quad i = 1,2.$

Then we can transform the problem as follows

$$\begin{cases} {}^cD_t^{\alpha} x(t) \in Ax(t) + F(t,x_t) + g(t)\frac{dS_Q^H}{dt}, t \in (s_i, t_{i+1}], i = 0,1,...,N \\ x(0) = \varphi \in \mathscr{B}, \\ x(t) = I_i(t,x_t), t \in (t_i, s_i], i = 1,...,N \end{cases}$$

From our assumptions on (i)-(ii) it follows that the multivalued function satisfy the conditions $(H_1) - (H_2)$.
All the assumptions in theorem (3.1) are satisfied so our inclusion has a mild solution.

Conclusion

The main goal of this thesis is to investigate the subject of fractional stochastic differential equations and inclusions in Hilbert spaces. We study some classes of stochastic differential equations and inclusions with caputo and Hilfer fractional derivative with an impulsive condition, Sufficient conditions for the existence of $\mathscr{PC}$-mild solution are established by using the theory of fixed point and the principle of fractional calculus. We studied the convex case and non convex case.

The main results are obtained by means of the theory of sectorial operators, semigroup analysis, fractional calculus, fixed point, and stochastic analysis theory and methods adopted directly from deterministic fractional inclusion.

Our future work will try to make some the above results and study the approximate controllability for impulsive fractional neutral stochastic inclusions with Hilfer derivative driven by sub-Fractional Brownian motion with infinite delay and sectorial operators.

Bibliography

[1] A. N. Kolmogorov, Wienersche Spiralen und einige andere interessant Kurven im Hilbertschen Raum, C. R. (Doklady) Acad. URSS (N. S.) 26 (1940), 115-118.

[2] A. Boudaoui, T. Caraballo, A. Ouahab *Existence of mild solutions to stochastic delay evolution equations with a fractional Brownian motion and impulse*, stoch. Anal. Appl.,33 (2015)244-258.

[3] A. N. Caravalho, T. Dlotko and M.J.D, *Nascimento, Nonautonomous semilinear evolution equations with almost sectorial operators*, J. Evol. Equ, 8 (2008), 631-659.

[4] A. Pazy *Semigroups of linear operators and applicationd to partial differential equations*, New York Springer.

[5] A. Boudaoui, T. Caraballo, A. Ouahab *Impulsive stochastic functional differential inclusion driven by a fractional Brownian motion with infinite delay*, Math. Meth. Appl. Scien., 39, 6, (2016) 1435-1451.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations* , Elsevier Science B.V, Amsterdam, 2006.

[7] A.A. Levakov, M.M. Vas'kovskii, *Existence of solutions of stochastic differential inclusions with standard and fractional Brownian motion*, Diffe. Equa., 51,8(2015)991-997.

[8] A. Lin, L. Hu, *Existence results for impulsive neutral stochastic functional integro differential inclusions with non local initial conditions*, Comput. Math. with Appl., 59(2010)64-73.

[9] A. Ouahab, *Fractional semilinear differential inclusions*, Comput. Math. Appl., 64(2012)3235-3252.

[10] A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values,* Studia Math. 90 (1988), 69-86.

[11] B. pasik- Duncan, *Semilinear Stochastic Equations in Hilbert Space with a Fractional Brownian Motion,* SIAM Journal on Mathematical analysis.

[12] B. Ahmad, R.P. Agarwal, A. Alsaedi, *fractional differential equation and inclusions with semiperiodic and three-point boundary conditions , Boundary Value Problems,*(20016) 2016:28.

[13] B. Boufoussi, S. Hajji *Functional differential equations driven by a fractional Brownian motion*, Comput. Math. With Appl. Scien., 62 (2011) 746-754.

[14] B.B. Mandelbrot, J.W.V Ness, *Fractional Brownian motions, fractional noises and applications,* SIAM. Rev., 10(1968)422-437.

[15] B. Gaveau. *Integral stochastique radonifiante,* C.R. Acad. Sci. Paris Ser. A 276, 617-612 (1973).

[16] C. Tudor, *Some properties of the sub-fractional Brownian motion,* Stochastics: An Intenational Journal of Probability and Stochastic Processes 79 (2007) 431-448.

[17] D. Nualart, *The Malliavin Calculus and related Topics*, 2nd edn. Springer, Berlin (2006).

[18] E. Bajlekova, *Fractional evolution equations in Banach spaces* (Ph.D. Thesis). Eindhoven University of Technology, 2011.

[19] F. Periago and B. Straub, *A functional calculus for almost sectorial operators and applications to abstract evolution equations*, J. Evol. Equ, 2 (2002), 41-68.

[20] G. Da prato and J. Zabczyk *Stochastic Equations In Dnfinite Dimensions* Cambridge Univ. Press, 1992.

[21] G. Shen, C. Chen, *Stochastic integration with respect to the sub-fractional Brownian* with $H \in (0, \frac{1}{2})$, Stat. Prob. Letters, 82 (2012) 240-251.

[22] H. Markus *The functional calculus for sectorial operators,* oper. Theory Adv. Appl., Birkhauser-Verlag, Basel , 169 (2006), 19-60.

[23] I Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, 1999.

[24] I. Mendy, *Parametric estimation for sub-fractional Ornstein- Uhlenbeck process*, J. Stat.plan. inference 143(2013)633-674.

[25] J.P. Aubin, H. Frankoeska, *Set-valued Analysis,*Birkhuser, Boston, Basel, Berlin 1990.

[26] J. Cao, Y. Luo, G, Liu *Some results for impulsive fractional differential inclusions with infinite delay and sectorial operators in Banach spaces,* Appl. Math. Comput., 273 (2016)237-257.

[27] J. Wang, A.G. Ibrahim, M. Feckan, *Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces,* Appl. Math. Comput, 257 (2015) 103-118.

[28] J.Y. Park, J.U. Jeong, *Existence results for impulsive neutral stochastic functional integro- differential inclusions with infinite delays* , Adv. Differ. Eq., 2014:17 (2014).

[29] J.K. Hale, J. Kato, *Phase spaces for retarded equations with infinite delay,* Funkc. Ekvacioj. 21 (1978) 11-41.

[30] J.R. Graef, J. Henderson, A.Ouahab, *Impulsive differential inclusions: A fixed point Approach, De Gruyter Series in Nonlinear Analysis and Applications*, 20, 2013.

[31] K. Dzhaparidze, H. Van Zanten *A series expansion of fractional Brownian motion*, Probab. Theory relat. Fields, 102(2004) 39-5.5.

[32] K. Yosida, *Functional Analysis,* Springer, New York (1980).

[33] K.S.Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,*John Wiley and Sons, New York. 1993.

[34] K. M. Furati, M.D. Kassim, N.e. Tatar *Existence and uniqueness for a problem involving Hilfer fractional derivative* computer and Mathematics with applications 64(2012) 1616-1626.

[35] L. Gawarecki. *Extension of a stochastic integral with respect to cylindrical martingales,* Stat. Probab. Lett. 34, 103-111 (1997).

[36] L. Gawarecki and V. Mandrekar *Stochastic Differential Equations in Infinite Dimensions* Springer-Verlag(2010).

[37] L. Yan, G. Shen, K. He, *Ito's formula for a sub-fractional Brownian motion,* Commun. Stoch. Anal. 5 (2011) 135-159.

[38] LC. Young, *An inequality of the Holder type connected with Stieltjes integration,* Acta Math, 67 (1936) 251-282.

[39] M. Benchohra, J. Henderson, S.K. Ntouyas, A.Ouahab, *Existence results for fractional order functional differential equations with infinite delay* , J. Math. Anal. Appl., 338, 2, (2008) 1340-1350.

[40] M. Benchohra, J. Henderson, S.K. Ntouyas, A.Ouahab, *Impulsive Differential Equations and Inclusions* ,Hindawi, Philadelphia, 2007.

[41] M. Benchohra, J. Henderson and S.K. Ntouyas, *Impulsive Differential Equations and inclusion,* New York. Hindawi Publishing coporation, 2006.

[42] M. Metivier and J. Pellaumail. *Stochastic Integration,* Academic Press, New York (1980).

[43] M. Ledoux and M. Talagrand. *Probability in Banach spaces,* Springers, Berlin (1991).

[44] M. Feckan, Y. Zhou, J. Wang, *On the concept and existence of solution for impulsive fractional differential equations,* Commun. Nonlinear Sci. Numer. Simul., 17,7(2012)3050-3060.

[45] P. Billingsley *Probability and measure,* Wiley, New York (1979).

[46] P. Balasubramaniam, S.K. Ntouyas, D. Vinayagam *Existence of solutions of semilinear stochastic delay evolution inclusions in a Hilbert space* J. Math. Anal. Appl. 305 (2005) 438–451.

[47] P. Billingsley. *Probability and Measure,* Wiley, New York (1979).

[48] R. S. Schatten *Norm Ideals of Continuous Operators,* Springer, New York (1970)

[49] R.P. Agarwal, M. Benchohra, S. Hamani, *Asurvey on existence results for bounday value problems of nonlinear fractional differential equations and inclusions,* Act Appl. Math.,109, 3(2010), 973-1033.

[50] R.Hilfer, *Application of fractional calculus in physics.* Singapore: world Scientific,2000.

[51] R.P. Agarwal, M. Belmekki, M. Benchohra, *Asurvey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative*, adv. Differ. Equ., 2009, 981728 (2009).

[52] S. Abbas, M. Benchohra, G.M. N'Guérékata *Topics in fractional Differential Equations*, Springer, New York, (2012).

[53] S.K. Ntouyas, S. Etemad *On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions,*Appl. Math. Comput, 266 (2015) 235-243.

[54] T. E. Duncan, J. Jakubowski and B. pasik- Duncan, *Stochastic integration for Fractional Brownian Motion in Hilbert Space,* Stoc. Dynamics, to appear.

[55] T.L.G. Bojdecki, L.G. Gorostiza, A. Talarczyk, *Some extensions of fractional Brownian motion and sub-fractional Brownian motion related to particle systems, Elect. Commun. Probab.*, 12, (2007) 161-172.

[56] T.L.G. Bojdecki, L.G. Gorostiza, A. Talarczyk, *Sub-fractional Brownian motion and its relation to occupation times, Stat. Prob. Lett.,* 69 (2004) 405-419.

[57] T. Caraballo, M.J. Garrido-Atienza, T. Taniguchi *The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion*, Nonl. Anal., 74(2011)3671-3684.

[58] T. Caraballo, K. Liu, *Exponential stability of mild solutions of stochastic partial differential equations with delays,* Stochastic Anal. Appl. 17 (1999) 743–763.

[59] T.Guendouzi, L. Bousmaha *Existence of solutions for fractional partial neutral stochastic function integro-differential inclusions with state-dependent delay and analytic resolvent operators* , Vietnam J. Math., 43,4(2015) 687-704.

[60] T.Guendouzi, L. Bousmaha *Approximate controlability of fractional neutral stochastic functional integro-differential inclusions with infinite delay Qual. Theory Dyn.* , Syst., 13,1 (2014)89-119.

[61] T.Guendouzi, O. Benzatout, *Existence of mild solutions for impulsive fractional stochastic differential inclusions with state-dependent delay* , Ch. J. Math., V 2014 (2014), Article ID 981714, 13 pages.

[62] W. Grecksch and V. V. Anh, *A parabolic stochastic differential equation with fractional Brownian motion input,* Statist. Probab. Letters 41 (1999) 337–345.

[63] Y.Mishura-*Stochastic Calculus for fractional Brownian Motion and Related Precess*-Springer 2008.

[64] Y. Zhou, *Fractional Evolution Equations and Inclusions: Analyse and Control,* San Diego, CA, Elsevier Science, 2016.

[65] Y.V. Rogovchenko, *Nonlinear impulse evolution systems and applications to population* models, J. Math. Anal. Appl., 207 (1997) 300-315.

[66] Y. Ren, X. Cheng, R. Sakthivel *On time-dependent stochastic evolution equations driven by fractional Brownian motion in Hilbert space with finite delay,* Math. meth. Appl. Scien., 37 (2013) 2177-2184.

[67] Y. Ren, X. Cheng, R. Sakthivel, *impulsive neutral stochastic functional integro-differential equations with infinite delay driven by fBm,* Appl. Math. Comput, 247 (2014) 205-212.

[68] Y.Mishura-*Stochastic Calculus for fractional Brownian Motion and Related Precess*-Springer 2008.

[69] Y. Ren, T. Hou, R. Sakthivel, *Non-densely defined impulsive neutral stochastic functional differential equations driven by fBm in Hilbert space with infinite delay,* Front. Math. China, 10, 2 (2015) 351-365.

[70] Z. Zhang, B. Liu, *Existence results of nondensely defined fractional evolution differential inclusions,* J. Appl. Math., 2012, 316850 (2012).

[71] Z. Yan, H. Zhang, *Existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay,* Elec. Journal of Differ. Equa, 81 (2013) 1-21.

[72] Z. Yan, X. Yan, *Existence of solutions for impulsive partial stochastic neutral integro differential equations with state-dependent delay,* Collect. Math, 64 (2013) 235-250.

Titre de la thèse ***: Une Contribution à l'étude de certaines classes d'inclusions différentielles stochastiques non-linéaires.***

Résumé: Dans cette thèse, nous avons étudié le problème des inclusions différentielles stochastique fractionnaire dirigé par le mouvement Brownien sous-fractionnaire dans l'espace de Hilbert. Nous avons étudié l'existence de la solution PC-mild en utilisant la théorie du point fixe. Un exemple est donné pour illustrer la théorie retenue.

Mots clé : Solution mild, Inclusions différentielle stochastique fractionnaire impulsive, Mouvement Brownien fractionnaire, Operateur sectoriel.

Thesis title ***: A contribution to the study of certain classes of nonlinear stochastic differential inclusions.***

Abstract: The research reported in this thesis deals with the problem of fractional stochastic differential inclusion driven by Sub-fractional Brownian motion in Hilbert space. We have study the existence of PC-mild solution by using the fixed point theory. An example is given to illustrate the obtained theory.

Keywords: Mild solution, Impulsive fractional stochastic differential inclusions, Fractional Brownian motion, Fractional sectorial operators, Infinite delay.

عنوان الاطروحة مساهمة فى دراسة فئات معينة من الاحتواءات التفاضلية العشوائية غير خطية

الملخص في هذه الأطروحة درسنا مشكلة الاحتواءات التفاضلية العشوائية الجزئية التي تحركها الحركة البراونية الجزئية في فضاء هلبرت. لقد درسنا وجود الحل معتدل باستخدام نظرية النقطة الثابتة يتم إعطاء مثال لتوضيح النظرية المعتمدة.

الكلمات المفتاحية الاحتواءات التفاضلية العشوائية جزئية اندفاعية حركة براونية جزئية-عوامل تشغيل قطاعية جزئية-تاخير لانهائى.