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L'étude asymptotique d'estimation récursive pour des données fonctionnelles.

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Dedication

To my father, the man of my life, my eternal example, my moral support, and source of joy and happiness, the one who has always sacrificed to see me succeed.
To my mother, the light of my days, the source of my efforts, the flame of my heart, my life and my happiness; Mom I love. May God keep you in his vast paradise my parents.
To the people I loved today, to my dear brother and my dear sisters, my nieces Fatima, I dedicate this work whose great pleasure comes first and foremost to their advice, help, and encouragement.
To the people who have always helped and encouraged me, who was always at my side, and who accompanied me during my path of graduate studies, my kind friends, study colleagues, and brothers of hearts.

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To all, I offer my thanks, my respect, and my gratitude.

ملخص :

نتناول في هذه الاطروحة بعض النماذج الوظيفية ذات متغير عشوائي بحيث نقوم بتنبؤات انطلاقا من متغير عشوائي توضيحي يأخذ قيمه في فضاء غير منتهي(فضاء دالي) ونبحث عن تطوير بدائل لطريقة الانحدار. فقد تطرقنا الى دراسة محوري بحث عن التقدير المعلمي للبيانات الوظيفية. بحيث قمنا في المحور الاول بدراسة التقارب المنتظم و التقارب الطبيعي لتقدير القيمة العظمى لدالة الاخفاق او دالة الفشل الشرطية. اما المحور الثاني يتعلق بالتقدير التراجعي فقد قمنا في المقام الاول بتركيب التقدير التراجعي اللا معلمي الشرطي للبيانات الوظيفية. اما في المقام الثاني قمنا بدراسة التقارب نسبة المتوسط و التربيعي في حالة البيانات المرتبطة.
الكلمات المفتاحية: فضاء شبه متري , البيانات الوظيفية , التقدير التراجعي ,التقدير اللا معلمي , البيانات المرتبطة بقوة.

Résumé :

Dans cette thèse, nous traitons quelques modèles fonctionnels avec une variable aléatoire afin de faire des prévisions à partir d'une variable explicative à valeurs dans un espace de dimension infinie (espace fonctionnel), et nous cherchons à développer des alternatives à la méthode de régression, en effet nous avons étudié deux axes de recherche d'estimation non paramétrique pour des données fonctionnelles.

Le premier axe concerne de l'étude de la convergence uniforme et la normalité asymptotique d'estimateur du maximum de la fonction de hasard conditionnelle.

Le deuxième axe penche sur l'estimation récursive. En premier lieu nous avons construit un nouvel estimateur des paramètres conditionnels pour des données fonctionnelles.

En deuxième lieu nous sommes intéressés par la convergence presque sûre et en moyenne quadratique de notre estimateur où les données sont fortement mélangées.

Mots clés: Espace semi métrique, les données fonctionnelles, l'estimation récursive, l'estimation non paramétrique, données fortement mélangées.

Abstract:

In this thesis, we treat some functional models with a random variable to make predictions from an explanatory variable with values in an infinite dimensional space (functional space), and we try to develop alternatives to the regression method. Indeed, we have studied two research axes in nonparametric estimation for functional data.

The first axis concerns the study of uniform convergence and the asymptotic normality of the maximum estimator of conditional hazard function.

The second axis focuses on recursive estimation. Firstly, we built a new conditional parameter estimator for functional data.

Secondly, we are interested in the almost sure and mean quadratic convergence of our estimator, where the data are strongly mixed.

Keywords:

Semi metric space, the functional data, the recursive estimation, the nonparametric estimation, data strongly mixed.

Contents

Introduction

Functional statistics are a current area of research that plays a very important role in statistical research. Recently, it has experienced important developments, particularly in several statistical approaches, which are mixed and complete. This branch of statistics has studied the data, which are on very fine grids, and can be compared to curves or surfaces, for example the functions of time or space. This is statistical modeling of the data that can be observed on all their trajectories.

It is easy to obtain a very fine discretization of mathematical objects such as curves, surfaces and temperatures observed by satellite images. This type of variables can be found in many areas, such as meteorology, quantitative chemistry, biometrics, econometrics or medical imaging.

Among reference books on this subject, we can refer to monographs of Ramsay and Silverman [107],[108] for the applied aspects, Bosq[22] for the theoretical aspects, Ferraty and Vieu[62] for the nonparametric study and Ferraty and Romain[69] for the recent developments. In the same context, we refer to Manteiga and Vieu[93] as well as Ferraty[80].

In this thesis we are interested in some functional models with random variable in order to give predictions from an explanatory variable with values in functional space.

This thesis is presented in four chapters:

Chapter one is devoted to the functional statistics, where we give important works and concrete problems in this context. In addition, we give an overview on the conditional models and present a bibliographical list of major work on this models.

In chapter two we give some asymptotic notations and definitions, where we present importance tools (almost complete convergence, properties of different kernels), some results of strongly mixing conditions are given.

In chapter three we present some asymptotic properties related to the nonparametric estimation of the maximum of the conditional hazard function with functional data. Firstly we calculate the estimator of the maximum of the conditional hazard function from the estimates of the conditional distribution and the conditional density. Secondly we study the almost complete (uniform) convergence and the normality asymptotic of our estimator. we give also some comment and remark, tools that will used in this works.

In the fourth chapter we use the recursive methods of nonparametric estimation for construct a new estimator of the conditional distribution function with functional data and we also study the almost sure and mean quadratic convergence under strong mixing conditions of our estimator that we be calculate from the recursive estimator of the regression function.

We finish the thesis by a conclusion, in which, we summarize all our results. We also give some points prospects.

Chapter 1

Functional variables and conditional models

1.1 Functional variables

The statistical problems involved in the modeling and the study of functional random variables have a large advantage in statistics. The first work in this context is based on the discretization of these functional observations in order to be able to adapt traditional multivariate statistical techniques. But, thanks to the progress of the data-processing tool allowing the recovery of increasingly data, an alternative was recently elaborated in treating this type of data in its own dimension, i.e. by preserving the functional character. Indeed, since the 1960s, the handling of the observations in the form of trajectories was the object of several studies in various scientific disciplines such Obhukov[99], Holmstrom[82] in climatic, Deville [46] in econometrics, Molenaar and Boosma[95] and then Kirkpatrick and Heckman[85] in genetics.

The functional models of regression (parametric or nonparametric) are topics which were privileged these last years. Within the linear framework, the contribution of Ramsay and Silverman[107],[108] presents an important collection of statistical methods for the functional variables. Bosq[22] significantly contributed to the development of statistical methods within the framework of functional linear auto-regression process. By using functional principal components analysis, Cardot *ad al*[114] built an estimator for the model of the Hilbertien linear regression similar to Bosq[21] in the case of Hilbertian auto-regressive process . This estimator is defined using the spectral properties of the empirical version of the variance-covariance operator of the functional explanatory variable. They obtained convergence of probability for some cases and almost complete convergence of the built estimator for other cases.

1.1.1 Concrete problem in statistics for functional variables

In this part we mention a few areas wherein appear the functional data, to give the idea were functional statistics solves the type of problems.

• In biology, we find the first precursor work in (1958) concerning a study of increasing curves. More recently, another example is the study of variations of the angle of the knee during walking (Ramsay and Silverman[108]) and knee movement during exercise under constraint (Abramovich and Angelini[1], and Antoniadis and Sapatinas[11]). Concerning animal biology, the medley oviposition have been studied by several authors (Chiou*ad al*[34],[35], Cardot[29] and Chiou and Müller[33]). The data consist of curves giving the spawn for each quantity of eggs over time. • Chemometrics is part of the fields of study that promote the use of methods of functional statistical. Many works exists on the subject, include Frank and Friedman[73] , Hastie and Mallows[80] who have commented on the article by Frank and Friedman[73] providing an example of the measuring curves log-intensity of a laser radius refracted depending on the angle of refraction. Ferraty and Vieu[55] were interested in the study of the percentage of fat in the piece of meat (response variable) given the absorption curves of infrared wavelengths of these pieces of meat (explanatory variable).

• Environment-related applications have been particularly studied by Aneiros-Perez *ad al*(2004) who have worked on a forecasting problem of pollution. These data consist of measurements of peak ozone pollution every day (response variable) given pollutants curves and meteorological curves before (explanatory variables).

• Climatology is an area where functional data appear naturally. A study of the phenomenon El Niño (hot current in Pacific Ocean) has been realized by Besse *ad al.* [62]; Ramsay and Silverman [109], Ferraty *a*d al.[59] and Hall and Vial[78].

1.2 Conditional models

The estimation of the conditional distribution function in a functional framework was introduced by Ferraty *ad al*[61]) who is built a estimator of double kernel for the conditional distribution function and he specified the rate of almost complete convergence of this estimation when the observation are independent (i.i.d). The case of $\alpha-$ mixing observations has studied by Ferraty *ad al*[60].

Another authors have addressed the estimation of the conditional distribution function as a preliminary study of quantile estimation. For example Ezzahrioui and Ould Saïd[118], [101] who studied the asymptotic normality of this estimator in i.i.d and $\alpha-$ mixing cases.

We refer to Cordot*ad al*[30] for a linear approach of conditional quantile in functional statistics. The estimation of the conditional density function and its derivatives in functional statistics was introduced by Ferraty *ad al*[61], these authors have achieved almost complete convergence in the independent case. Since this article, an abundant literature has developed on a conditional estimation and its derivatives; in particular in order to use it to estimate conditional mode. Indeed considering $\alpha-$ mixing observations. Ferraty *ad al*[60] have established the almost complete convergence of kernel estimator of conditional mode defined by maximizing random the conditional density.

1.2.1 Conditional hazard function

The estimation of the hazard function is a problem of considerable interest, especially to inventory theorists, medical researchers, logistics planners, reliability engineers and seismologists. The non-parametric estimation of the hazard function has been extensively discussed in the literature. Beginning with Watson and Leadbetter[121], there are many papers on these topics: Ahmad[3], Singpurwalla and Wong[118], etc. We can cite Quintela[101] for a survey. The literature on the estimation of the hazard function is very abundant, when observations are vectorial. Cite, for instance, Watson and Leadbetter[121], Roussas[113], Lecoutre and Ould-Saïd[88], Estèvez *ad al*[49] and Quintela-del-Rio[100] for recent references. In all these works the authors consider independent observations or dependent data from time series. The first results on the nonparametric estimation of this model, in functional statistics were obtained by Ferraty *ad al*[66]. They studied the almost complete convergence of a kernel estimator for hazard function of a real random variable dependent on a functional predictor. Asymptotic normality of the latter estimator was obtained, in the case of α- mixing, by Quintela-del-Rio [102]. We refer to Ferraty *ad al*[80] and Mahhiddine *ad al*[92] for uniform almost complete convergence of the functional component of this nonparametric model.

1.2.2 Recursive models

The idea of recursive methods is to use the estimates calculated on the basis of the initial data and to update them with only new observations arriving in the database. A major advantage of these methods is that it is not necessary to restart all the calculation calculations of the model parameters whenever the data base is completed by new observations. In general, the advantage of these methods is to take into account the successive arrival of the data and to refine, as time goes by, the estimation algorithms implemented, the applications of a Such approach are numerous. The gain in terms of computation time can be very interesting.

Historically, the recursive estimation with rate was introduced by Wolverten and Wagner[123]. Later, Baltagi and Li[13] proposed a simple recursive estimation method for linear regression models with AR(p) disturbances. As a recent application of recursive methods we cite Amiri and Thiam[8] who studied regression estimation by local polynomial fitting for multivariate data streams. The objective of our work is to propose a parametric family of recursive kernel estimator of the *cdf* by adopting to functional case the result given by Roussas[114].

The estimate of the *cdf* in a functional setting has been introduced by Ferraty *ad al*[62]. The authors built a double kernel estimator for the *cdf* and they established the almost complete convergence rate of the estimator when observations are independent and identically distributed (i.i.d). The case of α-mixing observations has been studied earlier by Ferraty *ad al*[59]. The first uniforms results available in the literature on the estimation of the distribution function conditionally to a functional variable were established in Ferraty *ad al.* [80]. More recently, Amiri and Kherdani[10] who studied a recursive kernel regression method adapted to censored data, the asymptotic normality of the kernel estimator of the *cdf* was studied by Bouadjemi Abdelkader[25], the author introduced a new nonparametric estimator of the *cdf* of a scalar response variable Y given a functional random variable X. This estimate was based on recursive approach. Under certain terms and conditions, he proved the asymptotic normality of the built model. Keddani *ad al*[84] built an estimator of the *cdf* when the explanatory variable takes its values in a functional space by using the recursive estimation method when the sample is considered as an i.i.d sequence. Authors proposed a technique based on a multivariate counterpart of the stochastic approximation method for successive experiments for the local polynomial estimation issue.

Chapter 2

Some asymptotic notations and results

2.1 Definitions and tools

2.1.1 Types of convergence

All through this part, $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are sequences of real random variables, while $(u_n)_{n\in\mathbb{N}}$ is a deterministic sequence of positive real numbers. We will use the notation $(Z_n)_{n\in\mathbb{N}}$ for a sequence of independent and centered r.r.v.

The following definitions and results can be found in (Ferraty *and Vieu.*[62])

Definition 2.1.1 *One says that* $(X_n)_{n\in\mathbb{N}}$ *converges almost completely (a.co.) to some r.r.v.* X*, if and only if*

$$\forall \varepsilon > 0, \quad \sum_{n\in\mathbb{N}} \mathbb{P}\left(|X_n - X| > \varepsilon\right) < \infty,$$

and the almost complete convergence of $(X_n)_{n\in\mathbb{N}}$ *to* X *is denoted by*

$$\lim_{n\to\infty} X_n = X, \ a.co.$$

Definition 2.1.2 *One says that the rate of almost complete convergence of* $(X_n)_{n\in\mathbb{N}}$ *to* X *is of order* u_n *if and only if*

$$\exists \varepsilon_0 > 0, \qquad \sum_{n\in\mathbb{N}} \mathbb{P}\left(|X_n - X| > \varepsilon_0 u_n\right) < \infty,$$

and we write

$$X_n - X = O_{a.co.}(u_n)$$

Proposition 2.1.1 *Assume that* $\lim_{n\to\infty} u_n = 0,\ X_n = O_{a.co.}(u_n)$ *and* $\lim_{n\to\infty} Y_n = l_0$, *a.co., where* l_0 *is a deterministic real number.*

i) We have $X_nY_n = O_{a.co.}(u_n)$;

ii) We have $\dfrac{X_n}{Y_n} = O_{a.co.}(u_n)$ *as long as* $l_0 \neq 0$.

Remark 2.1.1 *The almost convergence of* Y_n *to* l_0 *implies that there exists some* $\delta > 0$ *such that*

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(|Y_n| > \delta\right) < \infty.$$

Now, one suppose that $Z_1, \ldots, Z_n$ are independent r.r.v. with zero mean. As can be seen throughout this part, the statement of almost complete convergence properties needs to find an upper bound for some probabilities involving sum of r.r.v. such as

$$\mathbb{P}\left(\Big|\sum_{i=1}^{n} Z_i\Big| > \varepsilon\right),$$

where, eventually, the positive real ε decreases with n. In this context, there exists powerful probabilistic tools, generically called Exponential Inequalities. The literature contains various versions of exponential inequalities. These inequalities differ according to the various hypotheses checked by the variables Z_i's. We focus here on the so-called Bernstein's inequality. This choice was made because the from of Bernstein's inequality is the easiest for the theoretical developments on functional statistics that have been stated throughout our thesis. Other forms of such exponential inequality can be found in (see Nagaev ([96],[97])).

Proposition 2.1.2 *Assume that*

$$\forall m \geq 2,\ |\mathbb{E}Z_i^m| \leq (m!/2)(a_i)^2 b^{m-2},$$

and let $(A_n)^2 = (a_1)^2 + \ldots + (a_n)^2$. *Then, we have:*

$$\forall \varepsilon \geq 0,\quad \mathbb{P}\left(\Big|\sum_{i=1}^{\infty} Z_i\Big| \geq \varepsilon A_n\right) \leq 2\exp\left\{-\frac{\varepsilon^2}{2\left(1+\frac{\varepsilon b}{A_n}\right)}\right\}.$$

Corollary 2.1.1 *i) If* $\forall m \geq 2, \exists C_m > 0,\ \mathbb{E}|Z_1^m| \leq C_m a^{2(m-1)}$, *we have*

$$\forall \varepsilon \geq 0,\quad \mathbb{P}\left(\Big|\sum_{i=1}^{\infty} Z_i\Big| \geq n\varepsilon\right) \leq 2\exp\left\{-\frac{n\varepsilon^2}{2a^2(1+\varepsilon)}\right\}.$$

ii) Assume that the variables depend on n *(that is,* $Z_i = Z_{i,n}$*). If* $\forall m \geq 2, \exists C_m > 0,\ \mathbb{E}|Z_1^m| \leq C_m a^{2(m-1)}$, *and if* $u_n = n^{-1}a_n^2 \log n$ *verifies* $\lim_{n\to\infty} u_n = 0$, *we have:*

$$\frac{1}{n}\sum_{i=1}^{n} Z_i = O_{a.co.}\left(\sqrt{u_n}\right).$$

Remark 2.1.2 *By applying Proposition 2.1.2 with* $A_n = a\sqrt{u_n}$, $b = a^2$ *and taking* $\varepsilon = \varepsilon_0\sqrt{u_n}$, *we obtain for some* $C' > 0$*:*

$$\mathbb{P}\left(\frac{1}{n}\Big|\sum_{i=1}^{\infty} Z_i\Big| > \varepsilon_0\sqrt{u_n}\right) \leq 2\exp\left\{-\frac{\varepsilon_0^2 \log n}{2(1+\varepsilon_0\sqrt{u_n})}\right\} \leq 2n^{-C'\varepsilon_0^2}.$$

Corollary 2.1.2 *i) If* $\exists M < \infty,\ |Z_1| \leq M$, *and denoting* $\sigma^2 = \mathbb{E}Z_1^2$, *we have*

$$\forall \varepsilon \geq 0,\quad \mathbb{P}\left(\Big|\sum_{i=1}^{\infty} Z_i\Big| \geq n\varepsilon\right) \leq 2\exp\left\{-\frac{n\varepsilon^2}{2\sigma^2(1+\varepsilon\frac{M}{\sigma^2})}\right\}.$$

ii) Assume that the variables depend on n (that is, $Z_i = Z_{i,n}$) and are such that $\exists M = M_n < \infty$, $|Z_1| \leq M$ and define $\sigma_n^2 = \mathbb{E}Z_1^2$. If $u_n = n^{-1}\sigma_n^2 \log n$ verifies $\lim_{n\to\infty} u_n = 0$, and if $M/\sigma_n^2 < C < \infty$, then we have:

$$\frac{1}{n}\sum_{i=1}^{n} Z_i = O_{a.co.}\left(\sqrt{u_n}\right).$$

Remark 2.1.3 *By applying Proposition 2.1.2 with $a_i^2 = \sigma^2$, $A_n = n\sigma^2$, and by choosing $\varepsilon = \varepsilon_0\sqrt{u_n}$, we obtain for some $C' > 0$:*

$$\mathbb{P}\left(\frac{1}{n}\Big|\sum_{i=1}^{\infty} Z_i\Big| > \varepsilon_0\sqrt{u_n}\right) \leq 2\exp\left\{-\frac{\varepsilon_0^2 \log n}{2(1+\varepsilon_0\sqrt{v_n})}\right\} \leq 2n^{-C'\varepsilon_0^2}.$$

where $v_n = \frac{Mu_n}{\sigma_n^2}$

2.1.2 The properties of kernel

Definition 2.1.3 *i) A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called a kernel of type I if there exist two real constants $0 < C_1 < C_2 < \infty$ such that:*

$$C_1 \mathbf{1}_{[0,1]} \leq K \leq C_2 \mathbf{1}_{[0,1]}.$$

ii) A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called a kernel of type II if its support is $[0,1]$ and if its derivative K' exists on $[0,1]$ and satisfies for two real constants $-\infty < C_2 < C_1 < 0$:

$$C_2 \leq K' \leq C_1.$$

The first kernel family contains the usual discontinuous kernels such as the asymmetrical box one while the second family contains the standard asymmetrical continuous ones (as the triangle, quadratic, ...). Finally, to be in harmony with this definition and simplify our purpose, for local weighting of real random variables we just consider the following kernel-type.

Definition 2.1.4 *A function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ with compact support* $[-1, 1]$ *and such that* $\forall u \in (0, 1),\ K(u) > 0$ *is called a kernel of type* 0*.*

We can now build the bridge between local weighting and the notation of small ball probabilities. To fix the ideas, consider the simplest kernel among those of type I namely the asymmetrical box kernel. Let x be f.r.v. valued in $\mathcal{F}$ and x be again a fixed element of $\mathcal{F}$. We can write:

$$\mathbb{E}\left(\mathbf{1}_{[0,1]}\left(\frac{d(x, X)}{h}\right)\right) = \mathbb{E}(\mathbf{1}_{B(x,h)}(X)) = \mathbb{P}(X \in B(x, h)).$$

The probability of the ball $B(x, h)$ appears clearly in the normalization. At this stage it is worth telling why we are saying *small* ball probabilities. In fact, as we will see later on, the smoothing parameter h (also called the *bandwith*) decreases with the size of the sample of the functional variables (more precisely, h tends to zero when n tends to ∞). Thus, when we take n very large, h is close to zero and then $B(x, h)$ is considered as a small ball and $P(X \in B(x, h))$ as a small ball probability.
From now, for all x in $\mathcal{F}$ and for all positive real h, we will use the notation:

$$\phi_x(h) = \mathbb{P}(X \in B(x, h)).$$

This notion of small ball probabilities will play a major role both from theoretical and piratical points of view. Because the notion of ball is strongly linked with the semi-metric d, the choice of this semi-metric will become an important stage.
Now, let X be a f.r.v. taking its values in the semi-metric space $(\mathcal{F}, d)$, let x be a fixed element of $\mathcal{F}$, let h be a real positive number and let K be a kernel function.

Lemma 2.1.1 *If K is a kernel of type I, then there exist nonnegative finite real constant C and C' such that:*

$$C\phi_x(h) \leq \mathbb{E}K\left(\frac{d(x, X}{h}\right) \leq C'\phi_x(h).$$

Lemma 2.1.2 *If K is a kernel of type II and if $\phi_x(.)$ satisfies*

$$\exists C_3 > 0,\ \exists \epsilon_0,\ \forall \epsilon < \epsilon_0,\ \int_0^{\epsilon} \phi_x(u)du > C_3 \epsilon \phi_x(\epsilon),$$

then there exist nonnegative finite real constant C and C' such that, for h small enough:

$$C\phi_x(h) \leq \mathbb{E}K\left(\frac{d(x,X)}{h}\right) \leq C'\phi_x(h).$$

Lemma 2.1.3 *[65] We have*

$$\frac{1}{F(h)}\int_0^1 tK(t)\,dP^{\|x-x_i\|/h}(t) \longrightarrow M_0 \text{ as } n \longrightarrow \infty;$$

$$\frac{1}{F(h)}\int_0^1 K(t)\,dP^{\|x-x_i\|/h}(t) \longrightarrow M_1 \text{ as } n \longrightarrow \infty;$$

$$\frac{1}{F(h)}\int_0^1 K^2(t)\,dP^{\|x-x_i\|/h}(t) \longrightarrow M_2 \text{ as } n \longrightarrow \infty.$$

Proof.

We note that

$$tK(t) = K(1) - \int_t^1 (sK(s))'\,ds.$$

Applying Fubini's Theorem, we get

$$\begin{aligned}\int_0^1 tK(t)\,\mathrm{dP}^{\|x-x_i\|/h}(t) &= K(1)F(h) - \int_0^1 \left(\int_t^1 (sK(s))'\,ds\right)\mathrm{dP}^{\|x-x_i\|/h}(t)\\ &= K(1)F(h) - \int_0^1 (sK(s))'\,F(hs)\,ds.\end{aligned}$$

Similarly, we have

$$\int_0^1 K(t)\,\mathrm{dP}^{\|x-x_i\|/h}(t) = K(1)F(h) - \int_0^1 (K(s))'\,F(hs)\,ds$$

and

$$\int_0^1 K^2(t)\,\mathrm{dP}^{\|x-x_i\|/h}(t) = K^2(1)\,F(h) - \int_0^1 \left(K^2(s)\right)' F(hs)\,ds.$$

This proof is finished by applying Lebesgue's dominated convergence theorem.

Lemma 2.1.4 ***Toeplitz's Lemma****[24] Let $(a_{n,k})_{n\geq 1,k\geq 1}$ be a real sequence and $(w_n)_{n\geq 1}$ a sequence which converges to w. On suppose that:*

(i) for any $k \geq 1$ $\lim_{n\to\infty} a_{n,k} = 0$;

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = A < \infty$;

(iii) there exists a constant $C > 0$ such that for any $n > 1$, $\sum_{k=1}^{\infty} |a_{n,k}| < C < \infty$.

Thus we have:

$$\sum_{k=1}^{\infty} a_{n,k} w_k \longrightarrow \infty,$$

as $n \longrightarrow \infty$.

2.1.3 Approximation theorem

The following theorem allows to approximate independent random variables using Brownian motion to exploit the law of iterated logarithm checked by Brownian motion (see Bosq[24]) then we will give the strongly mixing conditions.

Theorem 2.1.1 *Let X_n a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ such that for any $n \geq 0$, EX_n^2 exists and $EX_n = 0$.*

Let:

$$S_n = \sum_{i=1}^{n} X_i,\ S_0 = 0\ and V_n = \sum_{i=1}^{n} EX_i^2\ if n \geq 1,\ V_0 = 0.$$

for any $\alpha \geq 0$, *suppose that* $V_n \longrightarrow \infty$ *and:*

$$\sum_{k=1}^{\infty} \frac{(\ln_2 V_k)^{\alpha}}{V_k} E\left(X_k^2 1_{\left\{X_k^2 > \frac{V_k}{\ln V_k (\ln_2 V_k)^{2(\alpha+1)}}\right\}}\right) < \infty.$$

Let S *a random function defined on* $[0, +\infty[$ *such that:*

$$\forall\, t \epsilon \left[V_n,\, V_{n+1}\right[,\; S\left(t\right) = S_n.$$

So, defining $\{S\left(t\right),\, t \geq 0\}$ *if necessary on a new probability space, there exists Brownian motion* ζ *such that*

$$\left|S\left(t\right) - \zeta\left(t\right)\right| = o\left(t^{\frac{1}{2}} \left(\ln\ln t\right)^{\frac{1-\alpha}{2}}\right).$$

law of iterated logarithm for Brownian motion

Theorem 2.1.2 *If* ζ *is Brownian motion, then we have:*

$$\overline{\lim_{t\to\infty}} \frac{\zeta\left(t\right)}{\sqrt{2t\, \ln\ln t}} = 1\, a.s.$$

2.1.4 The mixing conditions

The $\alpha - mixing$ or strong mixing notion witch is one of the most general among the different mixing structures introduced in literature (see Ferraty and Vieu[62] for definitions of various other mixing structures and link between them the strong mixing notion is defined in the following way:

We consider a sequence of random variables $(\Delta_n)_{n\in\mathbb{N}}$ defined on probabilistic space $(\omega, \mathcal{F}, \mathbb{B})$ in some space $(\omega, \mathcal{F}')$. let us denote for $-\infty \leq j \leq k \leq +\infty$ and for $\mathcal{F}_j^k$ the σ algebra generated by the random variables $(\Delta_i, j \leq i \leq k)$.

The strong mixing coefficients are defined by the following quantities

$$\alpha(n) = \sup_{k\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^{k}} \sup_{\mathcal{F}_{k+n}^{+\infty}} \left|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{B})\right|$$

Definition 2.1.5 *The sequence* $(\Delta_n)_{n\in\mathbb{Z}}$ *is said* $\alpha-$*mixing (or strongly mixing) if*

$$\lim_{n\longrightarrow\infty} \alpha(n) = 0$$

Definition 2.1.6 *The sequence* $(\Delta_n)_{n\in\mathbb{Z}}$ *is said arithmetically equivalently algebraically* $\alpha-$*mixing with rate* $\alpha > 0$ *if:*

$$\exists C > 0, \quad \alpha(n) \leq Cn^{-\alpha}.$$

it is called geometrically $\alpha-$*mixing if*

$$\exists C > 0, \quad \exists t \in (0,1), \alpha(n) \leq Ct^n.$$

Lemma 2.1.5 *Let* $\Delta_{i i\in\mathbb{N}}$ *the family of random variables valued in* $\mathbb{R}$ *that verified the of strongly mixing we put:*

$$S_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \big|Cov(\Delta_i, \Delta_j\big|$$

If $\|\Delta\| \leq \infty, \forall i \in \mathbb{N}$ *then they are for all* $\varepsilon > 0$ *and for all* $x > 1$

$$\mathbb{P}\big(|\sum \Delta_i| > 4\varepsilon\big) \leq \big(1 + \frac{\varepsilon^2}{rS_n^2}\big)^{\frac{-r}{2}} + 2nCr^{-1}\big(\frac{2r}{\varepsilon}\big)^{a+1}$$

$a > 0$ *and* $(\alpha = n^{-a})$

Lemma 2.1.6 *We consider a family of random variables* $\Delta_{i\in\mathbb{N}}$ *valued in* $\mathbb{R}$. *If the condition of strongly mixing is verified and if* $\|\Delta\| < \infty$ *there are for all* $i \neq j$

$$\big|Cov(\Delta_i, \Delta_j)\big| \leq 4\alpha(|i-j|).$$

Chapter 3

Consistency rates and asymptotic normality of the high risk conditional for functional data

In this chapter we will present some asymptotic properties related to the non-parametric estimation of the maximum of the conditional hazard function for functional data. In a functional data setting, the conditioning variable is allowed to take its values in some abstract semi-metric space. In this case, Ferraty *ad al.* (2008) define non-parametric estimators of the conditional density and the conditional distribution. They give the rates of convergence (in an almost complete sense) to the corresponding functions, in a independence and dependence (α-mixing) context. We extend their results by calculating the maximum of the conditional hazard function of these estimates, and establishing their asymptotic normality, considering a particular type of kernel for the functional part of the estimates. Because the hazard function estimator is naturally constructed using these two last estimators, the same type of properties is easily derived for it. Our results are valid in a real (one- and multi-dimensional) context.

If X is a random variable associated to a lifetime (ie, a random variable with values in $\mathbb{R}^+$, the hazard rate of X (sometimes called hazard function, failure or survival rate) is defined at point x as the instantaneous probability that life ends at time x. Specifically, we have:

$$h(x) = \lim_{dx \to 0} \frac{\mathbb{P}\left(X \leq x + dx | X \geq x\right)}{dx}, \quad (x > 0).$$

When X has a density f with respect to the measure of Lebesgue, it is easy to see that the hazard rate can be written as follows:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}, \text{ for all } x \text{ such that } F(x) < 1,$$

where F denotes the distribution function of X and $S = 1 - F$ the survival function of X.

In many practical situations, we may have an explanatory variable Z and the main issue is to estimate the conditional random rate defined as

$$h^Z(x) = \lim_{dx \to 0} \frac{\mathbb{P}\left(X \leq x + dx | X > x, \quad Z\right)}{dx}, \text{ for } x > 0,$$

which can be written naturally as follows:

$$h^Z(x) = \frac{f^Z(x)}{S^Z(x)} = \frac{f^Z(x)}{1 - F^Z(x)}, \text{ once } F^Z(x) < 1. \tag{3.1}$$

Study of functions h and h^Z is of obvious interest in many fields of science (biology, medicine, reliability , seismology, econometrics, ...) and many authors are interested in construction of nonparametric estimators of h.

[1] (with rates of convergence[2]) for nonparametric estimates of the derivative of the conditional hazard and the maximum risk.

---

[1]Recall that a sequence $(T_n)_{n \in \mathbb{N}}$ of random variables is said to converge almost completely to some variable T, if for any $\epsilon > 0$, we have $\sum_n \mathbb{P}(|T_n - T| > \epsilon) < \infty$. This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre[24]).

[2]Recall that a sequence $(T_n)_{n \in \mathbb{N}}$ of random variables is said to be of order of complete convergence u_n, if there exists some $\epsilon > 0$ for which $\sum_n \mathbb{P}(|T_n| > \epsilon u_n) < \infty$. This is denoted by $T_n = \mathcal{O}(u_n)$, *a.co.* (or equivalently by $T_n = \mathcal{O}_{a.co.}(u_n)$).

3.1 Nonparametric estimation with functional data

Let $\{(Z_i, X_i),\ i = 1, \dots, n\}$ be a sample of n random pairs, each one distributed as (Z, X), where the variable Z is of functional nature and X is scalar. Formally, we will consider that Z is a random variable valued in some semi-metric functional space $\mathcal{F}$, and we will denote by $d(\cdot,\cdot)$ the associated semi-metric. The conditional cumulative distribution of X given Z is defined for any $x \in \mathbb{R}$ and any $z \in \mathcal{F}$ by

$$F^Z(x) = \mathbb{P}(X \leq x | Z = z),$$

while the conditional density, denoted by $f^Z(x)$ is defined as the density of this distribution with respect to the Lebesgue measure on $\mathbb{R}$. The conditional hazard is defined as in the non-infinite case (3.1).

In a general functional setting, f, F and h are not standard mathematical objects. Because they are defined on infinite dimensional spaces, the term operators may be a more adjusted in terminology.

3.1.1 The functional kernel estimates

We assume the sample data $(X_i, Z_i)_{1 \leq i \leq n}$ is i.i.d.

Following in (Ferraty *ad al.*[66]), the conditional density operator $f^Z(\cdot)$ is defined by using kernel smoothing methods

$$\widehat{f}^Z(x) = \frac{\sum_{i=1}^{n} h_H^{-1} K\left(h_K^{-1} d(z, Z_i)\right) H'\left(h_H^{-1}(x - X_i)\right)}{\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right)},$$

where k and H' are kernel functions and h_H and h_K are sequences of smoothing parameters. The conditional distribution operator $F^Z(\cdot)$ can be estimated by

$$\widehat{F}^Z(x) = \frac{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right) H\left(h_H^{-1}(x - X_i)\right)}{\displaystyle\sum_{i=1}^{n} K\left(h_K^{-1} d(z, Z_i)\right)},$$

with the function $H(\cdot)$ defined by $H(x) = \int_{-\infty}^{x} H'(t)dt$. Consequently, the conditional hazard operator is defined in a natural way by

$$\widehat{h}^Z(x) = \frac{\widehat{f}^Z(x)}{1 - \widehat{F}^Z(x)}.$$

For $z \in \mathcal{F}$, we denote by $h^Z(\cdot)$ the conditional hazard function of X_1 given $Z_1 = z$. We assume that $h^Z(\cdot)$ is unique maximum and its high risk point is denoted by $\theta(z) := \theta$, which is defined by

$$h^Z(\theta(z)) := h^Z(\theta) = \max_{x \in \mathcal{S}} h^Z(x). \tag{3.2}$$

A kernel estimator of θ is defined as the random variable $\widehat{\theta}(z) := \widehat{\theta}$ which maximizes a kernel estimator $\widehat{h}^Z(\cdot)$, that is,

$$\widehat{h}^Z(\widehat{\theta}(z)) := \widehat{h}^Z(\widehat{\theta}) = \max_{x \in \mathcal{S}} \widehat{h}^Z(x), \tag{3.3}$$

where h^Z and $\widehat{h}^Z$ are defined above.

Note that the estimate $\widehat{\theta}$ is note necessarily unique and our results are valid for any choice satisfying (3.3). We point out that we can specify our choice by taking

$$\widehat{\theta}(z) = \inf\left\{t \in \mathcal{S} \text{ such that } \widehat{h}^Z(t) = \max_{x \in \mathcal{S}} \widehat{h}^Z(x)\right\}.$$

As in any non-parametric functional data problem, the behavior of the estimates is controlled by the concentration properties of the functional variable Z.

$$\phi_z(h) = \mathbb{P}(Z \in B(z, h)),$$

where $B(z,h)$ being the ball of center z and radius h, namely $B(z,h) = \mathbb{P}(f \in \mathcal{F}, d(z,f) < h)$ (for more details, see Ferraty and Vieu[62], Chapter 6).

In the following, z will be a fixed point in $\mathcal{F}$, $\mathcal{N}_z$ will denote a fixed neighborhood of z, $\mathcal{S}$ will be a fixed compact subset of $\mathbb{R}^+$. We will led to the hypothesis below concerning the function of concentration ϕ_z

(H1) $\forall h > 0,\ 0 < \mathbb{P}(Z \in B(z,h)) = \phi_z(h)$ and $\lim_{h \to 0} \phi_z(h) = 0$

Note that (H1) can be interpreted as a concentration hypothesis acting on the distribution of the *f.r.v.* of Z.

Our nonparametric models will be quite general in the sense that we will just need the following simple assumption for the marginal distribution of Z, and let us introduce the technical hypothesis necessary for the results to be presented. The non-parametric model is defined by assuming that

$$\text{(H2)}\begin{cases} \forall (x_1,x_2) \in \mathcal{S}^2, \forall (z_1,z_2) \in \mathcal{N}_z^2, \text{ for some } b_1 > 0,\ b_2 > 0 \\ |F^{z_1}(x_1) - F^{z_2}(x_2)| \leq C_z(d(z_1,z_2)^{b_1} + |x_1 - x_2|^{b_2}), \end{cases}$$

$$\text{(H3)}\begin{cases} \forall (x_1,x_2) \in \mathcal{S}^2, \forall (z_1,z_2) \in \mathcal{N}_z^2, \text{ for some } j = 0,1,\ \nu > 0,\ \beta > 0 \\ |f^{z_1\,(j)}(x_1) - f^{z_2\,(j)}(x_2)| \leq C_z(d(z_1,z_2)^{\nu} + |x_1 - x_2|^{\beta}), \end{cases}$$

(H4) $\exists \gamma < \infty, f'^Z(x) \leq \gamma,\ \forall (z,x) \in \mathcal{F} \times \mathcal{S}$,

(H5) $\exists \tau > 0, F^Z(x) \leq 1 - \tau,\ \forall (z,x) \in \mathcal{F} \times \mathcal{S}$.

(H6) H' is twice differentiable such that

$$\begin{cases} \text{(H6a) } \forall (t_1,t_2) \in \mathbb{R}^2;\ |H^{(j)}(t_1) - H^{(j)}(t_2)| \leq C|t_1 - t_2|, \text{ for } j = 0,1,2 \\ \text{and } H^{(j)} \text{are bounded for } j = 0,1,2; \\ \text{(H6b) } \int_{\mathbb{R}} t^2 H'^2(t)dt < \infty; \\ \text{(H6c) } \int_{\mathbb{R}} |t|^{\beta}(H''(t))^2 dt < \infty. \end{cases}$$

(H7) The kernel K is positive bounded function supported on $[0, 1]$ and it is of class $\mathcal{C}^1$ on $(0, 1)$ such that $\exists C_1,\ C_2,\ -\infty < C_1 < K'(t) < C_2 < 0$ for $0 < t < 1$.

(H8) There exists a function $\zeta_0^z(\cdot)$ such that for all $t \in [0, 1]$

$$\lim_{h_K \to 0} \frac{\phi_z(th_K)}{\phi_z(h_K)} = \zeta_0^z(t) \text{ and } nh_H\phi_x(h_K) \to \infty \text{ as } n \to \infty.$$

(H9) The bandwidth h_H and h_K and small ball probability $\phi_z(h)$ satisfying

$$\begin{cases} \text{(H9a)} \lim_{n\to\infty} h_K = 0,\ \lim_{n\to\infty} h_H = 0; \\ \text{(H9b)} \lim_{n\to\infty} \dfrac{\log n}{n\phi_x(h_K)} = 0; \\ \text{(H9c)} \lim_{n\to\infty} \dfrac{\log n}{nh_H^{2j+1}\phi_x(h_K)} = 0,\ j = 0, 1. \end{cases}$$

Remark 3.1.1 *Assumption (H1) plays an important role in our methodology. It is known as (for small h) the "concentration hypothesis acting on the distribution of X" in infinite dimensional spaces. This assumption is not at all restrictive and overcomes the problem of the non-existence of the probability density function. In many examples, around zero the small ball probability$\phi_z(h)$ can be written approximately as the product of two independent functions $\psi(z)$ and $\varphi(h)$ as $\phi_z(h) = \psi(z)\varphi(h) + o(\varphi(h))$. This idea was adopted by Masry[94] who reformulated the Gasser ad al.[75] one. The increasing propriety of $\phi_z(\cdot)$ implies that $\zeta_h^z(\cdot)$ is bounded and then integrable (all the more so $\zeta_0^z(\cdot)$ is integrable).*

Without the differentiability of $\phi_z(\cdot)$, this assumption has been used by many authors where $\psi(\cdot)$ is interpreted as a probability density, while $\varphi(\cdot)$ may be interpreted as a volume parameter. In the case of finite-dimensional spaces, that is $\mathcal{S} = \mathbb{R}^d$, it can be seen that $\phi_z(h) = C(d)h^d\psi(z) + oh^d)$, where $C(d)$ is the volume of the unit ball in $\mathbb{R}^d$. Furthermore, in infinite dimensions, there exist many examples fulfilling the decomposition mentioned above. We quote the following (which can be found in Ferraty et al.[64]):

1. $\phi_z(h) \approx \psi(h)h^\gamma$ *for som* $\gamma > 0$.

2. $\phi_z(h) \approx \psi(h)h^\gamma \exp\{C/h^p\}$ *for some* $\gamma > 0$ *and* $p > 0$.

3. $\phi_z(h) \approx \psi(h)/|\ln h|$.

The function $\zeta_h^z(\cdot)$ *which intervenes in Assumption (H9) is increasing for all fixed h. Its pointless limit* $\zeta_0^z(\cdot)$ *also plays a determinant role. It intervenes in all asymptotic properties, in particular in the asymptotic variance term. With simple algebra, it is possible to specify this function (with* $\zeta_0(u) := \zeta_0^z(u)$ *in the above examples by:*

1. $\zeta_0(u) = u^\gamma$,

2. $\zeta_0(u) = \delta_1(u)$ *where* $\delta_1(\cdot)$ *is Dirac function,*

3. $\zeta_0(u) = \mathbf{1}_{]0,1]}(u)$.

Remark 3.1.2 *Assumptions (H2) and (H3) are the only conditions involving the conditional probability and the conditional probability density of* Z *given* X*. It means that* $F(\cdot|\cdot)$ *and* $f(\cdot|\cdot)$ *and its derivatives satisfy the Hölder condition with respect to each variable. Therefore, the concentration condition (H1) plays an important role. Here we point out that our assumptions are very usual in the estimation problem for functional regressors (see, e.g., Ferraty ad al.[66]).*

Remark 3.1.3 *Assumptions (H6), (H7) and (H9) are classical in functional estimation for finite or infinite dimension spaces.*

3.2 Nonparametric estimate of the maximum of the conditional hazard function

Let us assume that there exists a compact $\mathcal{S}$ with a unique maximum θ of h^Z on $\mathcal{S}$. We will suppose that h^Z is sufficiently smooth (at least of class $\mathcal{C}^2$) and verifies that $h'^Z(\theta) = 0$ and $h''^Z(\theta) < 0$.

Furthermore, we assume that $\theta \in \mathcal{S}^\circ$, where $\mathcal{S}^\circ$ denotes the interior of $\mathcal{S}$, and that θ satisfies the uniqueness condition, that is; for any $\varepsilon > 0$ and $\mu(z)$, there exists $\xi > 0$ such that $|\theta(z) - \mu(z)| \geq \varepsilon$ implies that $|h^Z(\theta(z)) - h^Z(\mu(z))| \geq \xi$.

We can write an estimator of the first derivative of the hazard function through the first derivative of the estimator. Our maximum estimate is defined by assuming that there is some unique $\widehat{\theta}$ on $\mathcal{S}^\circ$.

It is therefore natural to try to construct an estimator of the derivative of the function h^Z on the basis of these ideas. To estimate the conditional distribution function and the conditional density function in the presence of functional conditional random variable Z.

The kernel estimator of the derivative of the function conditional random functional h^Z can therefore be constructed as follows:

$$\widehat{h'}^Z(x) = \frac{\widehat{f'}^Z(x)}{1 - \widehat{F}^Z(x)} + (\widehat{h}^Z(x))^2, \tag{3.4}$$

the estimator of the derivative of the conditional density is given in the following formula:

$$\widehat{f'}^Z(x) = \frac{\sum_{i=1}^{n} h_H^{-2} K(h_K^{-1} d(Z, Z_i)) H''(h_H^{-1}(x - X_i))}{\sum_{i=1}^{n} K(h_K^{-1} d(Z, Z_i))}. \tag{3.5}$$

Later, we need assumptions on the parameters of the estimator, ie on K, H, H', h_H and h_K are little restrictive. Indeed, on one hand, they are not specific to the problem estimate of h^Z (but inherent problems of F^Z, f^Z and f'^Z estimation), and secondly they consist with the assumptions usually made under functional variables.

We state the almost complete convergence (withe rates of convergence) of the maximum estimate by the following results:

Theorem 3.2.1 *Under assumptions (H1)-(H7) we have*

$$\widehat{\theta} - \theta \to 0 \quad a.co. \tag{3.6}$$

Remark 3.2.1 *The hypothesis of uniqueness is only established for the sake of clarity. Following our proofs, if several local estimated maxima exist, the asymptotic results remain valid for each of them.*

Proof. Because $h'^{Z}(\cdot)$ is continuous, we have, for all $\epsilon > 0$. $\exists\ \eta(\epsilon) > 0$ such that

$$|x - \theta| > \epsilon \Rightarrow |h'^{Z}(x) - h'^{Z}(\theta)| > \eta(\epsilon).$$

Therefore,

$$\mathbb{P}\{|\widehat{\theta} - \theta| \geq \epsilon\} \leq \mathbb{P}\{|h'^{Z}(\widehat{\theta}) - h'^{Z}(\theta)| \geq \eta(\epsilon)\}.$$

We also have

$$|h'^{Z}(\widehat{\theta}) - h'^{Z}(\theta)| \leq |h'^{Z}(\widehat{\theta}) - \widehat{h}'^{Z}(\widehat{\theta})| + |\widehat{h}'^{Z}(\widehat{\theta}) - h'^{Z}(\theta)| \leq \sup_{x \in \mathcal{S}} |\widehat{h}'^{Z}(x) - h'^{Z}(x)|, \tag{3.7}$$

because $\widehat{h}'^{Z}(\widehat{\theta}) = h'^{Z}(\theta) = 0$.

Then, uniform convergence of h'^{Z} will imply the uniform convergence of $\widehat{\theta}$. That is why, we have the following lemma.

Lemma 3.2.1 *Under assumptions of Theorem 3.2.1, we have*

$$\sup_{x \in \mathcal{S}} |\widehat{h}'^{Z}(x) - h'^{Z}(x)| \to 0 \quad a.co. \tag{3.8}$$

■

The proof of this lemma will be given in Appendix.

Theorem 3.2.2 *Under assumptions (H1)-(H7) and (H9a) and (H9c), we have*

$$\sup_{x \in \mathcal{S}} |\widehat{\theta} - \theta| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n h_H^3 \phi_z(h_K)}}\right). \tag{3.9}$$

Proof. By using Taylor expansion of the function h'^Z at the point $\widehat{\theta}$, we obtain

$$h'^Z(\widehat{\theta}) = h'^Z(\theta) + (\widehat{\theta} - \theta)h''^Z(\theta_n^*), \tag{3.10}$$

with θ^* a point between θ and $\widehat{\theta}$. Now, because $h'^Z(\theta) = \widehat{h}'^Z(\widehat{\theta})$

$$|\widehat{\theta} - \theta| \leq \frac{1}{h''^Z(\theta_n^*)} \sup_{x\in\mathcal{S}} |\widehat{h}'^Z(x) - h'^Z(x)|. \tag{3.11}$$

The proof of Theorem will be completed showing the following lemma.

Lemma 3.2.2 *Under the assumptions of Theorem 3.2.2, we have*

$$\sup_{x\in\mathcal{S}} |\widehat{h}'^Z(x) - h'^Z(x)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n h_H^3 \phi_z(h_K)}}\right). \tag{3.12}$$

The proof of lemma will be given in the Appendix.

■

3.3 Asymptotic normality

To obtain the asymptotic normality of the conditional estimates, we have to add the following assumptions:

(H6d) $\displaystyle\int_{\mathbb{R}} (H''(t))^2 dt < \infty$,

(H10) $0 = \widehat{h'}^Z(\widehat{\theta}) < |\widehat{h'}^Z(x)|), \forall x \in \mathcal{S},\ x \neq \widehat{\theta}$

The following result gives the asymptotic normality of the maximum of the conditional hazard function. Let

$$\mathcal{A} = \left\{(z,x) : (z,x) \in \mathcal{S} \times \mathbb{R},\ a_2^x F^Z(x)\left(1 - F^Z(x)\right) \neq 0\right\}.$$

Theorem 3.3.1 *Under conditions (H1)-(H10) we have* $(\theta \in \mathcal{S}/f^Z(\theta), 1 - F^Z(\theta) > 0)$

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{h}'^Z(\theta) - h'^Z(\theta)\right) \xrightarrow{\mathcal{D}} N\left(0, \sigma_{h'}^2(\theta)\right)$$

where $\rightarrow^{\mathcal{D}}$ *denotes the convergence in distribution,*

$$a_l^x = K^l(1) - \int_0^1 \left(K^l(u)\right)' \zeta_0^x(u)du \quad \text{for } l = 1, 2$$

and

$$\sigma_{h'}^2(\theta) = \frac{a_2^x h^Z(\theta)}{(a_1^x)^2 \left(1 - F^Z(\theta)\right)} \int (H''(t))^2 dt.$$

Proof. Using again (3.17), and the fact that

$$\frac{\left(1 - F^Z(x)\right)}{\left(1 - \widehat{F}^Z(x)\right)\left(1 - F^Z(x)\right)} \longrightarrow \frac{1}{1 - F^Z(x)};$$

and

$$\frac{\widehat{f'}^Z(x)}{\left(1 - \widehat{F}^Z(x)\right)\left(1 - F^Z(x)\right)} \longrightarrow \frac{f'^Z(x)}{\left(1 - F^Z(x)\right)^2}.$$

The asymptotic normality of $(nh_H^3\phi_z(h_K))^{1/2}\left(\widehat{h'}^Z(\theta) - h'^Z(\theta)\right)$ can be deduced from both following lemmas,

Lemma 3.3.1 *Under Assumptions (H1)-(H2) and (H6)-(H8), we have*

$$(n\phi_z(h_K))^{1/2}\left(\widehat{F}^Z(x) - F^Z(x)\right) \xrightarrow{\mathcal{D}} N\left(0, \sigma_{F^Z}^2(x)\right), \tag{3.13}$$

where

$$\sigma_{F^Z}^2(x) = \frac{a_2^x F^Z(x)\left(1 - F^Z(x)\right)}{\left(a_1^x\right)^2}.$$

Lemma 3.3.2 *Under Assumptions (H1)-(H3) and (H5)-(H9), we have*

$$(nh_H\phi_z(h_K))^{1/2}\left(\widehat{h}^Z(x) - h^Z(x)\right) \xrightarrow{\mathcal{D}} N\left(0, \sigma_{h^Z}^2(x)\right), \tag{3.14}$$

where

$$\sigma_{h^Z}^2(x) = \frac{a_2^x h^Z(x)}{(a_1^x)^2 \left(1 - F^Z(x)\right)} \int_{\mathbb{R}} (H'(t))^2 dt.$$

Lemma 3.3.3 *Under Assumptions of Theorem 3.3.1, we have*

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{f'}^Z(x)-f'^Z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{f'^Z}(x)\right); \tag{3.15}$$

where

$$\sigma^2_{f'^Z(x)}=\frac{a_2^x f^Z(x)}{(a_1^x)^2}\int_{\mathbb{R}}(H''(t))^2dt.$$

Lemma 3.3.4 *Under the hypotheses of Theorem 3.3.1, we have*

$$Var\left[\widehat{f'}_N^Z(x)\right]=\frac{\sigma^2_{f'^Z(x)}}{nh_H^3\phi_z(h_K)}+o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right),$$

$$Var\left[\widehat{F}_N^Z(x)\right]=o\left(\frac{1}{nh_H\phi_z(h_K)}\right);$$

and

$$Var\left[\widehat{F}_D^Z\right]=o\left(\frac{1}{nh_H\phi_z(h_K)}\right).$$

Lemma 3.3.5 *Under the hypotheses of Theorem 3.3.1, we have*

$$Cov(\widehat{f'}_N^Z(x),\widehat{F}_D^Z)=o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right),$$

$$Cov(\widehat{f'}_N^Z(x),\widehat{F}_N^Z(x))=o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right)$$

and

$$Cov(\widehat{F}_D^Z,\widehat{F}_N^Z(x))=o\left(\frac{1}{nh_H\phi_z(h_K)}\right).$$

Remark 3.3.1

It is clear that, the results of lemmas, Lemma 3.3.4 and Lemma 3.3.5 allows to write

$$Var\left[\widehat{F}_D^Z-\widehat{F}_N^Z(x)\right]=o\left(\frac{1}{nh_H\phi_z(h_K)}\right)$$

The proofs of lemmas, Lemma3.3.1 can be (seen in Belkhir *ad al.*[18]), Lemma lem2-4 and Lemma lem3-4 (see Rabhi *ad al.*[104]).

■

Finally, by this last result and (3.10), we have the following theorem:

Theorem 3.3.2 *Under conditions (H1)-(H10), we have* $(\theta \in \mathcal{S}/f^Z(\theta), 1 - F^Z(\theta) > 0)$

$$\left(nh_H^3\phi_z(h_K)\right)^{1/2}\left(\widehat{\theta} - \theta\right) \xrightarrow{\mathcal{D}} N\left(0, \frac{\sigma_{h'}^2(\theta)}{(h''^Z(\theta))^2}\right);$$

with $\sigma_{h'}^2(\theta) = h^Z(\theta)\left(1 - F^Z(\theta)\right)\int (H''(t))^2 dt.$

3.4 Proofs of technical lemmas

Proof. Proof of Lemma 3.2.1 and Lemma 3.2.2 Let

$$\widehat{h}'^Z(x) = \frac{\widehat{f}'^Z(x)}{1 - \widehat{F}^Z(x)} + (\widehat{h}^Z(x))^2, \tag{3.16}$$

with

$$\widehat{h}'^Z(x) - h'^Z(x) = \underbrace{\left\{\left(\widehat{h}^Z(x)\right)^2 - \left(h^Z(x)\right)^2\right\}}_{\Gamma_1} + \underbrace{\left\{\frac{\widehat{f}'^Z(x)}{1 - \widehat{F}^Z(x)} - \frac{f'^Z(x)}{1 - F^Z(x)}\right\}}_{\Gamma_2}; \tag{3.17}$$

for the first term of (3.17) we can write

$$\left|\left(\widehat{h}^Z(x)\right)^2 - \left(h^Z(x)\right)^2\right| \leq \left|\widehat{h}^Z(x) - h^Z(x)\right|.\left|\widehat{h}^Z(x) + h^Z(x)\right|, \tag{3.18}$$

because the estimator $\widehat{h}^Z(\cdot)$ converge a.co. to $h^Z(\cdot)$ we have

$$\sup_{x\in\mathcal{S}}\left|\left(\widehat{h}^Z(x)\right)^2 - \left(h^Z(x)\right)^2\right| \leq 2\left|h^Z(\theta)\right|\sup_{x\in\mathcal{S}}\left|\widehat{h}^Z(x) - h^Z(x)\right|; \tag{3.19}$$

for the second term of (3.17) we have

$$\begin{aligned}\frac{\widehat{f}'^Z(x)}{1 - \widehat{F}^Z(x)} - \frac{f'^Z(x)}{1 - F^Z(x)} &= \frac{1}{(1 - \widehat{F}^Z(x))(1 - F^Z(x))}\left\{\widehat{f}'^Z(x) - f'^Z(x)\right\} \\ &+ \frac{1}{(1 - \widehat{F}^Z(x))(1 - F^Z(x))}\left\{f'^Z(x)\left(\widehat{F}^Z(x) - F^Z(x)\right)\right\} \\ &+ \frac{1}{(1 - \widehat{F}^Z(x))(1 - F^Z(x))}\left\{F^Z(x)\left(\widehat{f}'^Z(x) - f'^Z(x)\right)\right\}.\end{aligned}$$

Valid for all $x \in \mathcal{S}$. Which for a constant $C < \infty$, this leads

$$\sup_{x\in\mathcal{S}} \left| \frac{\widehat{f'}^Z(x)}{1-\widehat{F}^Z(x)} - \frac{f'^Z(x)}{1-F^Z(x)} \right| \leq$$

$$C \frac{\left\{ \sup_{x\in\mathcal{S}} \left| \widehat{f'}^Z(x) - f'^Z(x) \right| + \sup_{x\in\mathcal{S}} \left| \widehat{F}^Z(x) - F^Z(x) \right| \right\}}{\inf_{x\in\mathcal{S}} \left| 1 - \widehat{F}^Z(x) \right|}. \tag{3.20}$$

Therefore, the conclusion of the lemma follows from the following results:

$$\sup_{x\in\mathcal{S}} |\widehat{F}^Z(x) - F^Z(x)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_z(h_K)}}\right), \tag{3.21}$$

$$\sup_{x\in\mathcal{S}} |\widehat{f'}^Z(x) - f'^Z(x)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H^3\phi_z(h_K)}}\right), \tag{3.22}$$

$$\sup_{x\in\mathcal{S}} |\widehat{h}^Z(x) - h^Z(x)| = \mathcal{O}\left(h_K^{b_1} + h_H^{b_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nh_H\phi_z(h_K)}}\right), \tag{3.23}$$

$$\exists \delta > 0 \ \text{ such that } \ \sum_1^\infty \mathbb{P}\left\{ \inf_{y\in\mathcal{S}} |1 - \widehat{F}^Z(x)| < \delta \right\} < \infty. \tag{3.24}$$

The proofs of (3.21) and (3.22) appear in (Ferraty *ad al.*[62]), and (3.23) is proven in (Ferraty *ad al.*[66]).

- Concerning (3.24) by equation (3.21), we have the almost complete convergence of $\widehat{F}^Z(x)$ to $F^Z(x)$. Moreover,

$$\forall \varepsilon > 0 \qquad \sum_{n=1}^\infty \mathbb{P}\left\{ |\widehat{F}^Z(x) - F^Z(x)| > \varepsilon \right\} < \infty.$$

  On the other hand, by hypothesis we have $F^Z < 1$, i.e.

$$1 - \widehat{F}^Z \geq F^Z - \widehat{F}^Z,$$

  thus,

$$\inf_{y\in\mathcal{S}} |1 - \widehat{F}^Z(x)| \leq (1 - \sup_{x\in\mathcal{S}} F^Z(x))/2 \Rightarrow \sup_{x\in\mathcal{S}} |\widehat{F}^Z(x) - F^Z(x)| \geq (1 - \sup_{x\in\mathcal{S}} F^Z(x))/2.$$

In terms of probability is obtained

$$\mathbb{P}\left\{\inf_{x\in\mathcal{S}}|1-\widehat{F}^Z(x)|<(1-\sup_{x\in\mathcal{S}}F^Z(x))/2\right\}\leq\mathbb{P}\left\{\sup_{x\in\mathcal{S}}|\widehat{F}^Z(x)-F^Z(x)|\geq(1-\sup_{x\in\mathcal{S}}F^Z(x))/2\right\}$$
$$<\infty.$$

Finally, it suffices to take $\delta=(1-\sup_{x\in\mathcal{S}}F^Z(x))/2$ and apply the results (3.21) to finish the proof of this Lemma.

■

Proof. Proof of Lemma 3.3.2 We can write for all $x\in\mathcal{S}$

$$\begin{aligned}\widehat{h}^Z(x)-h^Z(x) &= \frac{\widehat{f}^Z(x)}{1-\widehat{F}^Z(x)}-\frac{f^Z(x)}{1-F^Z(x)}\\ &= \frac{1}{\widehat{D}^Z(x)}\Big\{\left(\widehat{f}^Z(x)-f^Z(x)\right)+f^Z(x)\left(\widehat{F}^Z(x)-F^Z(x)\right)\\ &\qquad -F^Z(x)\left(\widehat{f}^Z(x)-f^Z(x)\right)\Big\},\\ &= \frac{1}{\widehat{D}^Z(x)}\Big\{\left(1-F^Z(x)\right)\left(\widehat{f}^Z(x)-f^Z(x)\right)\\ &\qquad -f^Z(x)\left(\widehat{F}^Z(x)-F^Z(x)\right)\Big\}; \end{aligned} \tag{3.25}$$

with $\widehat{D}^Z(x)=\left(1-F^Z(x)\right)\left(1-\widehat{F}^Z(x)\right)$.

As a direct consequence of the Lemma 3.3.1, the result (3.26) (see Belkhir *ad al.*[18]) and the expression (3.25), permit us to obtain the asymptotic normality for the conditional hazard estimator.

$$(nh_H\phi_z(h_K))^{1/2}\left(\widehat{f}^Z(x)-f^Z(x)\right)\xrightarrow{\mathcal{D}}N\left(0,\sigma^2_{f^Z}(x)\right); \tag{3.26}$$

where

$$\sigma^2_{f^Z(x)}=\frac{a_2^x f^Z(x)}{(a_1^x)^2}\int_{\mathbb{R}}(H'(t))^2dt.$$

■

Proof. Proof of Lemma 3.3.3 For $i = 1, \dots, n$, we consider the quantities $K_i = K\left(h_K^{-1} d(z, Z_i)\right)$, $H_i''(x) = H''\left(h_H^{-1}(x - X_i)\right)$ and let $\widehat{f'}_N^Z(x)$ (resp. $\widehat{F}_D^Z$) be defined as

$$\widehat{f'}_N^Z(x) \;=\; \frac{h_H^{-2}}{n\,\mathbb{E}K_1}\sum_{i=1}^{n} K_i H_i''(x) \qquad (\text{resp. } \widehat{F}_D^Z \;=\; \frac{1}{n\,\mathbb{E}K_1}\sum_{i=1}^{n} K_i).$$

This proof is based on the following decomposition

$$\begin{aligned}\widehat{f'}^Z(x) - f'^Z(x) \;=\; & \frac{1}{\widehat{F}_D^Z}\left\{\left(\widehat{f'}_N^Z(x) - \mathbb{E}\widehat{f'}_N^Z(x)\right) - \left(f'^Z(x) - \mathbb{E}\widehat{f'}_N^Z(x)\right)\right\} + \\ & \frac{f'^Z(x)}{\widehat{F}_D^Z}\left\{\mathbb{E}\widehat{F}_D^Z - \widehat{F}_D^Z\right\},\end{aligned} \tag{3.27}$$

and on the following intermediate results.

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{f'}_N^Z(x) - \mathbb{E}\widehat{f'}_N^Z(x)\right) \xrightarrow{\mathcal{D}} N\left(0, \sigma_{f'^Z}^2(x)\right); \tag{3.28}$$

where $\sigma_{f'^Z}^2(x)$ is defined as in Lemma 3.3.3.

$$\lim_{n\to\infty}\sqrt{nh_H^3\phi_z(h_K)}\left(\mathbb{E}\widehat{f'}_N^Z(x) - f'^Z(x)\right) = 0. \tag{3.29}$$

$$\sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{F}_D^Z(x) - 1\right) \xrightarrow{\mathbb{P}} 0, \quad \text{as} \quad n \to \infty. \tag{3.30}$$

- Concerning (3.28).

  By the definition of $\widehat{f'}_N^Z(x)$, it follows that

$$\begin{aligned}\Omega_n \;&=\; \sqrt{nh_H^3\phi_z(h_K)}\left(\widehat{f'}_N^Z(x) - \mathbb{E}\widehat{f'}_N^Z(x)\right) \\ &=\; \sum_{i=1}^{n}\frac{\sqrt{\phi_z(h_K)}}{\sqrt{nh_H}\mathbb{E}K_1}\left(K_iH_i'' - \mathbb{E}K_iH_i''\right) \\ &=\; \sum_{i=1}^{n}\Delta_i,\end{aligned}$$

which leads

$$Var(\Omega_n) = nh_H^3\phi_z(h_K)Var\left(\widehat{f'}_N^Z(x) - \mathbb{E}\left[\widehat{f'}_N^Z(x)\right]\right). \qquad (3.31)$$

Now, we need to evaluate the variance of $\widehat{f'}_N^Z(x)$. For this we have for all $1 \leq i \leq n$, $\Delta_i(z,x) = K_i(z)H_i''(x)$, so we have

$$\begin{aligned} Var(\widehat{f'}_N^Z(x)) &= \frac{1}{(nh_H^2\mathbb{E}[K_1(z)])^2}\sum_{i=1}^n\sum_{j=1}^n Cov\left(\Delta_i(z,x),\Delta_j(z,x)\right) \\ &= \frac{1}{n\left(h_H^2\mathbb{E}[K_1(z)]\right)^2}Var\left(\Delta_1(z,x)\right). \end{aligned}$$

Therefore

$$Var\left(\Delta_1(z,x)\right) \leq \mathbb{E}\left(H_1''^2(x)K_1^2(z)\right) \leq \mathbb{E}\left(K_1^2(z)\mathbb{E}\left[H_1''^2(x)|Z_1\right]\right).$$

Now, by a change of variable in the following integral and by applying (H4) and (H7), one gets

$$\begin{aligned} \mathbb{E}\left(H_1''^2(y)|Z_1\right) &= \int_{\mathbb{R}} H''^2\left(\frac{d(x-u)}{h_H}\right)f^Z(u)du \\ &\leq h_H\int_{\mathbb{R}} H''^2(t)\left(f^Z(x-h_Ht,z) - f^Z(x)\right)dt + h_Hf^Z(x)\int_{\mathbb{R}} H''^2(t)dt \\ &\leq h_H^{1+b_2}\int_{\mathbb{R}}|t|^{b_2}H''^2(t)dt + h_Hf^Z(x)\int_{\mathbb{R}} H''^2(t)dt \\ &= h_H\left(o(1) + f^Z(x)\int_{\mathbb{R}} H''^2(t)dt\right). \qquad (3.32) \end{aligned}$$

By means of (3.32) and the fact that, as $n \to \infty$, $\mathbb{E}\left(K_1^2(z)\right) \longrightarrow a_2^x\phi_z(h_K)$, one gets

$$Var\left(\Delta_1(z,x)\right) = a_2^x\phi_z(h_K)h_H\left(o(1) + f^Z(x)\int_{\mathbb{R}} H''^2(t)dt\right).$$

So, using (H8), we get

$$\begin{aligned}\frac{1}{n\left(h_H^2\mathbb{E}[K_1(z)]\right)^2}Var\left(\Delta_1(z,x)\right) &= \frac{a_2^x\phi_z(h_K)}{n\left(a_1^x h_H^2\phi_z(h_K)\right)^2}h_H\left(o(1)+f^Z(x)\int_{\mathbb{R}}H''^2(t)dt\right)\\ &= o\left(\frac{1}{nh_H^3\phi_z(h_K)}\right)+\frac{a_2^x f^Z(x)}{(a_1^x)^2 nh_H^3\phi_z(h_K)}\int_{\mathbb{R}}H''^2(t)dt.\end{aligned}$$

Thus as $n\to\infty$ we obtain

$$\frac{1}{n\left(h_H^2\mathbb{E}[K_1(z)]\right)^2}Var\left(\Delta_1(z,x)\right)\longrightarrow\frac{a_2^x f^Z(x)}{(a_1^x)^2 nh_H^3\phi_z(h_K)}\int_{\mathbb{R}}H''^2(t)dt. \quad (3.33)$$

Indeed

$$\sum_{i=1}^n\mathbb{E}\Delta_i^2=\frac{\phi_z(h_K)}{h_H\mathbb{E}^2K_1}\mathbb{E}K_1^2(H_1'')^2-\frac{\phi_z(h_K)}{h_H\mathbb{E}^2K_1}\left(\mathbb{E}K_1H_1''\right)^2=\Pi_{1n}-\Pi_{2n}. \quad (3.34)$$

As for Π_{1n}, by the property of conditional expectation, we get

$$\Pi_{1n}=\frac{\phi_z(h_K)}{\mathbb{E}^2K_1}\mathbb{E}\left\{K_1^2\int H''^2(t)\left(f'^Z(x-th_H)-f'^Z(x)+f'^Z(x)\right)dt\right\}.$$

Meanwhile, by (H1), (H3), (H7) and (H8), it follows that:

$$\frac{\phi_z(h_K)\mathbb{E}K_1^2}{\mathbb{E}^2K_1}\underset{n\to\infty}{\longrightarrow}\frac{a_2^x}{(a_1^x)^2},$$

which leads

$$\Pi_{1n}\underset{n\to\infty}{\longrightarrow}\frac{a_2^x f^Z(x)}{(a_1^x)^2}\int(H''(t))^2dt, \quad (3.35)$$

Regarding Π_{2n}, by (H1), (H3) and (H6), we obtain

$$\Pi_{2n}\underset{n\to\infty}{\longrightarrow}0. \quad (3.36)$$

This result, combined with (3.34) and (3.35), allows us to get

$$\lim_{n\to\infty}\sum_{i=1}^n\mathbb{E}\Delta_i^2=\sigma_{f'^Z}^2(x) \quad (3.37)$$

Therefore, combining (3.33) and (3.36)-(3.37), (3.28) is valid.

- Concerning (3.29).

  The proof is completed along the same steps as that of Π_{1n}. We omit it here.

- Concerning (3.30). The idea is similar to that given by (Belkhir *ad al.*[18]).

  By definition of $\widehat{F}_D^Z(x)$, we have

$$\sqrt{nh_H^3\phi_z(h_K)}(\widehat{F}_D^Z(x)-1)=\Omega_n-\mathbb{E}\Omega_n,$$

where $\Omega_n = \frac{\sqrt{nh_H^3\phi_z(h_K)}\sum_{i=1}^n K_i}{n\mathbb{E}K_1}$. In order to prove (3.30), similar to (Belkhir *ad al.*[18]), we only need to prove $Var\ \Omega_n \to 0$, as $n\to\infty$. In fact, since

$$\begin{aligned} Var\ \Omega_n &= \frac{nh_H^3\phi_z(h_K)}{n\mathbb{E}^2K_1}(nVarK_1) \\ &\leq \frac{nh_H^3\phi_z(h_K)}{\mathbb{E}^2K_1}\mathbb{E}K_1^2 \\ &= \Psi_1, \end{aligned}$$

then, using the boundness of function K allows us to get that:

$$\Psi_1 \leq Ch_H^3\phi_z(h_K)\to 0, \quad \text{as } n\to\infty.$$

It is clear that, the results of (3.21), (3.22), (3.24) and Lemma 3.3.4 permits us

$$\mathbb{E}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + F^Z(x)\right) \longrightarrow 0,$$

and

$$Var\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + F^Z(x)\right) \longrightarrow 0;$$

then

$$\widehat{F}_D^x - \widehat{F}_N^Z(x) - 1 + F^Z(x) \xrightarrow{\mathbb{P}} 0.$$

Moreover, the asymptotic variance of $\widehat{F}_D^Z - \widehat{F}_N^Z(x)$ given in Remark 3.3.1 allows to obtain

$$\frac{nh_H\phi_z(h_K)}{\sigma^2_{F^Z}(x)} Var\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + \mathbb{E}\left(\widehat{F}_N^Z(x)\right)\right) \longrightarrow 0.$$

By combining result with the fact that

$$\mathbb{E}\left(\widehat{F}_D^Z - \widehat{F}_N^Z(x) - 1 + \mathbb{E}\left(\widehat{F}_N^Z(x)\right)\right) = 0,$$

we obtain the claimed result.

Therefore, the proof of this result is completed.

Therefore, the proof of this Lemma is completed.

■

Chapter 4

Recursive estimation of the conditional distribution function under strong mixing conditions for functional data

4.1 Framework of study

The idea of recursive methods is to use the estimates calculated on the basis of the initial data and to update them with only new observations arriving in the database.

A major advantage of these methods is that it is not necessary to restart all the calculation calculations of the model parameters whenever the data base is completed by new observations.

In general, the advantage of these methods is to take into account the successive arrival of the data and to refine, as time goes by, the estimation algorithms implemented, the applications of a Such approach are numerous. The gain in terms of computation time can be very interesting.

4.2 The model

Let $(X_1, Y_1), (X_2, Y_2), \ldots (X_n, Y_n)$ n pairs of random variables with the observations (X_i), $i = 1, \ldots, n$ are dependents of type strongly mixing, as (x, y) witch is a random pair valued in $\mathcal{F} \times \mathbb{R}$, where $(\mathcal{F}, d(;))$ is a semi-metric space and $d(x; X_i) = \|x - X_i\|$.

The conditional distribution function is defined by:

$$\widehat{F}^{[X]}(y) = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{[F(h_i)^l]} K\left(\frac{\|x - X_i\|}{h_i}\right) H\left(\frac{y - Y_i}{h_i}\right)}{\displaystyle\sum_{i=1}^{n} \frac{1}{[F(h_i)^l]} K\left(\frac{\|x - X_i\|}{h_i}\right)},$$

where K is a kernel, H is a distribution function h_n a sequence of positive reals and l is a parameter in $[0, 1]$, $F(h_i) = \mathbb{P}(\|x - X_i\| \leq h_i)$.

Our family of recursive estimators is defined by:

$$\widehat{F}_n^{[X]}(y) = \frac{\left[\displaystyle\sum_{i=1}^{n} F(h_i)\right]^{1-l} \varphi_n^l(y) + \left[\displaystyle\sum_{i=1}^{n+1} F(h_i)\right]^{1-l} H\left(\frac{y - Y_i}{h_i}\right) K_{n+1}^{[l]}(\|x - X_i\|)}{\left[\displaystyle\sum_{i=1}^{n} F(h_i)\right]^{1-l} G_n^l(y)(x, y) \left[\displaystyle\sum_{i=1}^{n+1} F(h_i)\right]^{1-l} K_{n+1}^{[l]}(\|x - X_i\|)},$$

with

$$\varphi_n^{[l]}(x, y) = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{[F(h_i)]^{1-l}} H\left(\frac{y - Y_i}{h_i}\right) K\left(\frac{\|x - X_i\|}{h_i}\right)}{\displaystyle\sum_{i=1}^{n} [F(h_i)]^{1-l}},$$

$$G_n^{[l]}(x) = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{[F(h_i)]^{1-l}} K\left(\frac{\|x - X_i\|}{h_i}\right)}{\displaystyle\sum_{i=1}^{n} [F(h_i)]^{1-l}},$$

and

$$K_i^{[l]}(\cdot) = \frac{1}{[F(h_i)]^l \sum_{j=1}^{i} [F(h_i)]^{1-l}} K\left(\frac{\cdot}{h_i}\right)$$

4.2.1 The hypothesis

(H_0)
(i) $\forall h_i > 0, \mathbb{P}(X \in B(x, h_i)) =: F(h_i)$ where $B(x, h_i) = \{x' \in \mathcal{F}/d(x, x' < h_i)\}$
(ii)$(X_i)_{i \in \mathbb{N}^*}$ is an $\alpha-$mixing sequence where the coefficients of mixture verify:

$$\exists a > 0, \exists c > 0 : \forall n \in \mathbb{N}, \alpha(n) \leq cn^{-a}.$$

(iii)$0 < \sup_{i \neq j} \mathbb{P}((X_i, X_j) \in B(x, h_i) \times B(x, h_j)) = \partial\left(\frac{(F(h_i))^{\frac{a+1}{a}}}{n^{\frac{1}{a}}}\right).$

Note that $H_0(i)$ can be interpreted as a concentration hypothesis acting on the distribution of the f.r.v, X where as $H_0(iii)$ concerns the behavior of the joint distribution of the pairs (X_i, X_j). In the fact this hypothesis is equivalent to suppose that for n large enough

$$\sup_{i \neq j} \frac{\mathbb{P}((X_i, X_j) \in B(x, h_i) \times B(x, h_j))}{\mathbb{P}(X \in B(x, h))} \leq C\left(\frac{F(h_i)}{n}\right)^{\frac{1}{a}}.$$

(H_1) K is a bounded kernel on the compact support $[0, 1]$ such that

$$0 < c_1 < K(t) < c_2 < \infty.$$

(H_2)
(i)The sequence of bandwidths $\{h_i, i \geq 1\}$ satisfies $0 < h_i \downarrow 0$ as $i \longrightarrow \infty$.
(ii) If $h_n \longrightarrow 0$ then $F(h_n) \longrightarrow F(0) = 0$ as $n \longrightarrow \infty$ and $\forall s \in [0, 1] \tau_h(s) = \frac{F(hs)}{F(h)} \longrightarrow \tau_0 < \infty$ when h tends to 0.
H_3
$(i) h_n \longrightarrow 0; nF(h_n) \longrightarrow \infty$ and
$A_{n,l} := \frac{1}{n} \sum_{i=1}^{n} \frac{h_i}{h_n} [\frac{F(h_i)}{F(h_n)}]^{1-l} \longrightarrow \alpha_l < \infty$, as $n \longrightarrow \infty$;

$(ii) \forall r \leq 2, B_{n,r} := \frac{1}{n}\sum_{i=1}^{n}[\frac{F(h_i)}{F(h_n)}]^r \longrightarrow \beta_r < \infty$, when $n \longrightarrow \infty$.

(H_4)

$$\lim_{n\longrightarrow\infty} \frac{nF(h_n)(\ln n)^{-1-\frac{2}{\mu}}}{(\ln\ln n)^{2(\alpha+1)}} = \infty \quad and \lim_{n\longrightarrow\infty}(\ln n)^{\frac{2}{\mu}}F(h_n) = 0.$$

α is a real positive.

(H_5)

(i) $\int_{\mathbb{R}}[H(t)]^2dt < \infty; \int_{\mathbb{R}}[H(t)]^2|t|^{\beta_2}dt < \infty; \int_{\mathbb{R}}[H(t)]dt < \infty;$
$\int_{\mathbb{R}}[H'(t)]dt = 1; \int_{\mathbb{R}}[H'(t)]|t|^{\beta_2}dt.$

(ii) For any $y \in \mathbb{R} \quad \forall(x_1, x_2) \in N_x^2$

$$|F^{[x_1]}(y_1) - F^{[x_2]}(y_2)| \leq (d(x_1, x_2)^{\beta_1} + |y_1 - y_2|^{\beta_2}),$$

with $\beta_1 > 0, \beta_2 > 0; (d(x_1, x_2) = \|x_1 - x_2\|$ and N_x^2 a fixed neighborhood of x.

(H_6)

(i)$C_{n,l} := \frac{1}{n}\sum_{i=1}^{n} h_i^{2\beta_1}\left[\frac{F(h_i)}{F(h_n)}\right]^{1-l} \longrightarrow 0$ as $n \longrightarrow \infty$.

(ii) H is a square integrable function as:

$$\sigma_{\varepsilon_i}^2(X) = Var\left[H\left(\frac{y - Y_i}{h_i}\right)\right] \longrightarrow \sigma_{\varepsilon}^2(X) = F^{[X]}(y)(1-F^{[x]}(y)) \text{ when } i \longrightarrow \infty.$$

(iii) The function ϕ is derivable at 0:

$$\phi(\|x - X_i\|) = \mathbb{E}\left\{\left[\int_{\mathbb{R}} H'(t)F^{[X]}(y - h_i t)dt - F^{[x]}\right] |\|x - X_i\|\right\}.$$

4.3 Almost sure convergence of the recursive kernel estimate

Theorem 4.3.1 *Under hypothesis $H_0(i)(ii)(iii)$; $H_1 - H_4$; $H_5(i)(ii)$ and $H_6(i)(ii)(iii)$ and if $\lim_{n\longrightarrow\infty} nh_n^2 = 0$, then*

$$\limsup_{n\longrightarrow\infty}\left[\frac{nF(h_n)}{\ln\ln n}\right]^{\frac{1}{2}}[\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y)] = \frac{[2M_2\beta_{1-2l}F^{[x]}(y)(1-F^{[x]}(y))]^{\frac{1}{2}}}{M_1\beta_{1-l}}.$$

Proof. Let $F^{[x]}(y) = \dfrac{\phi(x,y)}{G(x)}$, this later can be written as

$$\widehat{F}_n^{[x,l]}(y) = \frac{\phi_n^{[l]}(x,y)}{G_n^{[l]}(x)},$$

let the following decomposition:

$$\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y) = \frac{\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x)}{G_n^{[l]}(x)}.$$

The idea is to show that $G_n^{[l]}(x)$ converges almost surely to $G^{[l]}(x)$ and that $\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x)$ converges almost surely to 0.

The numerator can be written

$$\begin{aligned}\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x) &= \{\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x) - \mathbb{E}[\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x)]\}\\ &+ \{\mathbb{E}[\phi_n^{[l]}(x,y) - F^{[x]}(y)G_n^{[l]}(x)]\}\\ &= I_1 + I_2.\end{aligned}$$

We starting by studying I_1. For this purpose, we set:

$$W_i = \frac{1}{[F(h_i)^l]}K\left(\frac{\|x - X_i\|}{h_i}\right)\left[H\left(\frac{y - Y_i}{h_i}\right) - F^{[x]}(y)\right].$$

$$Z_i = W_i - \mathbb{E}(W_i)$$

and

$$S_n = \sum_{i=1}^{n} Z_i.$$

Remark that

$$I_1 = \frac{S_n}{\sum_{i=1}^{n}[F(h_i)]^{1-l}}.$$

Let $V_n = \sum_{i=1}^{n} \mathbb{E}(Z_i)^2$

$$\begin{aligned}
V_n &= \sum_{i=1}^{n} Var(W_i) \\
&= \sum_{i=1}^{n}[F(h_i)]^{-2l}\Bigg\{\mathbb{E}\Bigg(K^2\Bigg(\frac{\|x-X_i\|}{h_i}\Bigg)\Bigg[H\Bigg(\frac{y-Y_i}{h_i}\Bigg) - F^{[x]}(y)\Bigg]^2\Bigg)\Bigg\} \\
&\quad -\sum_{i=1}^{n}[F(h_i)]^{-2l}\Bigg\{\mathbb{E}^2\Bigg(K\Bigg(\frac{\|x-X_i\|}{h_i}\Bigg)\Bigg[H\Bigg(\frac{y-Y_i}{h_i}\Bigg) - F^{[x]}(y)\Bigg]\Bigg)\Bigg\} \\
&\quad -\sum_{i\neq j}^{n} Cov\Bigg(W_i, W_j\Bigg) \\
&= \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3
\end{aligned}$$

A_1 is written as

$$A_1 = \sum_{i=1}^{n}[F(h_i)]^{-2l}\Bigg\{\mathbb{E}\Bigg(K^2\Bigg(\frac{\|x-X_i\|}{h_i}\Bigg)\Bigg)\mathbb{E}\Bigg(\Bigg[H\Bigg(\frac{y-Y_i}{h_i}\Bigg) - F^{[x]}(y)\Bigg]^2 \mid X_i\Bigg)\Bigg\}.$$

As

$$\begin{aligned}
\mathbb{E}\Bigg(\Bigg[H\Bigg(\frac{y-Y_i}{h_i}\Bigg) - F^{[x]}(y)\Bigg]^2 \mid X_i\Bigg)\Bigg\} &= Var\Bigg(\Bigg[H\Bigg(\frac{y-Y_i}{h_i}\Bigg)\Bigg] \mid X_i\Bigg) \\
&+ \mathbb{E}^2\Bigg(\Bigg[H\Bigg(\frac{y-Y_i}{h_i}\Bigg) - F^{[x]}(y)\Bigg] \mid X_i\Bigg) \\
&= \sigma_{\varepsilon_i}(X) + \mathbb{E}^2\Bigg(\Bigg[H\Bigg(\frac{y-Y_i}{h_i}\Bigg) - F^{[x]}(y)\Bigg] \mid X_i\Bigg).
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}\left(\left[H\left(\frac{y-Y_i}{h_i}\right)-F^{[x]}(y)\right] \mid X_i\right) &= \int_{\mathbb{R}} H'(t)\left[F^{[x_i]}(y-h_it)-F^{[x_i]}(y)\right]dt \\
&+ \int_{\mathbb{R}} H'(t)\left[F^{[x_i]}(y)-F^{[X]}(y)\right]dt \\
&\leq O(h_i^{\beta_2})+F^{[x_i]}(y)-F^{[X]}(y) \\
&\leq \|x-X_i\|^{\beta_i} as\ i \longrightarrow \infty.
\end{aligned}$$

Under l'hypothesis $5(i)(ii)$ we have

$$\mathbb{E}\left(K^2\left(\frac{\|x-X_i\|}{h_i}\right)\left[H\left(\frac{y-Y_i}{h_i}\right)-F^{[x]}(y)\right]^2\right) \leq \|x-X_i\|^{2\beta_1}+\sigma_{\varepsilon_i}.$$

In these case,

$$\begin{aligned}
\mathbb{E}\left(K^2\left(\frac{\|x-X_i\|}{h_i}\right)^2\left[H\left(\frac{y-Y_i}{h_i}\right)-F^{[x]}(y)\right]^2\right) &\leq \sigma_\varepsilon(X)\mathbb{E}\left[K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right] \\
&+\mathbb{E}\left[\|x-X_i\|^{2\beta_1}K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\left[\|x-X_i\|^{2\beta_1}K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right] &\leq \mathbb{E}\left[\sup_{X_i\in\mathcal{B}(x,h_i)}\|x-X_i\|^{2\beta_1}K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right] \\
&\leq h_i^{2\beta_1}\mathbb{E}\left[K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right],
\end{aligned}$$

with $\mathcal{B}(x,h_i)$ is the closed ball with center x ad radius h_i such that $\mathcal{B}(x,h_i)=\{x'\in\mathcal{F}/\|x-x'\|\leq h_i\}$, then we get

$$\begin{aligned}
A_1 &\leq \sum_{i=1}^{n}[F(h_i)]^{-2l}[\sigma_\varepsilon(X)+h_i^{2\beta_1}]\mathbb{E}\left[K^2\left(\frac{\|x-X_i\|}{h_i}\right)\right] \\
&= A_{11}+A_{12}.
\end{aligned}$$

We get

$$A_{11}\leq\sigma_\varepsilon(X)\sum_{i=1}^{n}[F(h_i)]^{1-2l}\left[K^2(1)-\int_0^1(K^2(s))'\tau_{h_i}(s)ds\right].$$

Under the hypothesis H_3 and applying Toeplitz lemma we obtain

$$\frac{A_{11}}{n[F(h_n)]^{1-2l}} \longrightarrow \beta_{[1-2l]}\sigma_\varepsilon^2(X)M_2,$$

and under $H_6(i)(ii)(iii)$, we get

$$\frac{A_{12}}{n[F(h_n)]^{1-2l}} \longrightarrow 0$$

and

$$\frac{A_2}{n[F(h_n)]^{1-2l}} \longrightarrow 0.$$

Now we studying A_3 .

$$\begin{aligned} A_3 &= \sum_{i\neq j}^{n} Cov(W_i, W_j) \\ &= \sum_{i\neq j}^{n} F(h_i)^{-2l} Cov(N_i, N_j), \end{aligned}$$

with $N_i = K_i H_i$ where $K_i = (h_i^{-1}K\big(\|x - X_i\|\big))$; $H_i = (h_i^{-1}K\big(y - Y_i\big))$.

Because H is a commultative kernel we have $H_i \leq 1$. By using systematically this fact to bound the variables H_i we get

$$Cov(N_i, N_j) = Cov(\Delta_i, \Delta_j),$$

with $\Delta_i = K_i - \mathbb{E}(K_i)$.

On one hand, we have by the hypothesis $H_0(i), H_0(iii)$ and H_1

$$|Cov(\Delta_i, \Delta_j)| = O\bigg(\bigg(\frac{\phi_x(h_i)}{n}\bigg)^{\frac{1}{a}} \phi_x(h_i)\bigg),$$

these covariance can be controlled by means of the usual Davydov's covariance inequality for mixing processes (see Rio [111], formula 1.12a) to get her with $H_0(ii)$ this inequality leads to:

$$\forall\forall i \neq j \quad |Cov(\Delta_i, \Delta_j)| \leq C|i-j|^{-a}.$$

By the fact

$$\sum_{K\geq C_n+1} K^{-a} \leq \int_{C_n}^{\infty} t^{-a}dt = \frac{C_n^{-a+1}}{a-1},$$

thus by using the following classical technique (see Bosq[20]) we can write

$$S_n^{cov} = \sum_{0<|i-j|<u_n} |Cov(\Delta_i,\Delta_j)| + \sum_{|i-j|>u_n} |Cov(\Delta_i,\Delta_j)|.$$

Thus

$$S_n^{cov} \leq C_n \left(\frac{\phi_x(h_i)}{n}\right)^{\frac{1}{a}} \phi_x(h_i) + \frac{C_n^{a+1}}{a-1},$$

choosing $C_n = \left(\frac{\phi_x(h_i)}{n}\right)^{\frac{-1}{a}}$ we can deduce

$$S_n^{cov} = O(nF(h_i)),$$

$$A_3 = \frac{S_n^{cov}}{nF(h_i)^{2l}} \longrightarrow 0 \text{ where } n \longrightarrow \infty.$$

Therefore we can conclude that
$V_n \sim n[F(h_n)]^{1-2l}\beta_{[1-2l]}\sigma_\varepsilon^2(x)M_2$ when $n \longrightarrow \infty$.
By assuming that $n[F(h_n)] \longrightarrow \infty$ we obtain $\frac{\ln F(h_n)}{\ln n} \longrightarrow 0$ when $n \longrightarrow \infty$
It is clear that

$$\mathbb{E}[\exp(\lambda|H|^{\mu})] < \infty,$$

for any λ and μ positive this implies

$$\mathbb{E}(\max_{1\leq i\leq n|U_i|^p}) = O[(\ln n)^{\frac{p}{\mu}}], \quad p\geq 1, n\geq 2,$$

where $H_i = H\left(\frac{y-Y_i}{h_i}\right)$.
By using the fact that

$$\lim_{n\longrightarrow\infty} \frac{nF(h_n)(\ln n)^{-1-\frac{2}{\mu}}}{(\ln\ln n)^{2(\alpha+1)}} = \infty \quad and \quad \lim_{n\longrightarrow\infty} (\ln n)^{\frac{2}{\mu}} F(h_n) = 0,$$

α is a positive real.

We deduce that:

$$\lim_{n\longrightarrow\infty}\frac{nF(h_n)(\ln n)^{-\frac{2}{\mu}}}{\ln[n(F(h_n))^{1-2l}]\{\ln\ln[nF(h_n))^{1-2l}]\}^{2(\alpha+1)}}=\infty.$$

Let $b_n=(\delta\ln n)^{\frac{1}{\mu}}$ with $\delta>0$. We will have the existence of $n_0\geq 1$ such that for all $i\geq n_0$

$$\frac{iF(h_i)(\ln i)^{-\frac{2}{\mu}}}{\ln[i(F(h_i))^{1-2l}]\{\ln\ln[iF(h_i))^{1-2l}]\}^{2(\alpha+1)}}>\frac{2\|K\|_\infty^2\max\{|F^{[x]}(y)|^2,(\delta\ln i)^{\frac{2}{\mu}}\}}{[F(h_i)]^{2l}}\geq Z_i^2.$$

As the event $Z_i^2>\dfrac{i[F(h_i)]^{1-2l}}{\ln[i(F(h_i))^{1-2l}]\{\ln\ln[iF(h_i))^{1-2l}]\}^{2(\alpha+1)}}$ is impossible, for $i\geq n_0$. From $V_n\sim n[F(h_n)]^{1-2l}\beta_{[1-2l]}\sigma_\varepsilon^2(x)M_2$, we deduce that

$$\sum_{i=1}^{n}\frac{\ln\ln V_i^\alpha}{V_i}\mathbb{E}\left(Z_i^2\mathbf{1}_{\left\{\frac{V_i}{\ln[V_i]\{\ln\ln[V_i]\}^{2(\alpha+1)}}\right\}}\right)\leq\infty.$$

Let S a random function defined on $[0,\infty[$, let

$$for\quad t\in[V_n,V_{n+1}[,S(t)=S_n.$$

Theorem A.2.1 in the annexe (Amiri *ad al.*[16]) imples that it exists a Brownian motion ξ such that

$$\left|\frac{S(t)-\xi(t)}{(2t\ln\ln t)^{\frac{1}{2}}}\right|=O[(\ln\ln t)^{-\frac{a}{2}}]\forall t\in[V_n,V_{n+1}[.$$

But since, by the theorem of the Brownian motion verifies the law of iterated logarithm so:

$$\overline{\lim_{t\to\infty}}\frac{S(t)}{(2t\ln\ln t)^{\frac{1}{2}}}=\overline{\lim_{t\to\infty}}\left[\frac{S(t)-\xi(t)}{(2t\ln\ln t)^{\frac{1}{2}}}+\frac{\xi(t)}{(2t\ln\ln t)^{\frac{1}{2}}}\right]=1\quad a.s.$$

Then, we have $\dfrac{S_n}{(2V_n\ln\ln V_n)^{\frac{1}{2}}}\longrightarrow 1\quad a.s.$

By using the fact that $S_n = I_1 \sum_{i=1}^{n} [F(h_i)]^{1-l}$ and $\dfrac{V_{n+1}}{V_n} \longrightarrow 1$ when $n \longrightarrow \infty$, we obtain

$$\overline{\lim_{t\to\infty}} \frac{\sum_{i=1}^{n} [F(h_i)]^{1-l} I_1}{(2V_n \ln\ln V_n)^{\frac{1}{2}}} \frac{nF(h_n)^{1-2l}\{\ln\ln[nF(h_n)^{1-2l}]\}^{\frac{1}{2}}}{nF(h_n)^{1-2l}\{\ln\ln[nF(h_n)^{1-2l}]\}^{\frac{1}{2}}} = 1 \quad a.s$$

But $\sum_{i=1}^{n} [F(h_i)]^{1-l} = B_{n,(1-l)} n [F(h_n)]^{1-l}$.

We have

$$\frac{\{\ln\ln[nF(h_n)^{1-2l}]\}^{\frac{1}{2}} B_{n,(1-l)} B_{n,(1-l)}}{(2V_n \ln\ln V_n)^{\frac{1}{2}}} \longrightarrow \frac{\beta_{[1-l]}}{\{2\beta_{1-2l}\sigma_\varepsilon^2(X)M_2\}^{\frac{1}{2}}},$$

when $n \longrightarrow \infty$. It comes, then:

$$\overline{\lim_{t\to\infty}} \left\{ \frac{nF(h_n)}{\ln\ln\left[n(F(h_n))^{1-2l}\right]} \right\}^{\frac{1}{2}} I_1 = \sigma_l \text{ a.s}$$

with $\sigma_l = \dfrac{\left\{2\beta_{[1-2l]}\sigma_\varepsilon^2\left(X\right)M_2\right\}^{\frac{1}{2}}}{\beta_{[1-l]}}$.

As $\ln\ln\left[n\left(F\left(h_n\right)\right)^{1-2l}\right] = \left(\ln\ln n\right)\left[1 + o\left(1\right)\right]$, we conclude that

$$\overline{\lim_{n\to\infty}} \left\{ \frac{nF\left(h_n\right)}{\ln\ln n} \right\}^{\frac{1}{2}} I_1 = \frac{\left\{2\beta_{[1-2l]}\sigma_\varepsilon^2\left(X\right)M_2\right\}^{\frac{1}{2}}}{\beta_{[1-l]}}$$

Studying I_2:

We have to prove that

$$\overline{\lim_{t\to\infty}} \left\{ \frac{nF(h_n)}{\ln\ln n} \right\}^{\frac{1}{2}} I_2 = 0.$$

We have

$$
\begin{aligned}
I_2 &= \mathbb{E}[\phi_n^{[l]}(x,y) - F^{[x]}(y) f_n^{[l]}(x)] \\
&= \frac{1}{\sum_{i=1}^{n} [F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{[F(h_i)]^{l}} \mathbb{E}\Bigg\{ \Bigg[H\Bigg(\frac{y - Y_i}{h_i}\Bigg) - F^{[x]} \Bigg] K\Bigg(\frac{\|x - X_i\|}{h_i}\Bigg) \Bigg\} \\
&= \frac{1}{\sum_{i=1}^{n} [F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{[F(h_i)]^{l}} \Bigg\{ h_i \varphi'(0) F(h_i) \Bigg[K(1) - \int_0^1 (sK(s))' \tau_{h_i}(s) ds \Bigg] + o(h_i) \Bigg\}.
\end{aligned}
$$

The last equality above was obtained using the equation (4.1) when n tends to l'infinity (in the vicinity of infinity), based on the hypothesis H_3, we have

$$
I_2 \simeq h_n \varphi(0) \frac{\alpha_{[l]}}{\beta_{1-l}} M_0 [1 + o(1)]
$$

Thus

$$
\left\{ \frac{nF(h_n)}{\ln \ln n} \right\} \frac{1}{2} I_2 = \left\{ \frac{nF(h_n)}{\ln \ln n} \right\} h_n \varphi(0) \frac{\alpha_{[l]}}{\beta_{1-l}} M_0 [1 + o(1)] = o(1).
$$

In witch is verified for $\lim_{n \to \infty} n h_n^2 = 0$, we conclude then

$$
\overline{\lim_{t \to \infty}} \left\{ \frac{nF(h_n)}{\ln \ln n} \right\}^{\frac{1}{2}} I_2 = 0.
$$

Thus

$$
\left\{ \frac{nF(h_n)}{\ln \ln n} \right\} \frac{1}{2} \left[\varphi_n^{[l]} - F^{[x]} G_n^{[l]} \right] \longrightarrow \frac{\left\{ 2 \beta_{[1-2l]} \sigma_{\varepsilon}(X) M_2 \right\}^{\frac{1}{2}}}{\beta_{[1-l]}}.
$$

Now we show the almost sure convergence $G_n^{[l]}(x)$ to $G^{[l]}(x)$ to decide that of $F_n^{x,l}(y)$ to $F^{[x]}$.

In the same way, by letting $Z_i = W_i - \mathbb{E}(W_i)$ we can prove

$$
G_n^{[l]}(x) - \mathbb{E} G_n^{[l]}(x) = O\left(\sqrt{\frac{\ln \ln n}{nF(h_n)}} \right) \quad as.
$$

As $\mathbb{E}\left[G_n^{[l]}(x)\right] = M_1\left[1+o(1)\right], \quad G_n^{[l]}(x)$ converge almost surely to M_1 because we can write

$$G_n^{[l]}(x) = \left[G_n^{[l]}(x) - \mathbb{E}G_n^{[l]}(x)\right] + \mathbb{E}\left[G_n^{[l]}(x)\right].$$

That makes the end of the proof. ■

4.4 Mean quadratic convergence of the recursive Kernel estimate

We consider the next theorem:

Theorem 4.4.1 *: suppose that $H_0(i)(ii)(iii)$; $H_1{-}H_4$; $H_5(i)(ii)$ and $H_6(i)(ii)(iii)$ and satisfied, if these is a constant $c > 0$ such that $nf(h_n)h_n^2 \to c$ when $n \to \infty$ then*

$$\lim_{n\to\infty} \mathbb{E}[F_n^{[x,l]}(y) - F^{[x]}(y)]^2 = \frac{B_{[1-2l]}}{B_{1-l}^2}\frac{M_2}{M_1^2}F_{(y)}^{[x]}(1-F_{(y)}^{[x]}) + c[\varphi'(0)]^2\frac{\alpha_{[l]}^2}{B_{[1-l]}^2}\frac{M_0^2}{M_1^2}$$

Proof. It is known

$$\mathbb{E}\left[F_n^{[x,l]}(y) - F^{[x]}(y)\right]^2 = Var\left[F_n^{[x,l]}\right] + \mathbb{E}^2\left[F_n^{[x,l]}(y) - F^{[x]}(y)\right] = E_1 + E_2,$$

we will use the following decomposition for the calculation of E_2

$$\mathbb{E}\left[F_n^{[x,l]}(y)\right] = \frac{\mathbb{E}\left\{\left[\varphi_n^{[l]}(x,y)\right]\right\}}{\mathbb{E}\left[f_n^{[l]}(x)\right]} - \frac{\mathbb{E}\left\{\left[G_n^{[l]}(x) - \mathbb{E}G_n^{l}(x)\right]\varphi_n^{l}(x,y)\right\}}{\left\{\mathbb{E}\left[G_n^{[l]}(x)\right]\right\}^2} + \frac{\mathbb{E}\left\{\left[G_n^{[l]}(x) - \mathbb{E}G_n^{[l]}(x)\right]^2 F_n^{x,l}(y)\right\}}{\left\{\mathbb{E}[G_n^{[l]}(x)]\right\}^2}.$$

For the calculation ofE_1we use the following de composition of the variance that can be fond in (Collombe[36].

$$Var\left[F_n^{[x,l]}(y)\right] = \frac{Var\left[\varphi_n^{[l]}(x,y)\right]}{\mathbb{E}\left[G_n^{[l]}(x)\right]^2} - 4\frac{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]Cov\left[G_n^{[l]}(x),\varphi_n^{[l]}(x,y)\right]}{\left\{\mathbb{E}\left[G_n^{[l]}(x)\right]\right\}^3}$$
$$+ \quad 3Var\left[G_n^l(x)\right]\frac{\left\{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]\right\}^2}{\mathbb{E}\left\{G_n^l(x)\right\}^4} + O\left[\frac{1}{nF(h_n)}\right]$$

Studying the convergence of E_2:

We start by studying $\dfrac{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]}{\mathbb{E}\left[G_n^l(x)\right]} - F^{[x]}(y)$ we observe that:

$$\frac{\mathbb{E}\left[\varphi_n^{[l]}(x,y)\right]}{\mathbb{E}\left[G_n^{[l]}(x)\right]} - F^{[x]}(y) = \frac{\sum_{i=1}^{n}\frac{1}{[F(h_i)]^{[l]}}\mathbb{E}\left\{\left[H\left(\frac{y-Y_i}{h_i}\right) - F^{[x]}(y)\right]K\left(\frac{\|x-X_i\|}{h_i}\right)\right\}}{\sum_{i=1}^{n}\frac{1}{[F(h_i)]^{[l]}}\mathbb{E}[K\left(\frac{\|x-X_i\|}{h_i}\right)]}$$

Let:

$$\varphi(t) = \mathbb{E}\Big[\int_{\mathbb{R}} H(t)F^{[x]}(y-h_it)dt - F^{[x]}(y)\Big]\|x-X_i\| = t$$

Suppose that the function φ is derivable at point $t=0$ by the hypothesis $H_6(ii)$ we have:

$$\begin{aligned}\mathbb{E}\left[H\left(\frac{y-Y_i}{h_i}\right) - F^{[x]}(y)\right]K\left(\frac{\|x-X_i\|}{h_i}\right) &= \mathbb{E}\left[\varphi\Big(\|x-X_i\|\Big)K\Big(\frac{\|x-X_i\|}{h_i}\Big)\right] \\ &= \int_0^1 \varphi(h_it)K(t)d\mathbb{P}^{\frac{\|x-X_i\|}{h_i}}(t)\end{aligned}$$

So using the taylor expansion for φ around 0, we obtains

$$\mathbb{E}\Big\{\Big[H\left(\frac{y-Y_i}{h_i}\right)-F^{[x]}(y)\Big]K\Big(\frac{\|x-X_i\|}{h_i}\Big)\Big\} = h_i\varphi'(0)\int_0^1 tK(t)d\mathbb{P}^{\frac{\|x-X_i\|}{h_i}}(t)+O[h_i], \tag{4.1}$$

based on the proof of lemma 2 in (Ferraty ad al[64]), H_1 and fubini theorem

$$\int_0^1 tK(t)d\mathbb{P}^{\frac{\|x-X_i\|}{h_i}}(t) = F(h_i)[K(1)-\int_0^1 (sK(s)')\tau_{h_i}(s)ds]$$

and by(H_1) we obtain:

$$\begin{aligned}\frac{\mathbb{E}\Big[\varphi_n^l(x,y)\Big]}{\Big[G_n^l(x,y)\Big]} - F^{[x]}(y) &= \frac{\sum_{i=1}^n h_i[F(h_i)]^{1-l}\Big\{\varphi'(0)\Big[K(1)-\int_0^1 (sk(s)')\tau_{h_i}(s)ds\Big]\Big\}+\gamma_i}{\sum_{i=1}^n [F(h_i)]^{1-l}[K(1)-\int_0^1 (sK(s)')\tau_{h_i}(s)ds]} \\ &= \frac{D_1}{D_2}\end{aligned}$$

Finally$(H_2)$and(H_3) and Toeplitz lemma (see Masry[94]) permit us have:

$$\frac{D_1}{nh_n[F(h_n)]^{1-l}} = \alpha_{[e]}\varphi'(0)M_0[1+0(1)]\frac{D_2}{n[F(h_n)]^{1-l}} = B_{[1-l]}M_1[1+0(1)]$$

$$\frac{E\varphi_n^l(x,y)}{EF_n^l(x)} - F^{[x]}(y) = h_n\varphi'(0)\frac{\alpha_{[l]}}{\beta_{[1-l]}}\frac{M_0}{M_1}[1+0(1)],$$

the convergence of the other terms of the composition for calculating E_2 is a consequence of the terms of the variance there fore, we establish the convergence of the variance we have:

$$\begin{aligned}\mathbb{E}[G_n^{[l]}(x)] &= \frac{1}{\sum_{i=1}^n [F(h_i)]^{1-l}}\sum_{i=1}^n \frac{1}{[F(h_i)]^{1-l}}\mathbb{E}\Big[K\left(\frac{\|x-X_i\|}{h_i}\right)\Big] \\ &= \frac{\sum_{i=1}^n \frac{[F(h_i)]^{1-l}}{n[F(h_n)]^{1-l}}[K(1)-\int_0^1 (K(s)\tau_{h_i}(s)ds)]}{\beta_{n,[1-l]}} \\ &= M_1[1+o(1)]\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\varphi_n^{[l]}(x,y)] &= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{[F(h_i)]^l} \mathbb{E}\Big[H\left(\frac{y-Y_i}{h_i}\right) K\left(\frac{\|x-x_i\|}{h_i}\right) \Big] \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \mathbb{E}\Big[\int_{\mathbb{R}} H'(t) F^x(y-h_i t)dt \\
&\quad -F^{[x]}(y) + F^{[x]}(y) \Big] K\left(\frac{\|x-X_i\|}{h_i}\right) \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{F(h_i)^l} \mathbb{E}[(h_i^{\beta_2}) + F^{[x]}(y)] K\left(\frac{\|x-X_i\|}{h_i}\right) \\
&= \frac{1}{\sum_{i=1}^{n}[F(h_i)]^{1-l}} \sum_{i=1}^{n} \frac{1}{F(h_i)^{1-l}} F(h_i) M_1 [F^{[x]}(y) + o(h_i^{\beta_2})] \\
&= F^{[x]}(y) M_1 [1+o(1)]
\end{aligned}$$

$$\begin{aligned}
Var\Big(\varphi_n^{[l)}(x,y)\Big) &= \left[\sum_{i=1}^{n}[F(h_i)]^{1-l}\right]^{-2} \sum_{i=1}^{n} \left[\frac{1}{F(h_i)^l}\right]^2 \Bigg[Var\Bigg(K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right) \Bigg) \\
&\quad -Cov(K_i H_i, KjH_j) \Bigg],
\end{aligned}$$

$$\begin{aligned}
Var\Bigg(K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right) \Bigg) &= \mathbb{E}\Bigg[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right) \Bigg]^2 \\
&\quad -\mathbb{E}^2 \Bigg(K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right) \Bigg),
\end{aligned}$$

as

$$\mathbb{E}^2\Bigg[K\left(\frac{\|x-X_i\|}{h_i}\right) H\left(\frac{y-Y_i}{h_i}\right) \Bigg] = o[\{F(h_i)\}^2],$$

$$\mathbb{E}\left[K\left(\frac{\|x-X_i\|}{h_i}\right)H\left(\frac{y-Y_i}{h_i}\right)\right]^2 = \mathbb{E}\left\{K^2\left(\frac{\|x-X_i\|}{h_i}\right)\mathbb{E}^2\left(H\left(\frac{y-Y_i}{h_i}\right)|X\right)\right\} + \mathbb{E}\left\{\sigma^2_{\varepsilon_i}(x)K\left(\frac{\|x-X_i\|}{h_i}\right)\right\},$$

and

$$\mathbb{E}^2\left[H\left(\frac{y-Y_i}{h_i}\right)|X\right] = o(h_i) + [F^{[x]}(y)]^2.$$

There are $\sigma^2_{\varepsilon_i}(x) = Var\left[H\left(\frac{y-y_i}{h_i}\right)|X\right]$ we have by $(h_6)(ii)$, because $H_i(y) \leq 1$ the distribution function

$$Cov(K_iH_i, K_jH_j) = Cov(\Delta_i, \Delta_j),$$

$$Cov(K_iH_i, K_jH_j) = o(nF(h_n)),$$

$$Var(\varphi^l_n(x,y)) = \left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^{-2}\sum_{i=1}^{n}\left(F(h_i)^l\right)^{-2}M_2F(h_i) \left(F^x(y)^2 + \sigma^2_{\varepsilon_i}(x)\right)(1+\gamma_i) - O(nF(h_n)),$$

with $\gamma_i = O(h_i)$, we get

$$Var(\varphi^{[l]}_n(x,y)) = \frac{\beta_{[1-2l]}}{\beta^2_{[1-l]}}\left[[F^{[x]}(y)]^2 + \sigma^2_\varepsilon(x)\right]\frac{1}{nF(h_n)}M_2[1+o(1)] - o\big(nF(h_n)\big).$$

$$\begin{aligned}
Var(G^l_n) &= \left\{\left[\sum_{i=1}^{n}[F(h_i)]^{1-l}\right]^{-2}\sum_{i=1}^{n}\left[\frac{1}{[F(h_i)]^l}\right]^2\right\}\left\{Var\left[K\left(\frac{\|x-X_i\|}{h_i}\right)\right] - Cov(K_i;K_j)\right\}\\
&= \left[\sum_{i=1}^{n}[F(h_i)]^{1-l}\right]^{-2}\sum_{i=1}^{n}\left[\frac{1}{[F(h_i)]^l}\right]^2\Big(M_2F(h_i)[1+\gamma_i] - Cov(\Delta_i;\Delta_j)\Big)\\
&= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2}\sum_{i=1}^{n}[F(h_i)]^{1-2l}M_2[1+\gamma_i] - o(nF(h_n))\\
&= \frac{\beta_{[1-2l]}}{\beta^2_{[1-l]}}M_2[1+o(1)] - o(nF(h_n))
\end{aligned}$$

$$
\begin{aligned}
Cov(\varphi_n^{[l]}, G_n^{[l]}) &= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2} Cov(K_i H_i; K_j) \\
&= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2} \left\{ \sum_{i,j=1}^{n} \mathbb{E}\Big[K_i H_i K_j\Big] - \Big[\sum_{i=1}^{n} \mathbb{E}\Big(K_i H_i\Big) \times \sum_{j=1}^{n} \mathbb{E}\Big(K_j\Big)\Big]\right\},
\end{aligned}
$$

because $H_i \leq 1$ is distribution cumulative function we get:

$$
\begin{aligned}
Cov(\varphi_n^{[l]}, G_n^{[l]}) &= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2} \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(K_i, K_j) \\
&= \frac{1}{\left(\sum_{i=1}^{n}[F(h_i)]^{1-l}\right)^2} \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(\Delta_i, \Delta_j) \\
&= O\left(\frac{1}{nF(h_n)}\right).
\end{aligned}
$$

Finally, we have

$$
Var\Big[\widehat{F}^{x,[l]}(y)\Big] = \frac{\beta_{1-2l}}{\beta_{1-l}} \frac{M_2}{M_1^2} \sigma_\varepsilon^2 \frac{1}{nF(h_n)}[1 + o(1)].
$$

Given

$$
\mathbb{E}\Big\{\Big[G_n^l(x) - \mathbb{E}G_n^l(x)\Big]\varphi_n^{[l]}(x,y)\Big\} = O\left(\frac{1}{nF(h_n)}\right),
$$

$$
\mathbb{E}\Big\{\Big[G_n^l(x) - G(x)\Big]^2 \widehat{F}_n^{[x,l]}(y)\Big\} = O\left(\frac{1}{nF(h_n)}\right),
$$

and

$$
\mathbb{E}\Big[\widehat{F}_n^{[x,l]}(y) - F^{[x]}(y)\Big] = h_n \varphi'(0) \frac{\alpha_{[l]}}{\beta_{[1-l]}} \frac{M_0}{M_1}[1 + o(1)] + O\left(\frac{1}{nF(h_n)}\right).
$$

The proof takes and here.

■

General conclusion

In our work we are realized an important subject of statistics nonparametric in functional case, we are also interested to prove the different results of the convergence as will as the asymptotic normality of the estimator of the maximum of the conditional hazard function and the almost sure and mean quadratic convergence of our estimator under the strong mixing condition.

The richness of this functional statistical research area offers many perspectives both theoretically and practically. In the following, we will comment on some results already obtained, with the major concern of focusing on all open issues some of which are under development.

Prospects

The work developed in this thesis offers many prospects in the short and long term. Regarding the short-term prospects:

- recursive estimation of the mode and conditional quantile.
- the recursive estimation of the conditional distribution for truncated or censored data.
- the recursive estimation of the conditional distribution and density for ergodic observations.
- The work on the estimation of conditional quantiles and the conditional hazard function for functional explanatory variable opens several perspectives. For example, another estimator may consider using a different method than the estimate by the kernel method as Fourier techniques: Fourier series decomposition, wavelet series decomposition, series decomposition of polynomials...

- Estimation with spatial functional data can be approached in several ways

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