N°d'ordre:

Université de Saida–Dr. Moulay Tahar
Faculté de MIT

Thèse

Présentée pour obtenir le diplôme de

Doctorat 3ème Cycle

Spécialité: Mathématiques Appliquées

Filière: Mathématiques Appliquées

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Thème :

Analyse stochastique via les processus non Gaussiens et Applications en Finance

Thèse soutenue le 01/07/2026 devant le jury composé de:

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Dedication

This thesis is dedicated to:
My beloved grandmother,
whose memory will always stay in my heart.
My dear parents,
who have always believed in me.
My sisters and my brother,
for their encouragement.
My family,
for their presence and strength.
My friends,
for their companionship and support throughout this journey.

Acknowledgments

In the name of Allah, the most merciful, the most compassionate all praise be to Allah and prayers and peace be upon Mohammed his servant and messenger.
First, I thank God for giving me the strength and courage to complete this work.

The completion of this thesis would not have been possible without the support and help of many people, to whom I would like to express my sincere gratitude.

I would like to express my deep appreciation to my supervisor Professor Abdeldjebbar Kandouci for his patience, availability and the valuable time he devoted to this work. I am truly grateful for his guidance, support, advice and encouragement, he was always helpful and comprehensive. I would thank him also for his precious help in the administrative side.

I would like to express my sincere gratitude to my co-supervisor, Professor Omar Kebiri, for his patience, guidance, and continuous support throughout these years. I am very thankful for the many valuable discussions we shared and for his insightful remarks, which greatly contributed to improving the quality of this work. I would also like to thank him for receiving me at the BTU Cottbus–Senftenberg Institute of Mathematics during my research stay.

It was a great honor for me to work under their guidance. Their valuable advice and thoughtful suggestions greatly contributed to the completion of this thesis.

My special thanks go to Professor Sâadia Rahmani for accepting to chair the thesis committee. I would also like to express my sincere gratitude to the committee members Professor Ciprian Tudor, Professor Kadi Mokhtar, Professor Fatima Benziadi and Professor Tomás Caraballo for accepting to evaluate my work.

I would like to sincerely thank Professor Tomás Caraballo for receiving me at the University of Sevilla, Spain, during my research internship. I also express my sincere gratitude to Professor Ciprian Tudor and Dr. Nachit Yassine for hosting me at the Paul Painlevé Laboratory, University of Lille. I am very thankful for their support during my stay and for their valuable help in my research.

I am deeply grateful to all my teachers throughout my study levels. I would especially like to express my appreciation to the professors of the University of Saida for the knowledge and guidance they provided, and for equipping me with the tools needed to succeed in my academic journey.

My sincere gratitude goes to all the members of the Laboratory of Stochastic Models, Statistics and Applications for their kindness, support, and assistance. I am also very grateful to my doctoral colleagues, as well as to everyone who contributed in one way or another to the completion of this work.

My heartfelt thanks go to my dear parents, who have always been there for me. I am deeply grateful for their endless support and encouragement, which helped me reach this stage of my studies. I also wish to thank my sisters and brothers for their encouragement.

Abstract

The aim of this thesis is to develop a framework for the stochastic analysis of the Rosenblatt process: a non-Gaussian, self-similar process with long-range dependence and non-semimartingale behavior. This process is a natural modeling tool for complex phenomena, particularly in finance where classical Gaussian assumptions may not be sufficient to explain empirical features. First, we define a stochastic integral with respect to the Rosenblatt process for non adapted processes. The construction uses the convergence of Riemann sums and a suitable decomposition of the integrand process into adapted process and instantly independent process. This approach extends the anticipative stochastic calculus from the classical Brownian setting to the non-Gaussian setting. Next, we investigate stochastic differential equations driven by the Rosenblatt process. Under appropriate assumptions, we prove existence and uniqueness results for the solutions and study their stability properties. A numerical scheme based on the Euler Maruyama method is also implemented in order to illustrate the behavior of the solutions. Finally, we establish the conditional full support (CFS) property for the Rosenblatt process. We show that it plays a central role in financial modeling. In particular, we highlight that the CFS property ensures the absence of arbitrage opportunities without requiring the explicit calculation of the risk-neutral probability. As an application, we verify the CFS property in the context of SDE driven by the Rosenblatt bridge.

Key words: Rosenblatt process, non adapted process, SDE, numerical analysis, conditional full support, Rosenblatt bridge, absence of arbitrage opportunities.

Résumé

L'objectif de cette thèse est de développer un cadre pour l'analyse stochastique du processus de Rosenblatt : un processus non gaussien, auto-similaire, présentant une dépendance à long terme et un comportement non semi-martingale. Ce processus constitue un outil de modélisation naturel pour les phénomènes complexes, en particulier en finance où les hypothèses gaussiennes classiques peuvent s'avérer insuffisantes pour expliquer les caractéristiques empiriques. Nous avons défini d'abord une intégrale stochastique par rapport au processus de Rosenblatt pour les processus non adaptés. La construction repose sur la convergence des sommes de Riemann et une décomposition appropriée du processus à intégrer en processus adapté et processus instantanément indépendants. Cette approche étend le calcul stochastique anticipatif du cadre brownien classique au cadre non gaussien. Nous étudions ensuite les équations différentielles stochastiques pilotées par le processus de Rosenblatt. Sous des hypothèses appropriées, nous démontrons l'existence et l'unicité des solutions et étudions leurs propriétés de stabilité. Un schéma numérique basé sur la méthode d'Euler-Maruyama est également mis en œuvre afin d'illustrer le comportement des solutions. Enfin, nous établissons la propriété de support complet conditionnel (CFS) pour le processus de Rosenblatt. Nous montrons qu'elle joue un rôle central dans la modélisation financière. En particulier, nous soulignons que la propriété CFS garantit l'absence d'opportunités d'arbitrage sans nécessiter le calcul explicite de la probabilité de risque neutre. Comme application, nous vérifions la propriété CFS dans le cadre d'une équation différentielle stochastique pilotée par le pont de Rosenblatt.

Mots clés: Processus de Rosenblatt, processus non adapté, indépendance instantanée, EDS, analyse numérique, support complet conditionnel, pont de Rosenblatt, absence d'opportunités d'arbitrage.

الملخص:

تهدف هذه الأطروحة إلى تطوير إطار عمل للتحليل العشوائي لعملية روزنبلات: وهي عملية غير غاوسية ذاتية التشابه ذات تبعية طويلة المدى وسلوك غير شبه مارتينجالي. تُعد هذه العمليات أداة نمذجة طبيعية للظواهر المعقدة، لا سيما في مجال التمويل حيث قد لا تكفي الافتراضات الغاوسية الكلاسيكية لتفسير السمات التجريبية. أولًا، نُعرّف تكاملًا عشوائيًا بالنسبة لعملية روزنبلات للعمليات غير المُتكيفة. يعتمد هذا البناء على تقارب مجاميع ريمان وتفكيك مناسب لعملية التكامل إلى عملية مُتكيفة وعملية مستقلة لحظيًا. يُوسّع هذا النهج حساب التفاضل والتكامل العشوائي الاستباقي من الإطار البراوني الكلاسيكي إلى الإطار غير الغاوسي. ثانيًا، ندرس المعادلات التفاضلية العشوائية التي تحركها عملية روزنبلات. في ظل افتراضات مناسبة، نُثبت وجود حلول فريدة وندرس خصائص استقرارها. تم تطبيق مخطط عددي قائم على طريقة أويلر ماروياما لتوضيح سلوك الحلول. وأخيرًا، أثبتنا خاصية الدعم الكامل المشروط (CFS) لعملية روزنبلات، وبيّنا دورها المحوري في النمذجة المالية. وعلى وجه الخصوص، نؤكد أن خاصية الدعم الكامل المشروط تضمن غياب فرص المراجحة دون الحاجة إلى حساب صريح لاحتمالية الحياد عن المخاطرة. كتطبيق عملي، تحققنا من خاصية الدعم الكامل المشروط في سياق المعادلات التفاضلية العشوائية المدفوعة بجسر روزنبلات.

الكلمات المفتاحية : عملية روزنبلات؛ عملية غير متكيفة؛ استقلالية آنية؛ المعادلات التفاضلية العشوائية؛ المحاكاة العددية؛ الدعم الكامل الشرطي؛ جسر روزنبلات؛ غياب فرص المراجحة.

List of works

Publications:

1. Benkaddour, S., Kandouci, A., and Kebiri, O. (2025). Stochastic integral for non-adapted processes with respect to the Rosenblatt process. Random Operators and Stochastic Equations, vol. 33, no 3, p. 271-281.

2. Benkaddour, S., Kandouci, A., Kebiri, O., and Caraballo, T. (2026). Stochastic differential equation driven by the Rosenblatt process. Acta Universitatis Sapientiae, Mathematica, vol. 18, no 1, p. 9. https://doi.org/10.1007/s44426-026-00034-2.

3. Benkaddour, S., Kandouci, A. and Kebiri, O. (2026). The study of no-arbitrage property in a non-Gaussian setting: The case of the Rosenblatt model. Submitted for publication in Finance and Stochastics (Springer Heidelberg). ISSN: 0949-2984, 1432-1122.

Communications:

1. A new approach of stochastic integral with respect to the Rosenblatt process. Recent Developments in Stochastics with Applications in Mathematical Physics and Finance. From 16 to 22 October 2022 in Hammamet, Tunisia.

2. Stochastic integral for non adapted process related to the Rosenblatt process. International Symposium MOAD'2022. From 15 to 17 November 2022 in Bejaia, Algeria.

3. Hurst index estimation for the Rosenblatt process. First international workshop of statistics and its application (IWSA'23) 1–2 March 2023 at University of Saida,

Algeria.

4. The Rosenblatt process. BTU STOCHASTIS Seminar, Brandenburg University of Technology Cottbus-Senftenberg, Germany, 20 December 2023.

5. Numerical solution of SDE driven by the Rosenblatt process. The first National Conference on Applied Mahematics, Statistics and Applications. 18–19 November 2024, in university of Batna, Algeria.

6. Stochastic differential equation driven by the Rosenblatt process. The 4th National Conference of Mathematics and Applications (CNMA'2024). 07–08 December 2024, in university of Mila, Algeria.

7. Existence of solution for SDE driven by the Rosenblatt process. The First National Symposium on Innovative Mathematics: Retrospective and Perspectives 2024. 11–12 December 2024, in university of Mostaganem, Algeria.

8. The study of SDE driven by a non Gaussian process. International Conference on Mathematics and its Applications in Science and Technology. 15–16 December 2024, in university of Setif, Algeria.

Research events

1. Online workshop: "Tips and Advice for Preparing Your Manuscript" (90 min), Researcher Academy On Campus program, Algeria, 01 November 2022.

2. BTU Stochastics Seminar, Recent Results on Weakly Self-Avoiding fractional Brownian Motion, Brandenburg University of Technology Cottbus-Senftenberg, Germany, 12 July 2023.

3. Autumn School on: Deep learning Methods for Stochastic Dynamics, Laboratory of Stochastic Models, Statistics and Applications, Saida, from 29 October to 2 November 2023.

4. Applied mathematics week, Laboratory of Stochastic Models, Statistics and Applications, Saida, 2–7 March 2024.
5. Stochastic optimal control of epidemic models, Laboratory of Stochastic Models, Statistics and Applications, Saida, 19 April 2024.
6. Artificial Intelligence day during the Student Week, Laboratory of Stochastic Models, Statistics and Applications, Saida, from 19 to 23 May 2024.
7. Attractors of Caputo fractional differential equations, Dpto. Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla-Spain, 15 January 2025.
8. Conference series on Epidemic Models: Covid-19, HIV, Ebola and Optimal Relaxed Control for a Decoupled G-FBSDE, Laboratory of Stochastic Models, Statistics and Applications, Saida, February 2025.
9. Artificial Intelligence School (AI School), Laboratory of Stochastic Models, Statistics and Applications, Saida, 17–19 June 2025.
10. Nonparametric Estimation of the "Smurf" Transition Rate and the Mortality Rate in a Two-Phase Aging Model, Laboratoire Paul Painlevé – UMR CNRS 8524, Université de Lille, France, 25 November 2025.
11. Conference series on Statistics and Artificial Intelligence (SAI'26), Laboratory of Stochastic Models, Statistics and Applications, Saida, 2–4 February 2026.

Presentations in the LMSSA Laboratory

1. The Application of non gaussian process, 16 May 2023.
2. An anticipating stochastic integral with respect to the Rosenblatt process, 20 September 2023.
3. On the non gaussian process, Applied Mathematics Week, 2 March 2024.

Contents

Introduction

Stochastic processes play a central role in modern financial modeling, providing mathematical tools to describe systems that evolve randomly over time. From asset prices and interest rates to market volatility and risk dynamics, these processes allow researchers to capture the uncertainty and irregular behavior observed in financial markets. Among them, self-similar processes have attracted particular attention because of their ability to model long-range dependence and scaling properties often present in financial data. A stochastic process is called self-similar if its distribution remains invariant under suitable scalings of time and space; more precisely, a process $(X_t)_{t\geq 0}$ is self-similar with index $H > 0$ if for every $a > 0$ the processes $(X_{ct})_{t\geq 0}$ and $(c^H X_t)_{t\geq 0}$ have the same finite-dimensional distributions. Such processes are useful in finance because they can capture behaviors observed across different time scales that are not well described by the classical Brownian motion model. For further exploration and applications of self-similar stochastic processes, interested readers may refer to bibliographical guides provided by Taqqu [78] and Willinger et al. [83], which offer comprehensive references and applications in various domains. The family of fractional Brownian motions $(B_t^H)_{t\geq 0}$ with the Hurst parameter $0 < H < 1$ stands out as one of the most extensively studied self-similar stochastic processes, because these processes are Gaussian, have stationary increments, and exhibit long-range dependence for $1/2 < H < 1$. These characteristics make them highly desirable for real-world modeling and applications.

Nevertheless, there have also been empirical observations of non-Gaussian data with fractal properties, such as in [27], where control error in single-input-single-output (SISO) loops is examined. Domanski [27] has demonstrated that the Gaussian assumption is not always suitable using data from various physical systems. In these situations, it doesn't appear reasonable to characterize these physical phenomena with a Gaussian process, such as fractional Brownian motion, and non-Gaussian alternatives can provide a more adequate framework.

In particular, several classes of stochastic processes have been introduced in the literature to capture long-range dependence beyond the Gaussian setting. Among them are stable processes, which are characterized by heavy-tailed distributions and have been extensively studied in connection with long-memory phenomena (see, for example, [60]). However, the strong tail behavior of stable processes may in some applications be too extreme, which motivates the search for intermediate models exhibiting long-range dependence with more moderate tail behavior. An interesting class of processes in this direction is given by the so-called Hermite processes, which arise naturally in non-central limit theorems for strongly dependent random variables, as established by Dobrushin and Major [26], and later developed by Taqqu [77]. These processes are neither Gaussian nor stable and exhibit intermediate tail behavior, in the sense that their distributions are heavier-tailed than the Gaussian law but lighter than stable laws. Within this class, the second-order Hermite process, known as the Rosenblatt process, occupies a central role.

In this thesis, we focus on this process as a representative non-Gaussian model exhibiting long-range dependence, and investigate its properties and applications in the context of stochastic modelling. The Rosenblatt process belongs to the category of Hermite processes, which are limits of normalized sums of long-range dependent random variables. The Gaussian fractional Brownian motion is one of the most studied Hermite processes due to its significant importance in modeling. In contrast, the Rosenblatt process represents the simplest non-Gaussian Hermite process. It is characterized by continuous non-differentiable paths and exhibits self-similarity with stationary increments. Consequently, it offers a richer framework for modeling complex systems where higher-order dependencies play a significant role. This process was first conceived by Murray Rosenblatt and was subsequently named the Rosenblatt process by Taqqu in [75]. The Rosenblatt process was studied later by Taqqu [79].

In [80], Tudor provides a stochastic analysis of Rosenblatt processes, whereas these references [1, 2, 5, 6, 18, 28, 67] investigate various structural, probabilistic, and statistical aspects of the Rosenblatt process. Maejima and Tudor [52] demonstrated that the multivariate Rosenblatt distribution is a member of the Thorin class, which is a subset of the class of self-decomposable distributions. Based on this finding, new properties of the Rosenblatt distribution are derived.

Despite its importance, the mathematical analysis of the Rosenblatt process presents substantial difficulties due to the loss of Gaussianity. In particular, the Rosenblatt process does not possess the characteristic of being a semimartingale, and this presents a limitation when attempting to apply the classical stochastic calculus framework developed by Itô . For the integral $\int_0^t u_s dX_s$ to have a meaningful interpretation, the integrator X is typically required to be a semimartingale. However, the Rosenblatt process does not satisfy this condition. As a result, we turn to two primary methods for developing a stochastic integration theory that accommodates such situations: the pathwise calculus approach and the Malliavin calculus/Skorohod integration approach. The first approach, which includes rough paths analysis [49] and the stochastic calculus via regularization [72], is directly applicable to the Rosenblatt process due to the regularity of its paths and its covariance structure; a pathwise Itô formula can be obtained. The Malliavin calculus is very related to the Gaussian character of the driven process, see [56].

The non-Gaussian nature of the Rosenblatt process significantly complicates the development of a stochastic calculus with respect to this process. Many classical techniques available in the Gaussian setting either fail or require substantial modifications. Consequently, several fundamental questions remain challenging, particularly concerning: the construction of stochastic integrals, the analysis of stochastic differential equations, and the investigation of the pathwise properties.

The present thesis is devoted to the study of stochastic analysis associated with the

Rosenblatt process. More specifically, this work focuses on three closely related aspects: stochastic integration for non-adapted processes, stochastic differential equations driven by the Rosenblatt process, and the conditional full support property of the Rosenblatt process.

The first part of this thesis concerns the construction and analysis of stochastic integrals of non-adapted processes with respect to the Rosenblatt process. Given the non-semimartingale nature of this process, classical Itô integration is not applicable, which necessitates the use of alternative approaches such as pathwise techniques and Malliavin calculus.

On the other hand, we have the problem of adaptation. In real life, a non-adapted process may depend on information that is not available at the current time, including future values of the driving noise. For instance, suppose that an investor has access to the future value of a stock price at a terminal time T. In this case, the corresponding process depends on future information and is therefore a non-adapted process. Such processes naturally arise in insider trading models, stochastic control problems with future information, and anticipative stochastic calculus. Another limitation of classical Itô integral is the adaptedness of the integrand to the natural filtration of the integrator i.e. $\int_0^t u_s dX_s$ is well defined if and only if u is adapted to the filtration generated by the process X. Then, the integral is defined like Riemann sums at which the evaluation points are the left endpoints of subintervals. The issue of non-adapted integrands was highlighted by Itô [35] at the International Symposium on Stochastic Differential Equations in Kyoto (1976), where he considered the problem of defining stochastic integrals when the integrand is not adapted to the natural filtration. To address this difficulty, Itô considered an enlargement of the filtration. Under this extended filtration, the integrand becomes adapted.

There have been several extensions in the literature on anticipating stochastic integration. We refer for instance to Hitsuda [33] and Skorokhod [62]. For more details on the anticipating integrals and their applications, see Buckdahn [16], Leon and Protter [46], Pardoux and Protter [66], and the references therein. Ayed and Kuo [4] introduced a different approach to the definition of anticipating stochastic integrals. Their method is based on decomposing an anticipating integrand into combinations of adapted processes and instantly independent processes. Using this decomposition, they defined the stochastic integral through a Riemann-sum construction derived from the classical stochastic integration framework developed by Kuo [41]. In this construction, the adapted component is evaluated at the left endpoints of the partition intervals, whereas the instantly independent component is evaluated at the right endpoints. This approach has stimulated further research on anticipating stochastic calculus. In particular, Kuo and Ayed [4] established an Itô formula for anticipating integrals, which was subsequently extended to more general settings by Kuo et al. [34, 43, 44]. The study of stochastic differential equations with anticipating initial conditions was later investigated by Khalifa et al. [39]. In addition, Kuo et al. [42] examined the corresponding Itô isometry associated with this class of anticipating integrals.

Belhadj et al. [7, 8] defined stochastic integrals for anticipating integrands with respect to the subfractional Brownian motion and the mixed fractional Brownian motion, and showed that the resulting anticipating integrals are a near-martingales under suitable conditions. Hamada et al. [32] studied stochastic integrals for non-adapted processes with respect to fractional Brownian motion.

Using the integrand decomposition method introduced by Ayed and Kuo [4], we define the stochastic integral of an anticipating process with respect to the Rosenblatt process. Once a stochastic integration framework with respect to the Rosenblatt process has been established, a natural continuation is the study of stochastic differential equations driven by this process. Stochastic Differential Equations (SDEs) represent a powerful framework for modeling dynamic systems influenced by both deterministic and random factors. Equations driven by a fractional Brownian motion (fBm) with a Hurst parameter $H \in (1/2, 1)$ can be analyzed using various approaches, including a pathwise approach based on p-variation techniques (as described in [48]), fractional calculus methods introduced by Zähle [84] and applied in [57], or approaches based on Hölder continuity [74].

Additionally, using the notion of p-variation and the rough path analysis developed by Lyons in [47, 48], Coutin and Qian demonstrated in [22] the existence of strong solutions for stochastic differential equations driven by fBm with $H > 1/4$ and investigated a Wong-Zakai approximation limit for these equations.

Due to the non-Gaussianity of the Rosenblatt process and the complexity of its dependence structures, it has received less research interest than fractional Brownian motion. So, studying the stochastic differential equations with the Rosenblatt process appears intriguing. Various aspects of stochastic evolution equations and related stochastic systems driven by Rosenblatt or more general Volterra-type noises, including existence, asymptotic behavior, and parameter estimation, have been investigated in the literature, see e.g. [15, 19, 23, 24, 63, 64, 61]. Coupek et al. [20] solved a linear-quadratic optimal control problem with an infinite time horizon for a scalar linear stochastic differential equation with additive Rosenblatt noise.

In our study of stochastic differential equations driven by the Rosenblatt process, we restrict ourselves to the case of a constant diffusion coefficient. Consequently, the stochastic integral involved in the equation is associated with a predictable integrand, which avoids some of the additional technical difficulties related to anticipating processes. This choice makes it possible to focus primarily on the influence of the Rosenblatt noise itself and on the specific features induced by its non-Gaussian structure. Although the stochastic differential equations considered in this thesis do not involve non-adapted coefficients, the stochastic integration framework developed in the first part remains an essential tool throughout this thesis. It also offers a natural framework for extending the analysis to more general stochastic equations involving random or anticipating diffusion terms.

The third part of this thesis addresses a different but complementary aspect related to the topological properties of the sample paths of the Rosenblatt process, namely the conditional full support (CFS) property by thinking of the problems of no arbitrage in non-Gaussian framework for asset prices driven by the Rosenblatt process. Conditional full support (CFS) is a regular property applied to asset prices, which states that the asset price path can continue arbitrarily near any given path with a positive conditional probability from any given time on. In the context of mathematical finance, specifically pricing models with transaction costs, Guasoni, Rasonyi, and Schachermayer [31] were the first who proposed the CFS property. According to their main finding, if a continuous price process has CFS property, then for any $\varepsilon > 0$, there exists a so-called ε-consistent price system. This system is a martingale (following an equivalent change of measure) that shadows the price process within the bid-ask spread implied by ε-sized proportional transaction costs (for a related study, see [37]). Under arbitrary small transaction costs, the price process does not accept arbitrage opportunities, as indicated by the existence of ε-consistent price systems for every $\varepsilon > 0$ because any arbitrage technique will produce arbitrage in a system of consistent prices as well, which is contradicted by the martingale property. Consistent price systems can be seen as generalizations of equivalent martingale measures (EMM's), because for every $\varepsilon > 0$, a price process is considered as a trivial ε-consistent price system if it admits an EMM.

Nonetheless, CFS condition is worth researching even for price processes that accept EMMs since it makes it possible to create a specific consistent price system that are helpful in solving superreplication issues with proportional transaction costs. This may be seen in [31] in the "face-lifting" outcome, it states that the superreplication price of any European-style vanilla contingent claim $g(P_T)$ with ε-sized proportional transaction costs tends to $\hat{g}(P_0)$ when $\varepsilon \downarrow 0$ if $(P_t)_{t\in[0,T]}$ is a price process with CFS, where $\hat{g}$ is the concave envelope of g (informally, the "smallest" concave function that majorizes g). This implies that superreplicating, for example, a European call option under small proportional transaction costs involves purchasing the underlying asset, which is the trivial superreplicating portfolio. According to the recent article by Bender, Sottinen, and Valkeila [9], CFS property has been found to be relevant to frictionless pricing models in addition to ones with transaction costs. According to their findings, a continuous price process does not admit arbitrage opportunities in a class of trading strategies if it possesses CFS property (also known as the conditional small ball property) and pathwise quadratic variation even though this is a little more limited than what classical models allow, it still includes most, if not all, of the practically applicable strategies.

CFS property is a fundamentally intriguing property from a purely mathematical standpoint, in addition to its applications in mathematical finance. The study of the CFS property, in particular, can be seen as a logical continuation of classical work on the supports of the laws of continuous Gaussian processes, by Kallianpur [38], and diffusions, initiated by Stroock and Varadhan [65] and continued by several other authors (see e.g. [54]). The CFS condition has been the subject of numerous studies, including those by: Guasoni et al. [31] where it was demonstrated that there is a CFS condition for fractional Brownian motion with an arbitrary Hurst parameter, then Cherny [17] later generalized this and demonstrated that any Brownian moving average fulfills the CFS property. Following that, Gasbarra [29] established this condition for Gaussian processes with stationary increments. Pakkanan [59] established conditions for the process $Q = L + H * W$ to have conditional full support, where W is a Brownian motion, L is a continuous process and the processes L and H are either progressive or independent of W.

Dani et al. [25] studied the conditional full support (CFS) property for the processes $S_t = L_t + \int_0^t h_s dB_s^H$ where $(B_t^H)_{t\in[0,T]}$ is a fractional Brownian motion, $(L_t)_{t\in[0,T]}$ is a continuous process, and the processes $(L_t)_{t\in[0,T]}$ and $(h_t)_{t\in[0,T]}$ are independent of $(B_t^H)_{t\in[0,T]}$.

Although the three topics considered in this thesis may appear distinct at first sight, they are in fact closely connected through the common objective of developing a deeper stochastic framework for the study of the Rosenblatt process. Stochastic integration provides the analytical foundation for the analysis of stochastic differential equations, while the study of pathwise properties, such as the conditional full support condition, contributes to a more comprehensive understanding of both the process itself and the systems it drives. Taken together, these elements offer a coherent approach to the study of non-Gaussian stochastic processes exhibiting long-range dependence.

The mathematical tools used throughout this thesis are mainly based on Malliavin calculus, Wiener chaos techniques, stochastic analysis for non-Gaussian processes, and functional analytic methods. These tools allow us to overcome the analytical difficulties associated with the non-Gaussian nature of the Rosenblatt process.

Outline of thesis

This thesis is organized as follows: In the first chapter, we recall the basic background on Rosenblatt process. The second chapter is devoted to the presentation of the anticipating stochastic integral introduced by Ayed and Kuo [4], whose construction is based on a suitable decomposition of the integrand. We also review several previous works related to anticipating stochastic integration for fractional Gaussian processes. Finally, we establish our main results by defining and studying anticipating stochastic integrals with respect to the Rosenblatt process.

The third chapter focuses on stochastic differential equations driven by the Rosenblatt process. We consider a class of equations with constant diffusion coefficient, corresponding to the predictable setting. We investigate the existence, uniqueness, and stability of the corresponding solutions. In addition, several illustrative examples and numerical simulations are provided in order to compare the exact solutions with their numerical approximations.

The fourth part of this thesis concerns the study of the conditional full support property. We begin by recalling the fundamental notions and main concepts related to this property. We then review several previous results established in the Gaussian framework. After this preliminary analysis, we examine the CFS property by thinking of the problems of no arbitrage in non-Gaussian framework for asset prices driven by the Rosenblatt process then, we build the absence of arbitrage opportunities without calculating the risk-neutral probability by the existence of the consistent price systems. In addition we prove the conditional full support property for a class of stochastic differential equations driven by the Rosenblatt bridge.

Finally, the conclusion summarizes the main contributions of the thesis and discusses several possible directions for future research.

Chapter 1

On the Rosenblatt process: Background and definitions

The limits appearing in the non-central limit theorem (see, for example, [77] or [26]) characterize an intriguing class of self-similar processes. We briefly review this background. Let us review the concept of Hermite rank. The Hermite polynomial of degree m is represented by $H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{\frac{-x^2}{2}}$ and let g be a function on $\mathbb{R}$ such that $\mathbf{E}[g(\xi_0)] = 0$ and $\mathbf{E}[g(\xi_0)^2] < \infty$. Assume that the expansion of g in Hermite polynomials is as follows.

$$g(x) = \sum_{l=0}^{\infty} c_l H_l(x),$$

where $c_l = \frac{1}{l!}\mathbf{E}[g(\xi_0)H_l(\xi_0)]$. The definition of the Hermite rank of g is

$$k = \min\{l \mid c_l \neq 0\}.$$

Since $\mathbf{E}[g(\xi_0)] = 0$, we have $k \geq 1$. Let g be a Hermite rank k function, and let $(\xi_n)_{n\in\mathbb{Z}}$ be a stationary Gaussian sequence with variance 1 and mean 0 that displays long range dependency in the sense that the correlation function satisfies

$$r(n) := \mathbf{E}(\xi_0 \xi_n) = n^{\frac{2H-2}{k}} L(n),$$

where $H \in (\frac{1}{2}, 1)$, $k \geq 1$ and L is a slowly varying function at infinity (see e.g. [50] for the definition). Then the following family of stochastic processes

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j)$$

converges to a self-similar stochastic process with stationary increments that resides in the k-th Wiener chaos as $n \to \infty$, in the sense of finite dimensional distributions. This process is called the Hermite process of order k. The only Gaussian process in the class of Hermite processes is fractional Brownian motion. Their practical significance is considerable, as they constitute a broad class of stochastic processes suitable for modeling long-range dependence, self-similarity, and Hölder regularity, while permitting substantial deviations from fractional Brownian motion and other Gaussian processes. Moreover, owing to their non-Gaussian nature and their self-similarity with stationary increments, Hermite processes provide appropriate inputs for models in which empirical data display self-similar behavior accompanied by non-Gaussian characteristics. Based on the analysis of stochastic modeling for river-flow time series in [45], the work [76] discusses the necessity of non-Gaussian self-similar processes in practice (e.g., in hydrology). The fundamental characteristics of the Hermite process are examined in this chapter, with a particular emphasis on the Rosenblatt process, which is the most well-known Hermite process after fBm.

1.1 The Hermite processes

Definition 1.1.1. *The Hermite process $(R_H^k(t))_{t\geq 0}$ of order k and with Hurst parameter $H \in (0,1)$ is defined by a multiple Wiener-Itô integral of order k with respect to the standard Brownian motion $(B(y))_{y\in\mathbb{R}}$ by*

$$R_H^k(t) = c(H,k) \int_{\mathbb{R}^k} \left(\int_0^t \prod_{j=1}^{k} (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}) ds \right) dB(y_1)...dB(y_k), \tag{1.1}$$

with $x_+ = max(x,0)$, and the constant $c(H,k)$ is a normalizing constant that ensures $\mathbf{E}(R_H^k(t))^2 = 1$.

Remark 1.1.1.
- *If $k=1$, we obtain the fractional Brownian motion.*
- *For $k=2$, we obtain the Rosenblatt process, which is introduced below.*

The following lemma will be used in the computation of the covariance of the Hermite process, and the constant $c(H,k)$.

Lemma 1.1.1. *[81] For $a+b<1$*

$$\int_{-\infty}^{u\wedge v} (u-y)^a (v-y)^b dy = \beta(-1-a-b, b+1)|u-v|^{a+b+1}.$$

We now recall some important properties of the Hermite process established in [81].

1.1.1 Some properties of the Hermite processes

1. $R_H^k(t)$ is H self-similar, so that $\forall c > 0$.

$$R_H^k(ct) \stackrel{d}{=} c^H R_H^k(t). \tag{1.2}$$

Proof. Let $c > 0$, we have:

$$\begin{aligned}
R_H^k(ct) &= c(H,k)\int_{\mathbb{R}^k}\int_0^{ct}(\prod_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})dsdB(y_1)...dB(y_k) \\
&= c\,c(H,k)\int_{\mathbb{R}^k}\int_0^{t}(\prod_{j=1}^k (cs-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})dsdB(y_1)...dB(y_k) \\
&= c\,c(H,k)\int_{\mathbb{R}^k}\int_0^{t}(\prod_{j=1}^k (cs-cy_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})dsdB(c^{-1}y_1)...dB(c^{-1}y_k) \\
&= c\,c^{-(\frac{1}{2}+\frac{1-H}{k})}c(H,k)\int_{\mathbb{R}^k}\int_0^{t}(\prod_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})dsdB(c^{-1}y_1)...dB(c^{-1}y_k) \\
&= c\,c^{-(\frac{1}{2}+\frac{1-H}{k})}c^{\frac{-k}{2}}c(H,k)\int_{\mathbb{R}^k}\int_0^{t}(\prod_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})})dsdB(y_1)...dB(y_k) \\
&\overset{(d)}{=} c^H R^k(t).
\end{aligned}$$

Recall that $\overset{(d)}{=}$ denotes the equivalence of finite dimensional distributions. □

2. The increments of the process R^k are stationary.

   Proof. It follows immediately from (1.1) that for every $h > 0$ the processes $(R^k(t+h) - R^k(t))_{t\geq 0}$ and $(R^k(t))_{t\geq 0}$ coincide in law. □

3. All moments of the Hermite process are finites and for every $p \geq 1$

$$\mathbb{E}(|R_H^k(t)|)^p = \mathbb{E}|R_H^k(1)|^p t^{2H}, \quad for\ every\ t \geq 0. \tag{1.3}$$

   Proof. This is a consequence of property 1. □

4. For every $s, t \geq 0$ the covariance of the Hermite processes is

$$\mathbf{E}\big[R_H^k(t)R_H^k(s)\big] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{1.4}$$

Proof. For all $t, s \in \mathbb{R}^+$ and $H \in (0,1)$, by using the isometry of multiple Wiener Itô integrals and Fubini's theorem, we have:

$$\begin{aligned}
\mathbb{E}(R_H^k(t)R_H^k(s)) &= k!c(H,k)^2 \int_{\mathbb{R}^k} \left(\int_0^t \int_0^s \prod_{j=1}^{k} (u-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} (v-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} dvdu \right) dy_1...dy_k \\
&= k!c(H,k)^2 \int_0^t \int_0^s \left(\int_{\mathbb{R}^k} \prod_{j=1}^{k} (u-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} (v-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} dy_1...dy_k \right) dvdu \\
&= \ k!c(H,k)^2 \int_0^t \int_0^s \left[\int_{\mathbb{R}} (u-y)_+^{-(\frac{1}{2}+\frac{1-H}{k})} (v-y)_+^{-(\frac{1}{2}+\frac{1-H}{k})} dy \right]^k dvdu
\end{aligned}$$

let the beta function be

$$\beta(p,q) = \int_0^1 z^{p-1}(1-z)^{q-1}dz, \quad p,q > 0 \tag{1.5}$$

by using lemma 1.1.1

$$\int_{\mathbb{R}} (u-y)_+^{a-1}(v-y)_+^{a-1}dy = \beta(a, 1-2a)|u-v|^{2a-1} \tag{1.6}$$

we obtain

$$\begin{aligned}
\mathbb{E}(R_H^k(t)R_H^k(s)) &= k!c(H,k)^2 \beta\left(\tfrac{1}{2} - \tfrac{1-H}{k}, \tfrac{2H-2}{k}\right)^k \int_0^t \int_0^s (|u-v|^{\frac{2H-2}{k}})^k dvdu \\
&= k!c(H,k)^2 \frac{\beta(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k})^k}{H(2H-1)} \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})
\end{aligned}$$

we will choose

$$c(H,k)^2 = \left(\frac{\beta(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k})^k}{k!H(2H-1)} \right)^{-1}$$

in order to obtain $\mathbb{E}(R_H^k(t))^2 = 1$. So by that we get:

$$\mathbb{E}(R_H^k(t)R_H^k(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

□

The self-similarity and the stationarity of increments imply that

5. The Hermite process has Hölder continuous paths of order δ with $0 < \delta < H$.

   Proof. The previous property follows from Kolmogorov's criterion and the fact that for any $p \geq 1$

$$\begin{aligned} \mathbb{E}(|R_H^k(t) - R_H^k(s)|^p) &= \mathbb{E}(|R_H^k(t-s)|)^p \\ &= \mathbb{E}(|(t-s)^H R_H^k(1)|)^p \\ &= \mathbb{E}(|R_H^k(1)|)^p |t-s|^{pH}. \end{aligned}$$

□

6. The Hermite process exhibits long-range dependence.

   Proof. This follows from Proposition A.2 in [81] because the self-similarity index is $H > \frac{1}{2}$. □

1.1.2 Additional Representations

The Hermite process on a finite time interval can be represented as a multiple integral with respect to a Wiener process. The kernel K^H of fractional Brownian motion is used in this representation, where

$$K^H(t,x) = d(H)(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{x}\right) \tag{1.7}$$

$d(H)$ is constant and

$$F_1(z) = d(H)\left(\frac{1}{2} - H\right)\int_0^{z-1} \theta^{H-\frac{3}{2}}(1-(\theta+1)^{H-\frac{1}{2}})d\theta.$$

If $H > \frac{1}{2}$, the kernel K_H has the simpler expression

$$K^H(t,x) = d(H)x^{\frac{1}{2}-H}\int_x^t (u-x)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du, \tag{1.8}$$

where $t > s$ and $d(H) = \left(\frac{H(H-1)}{\beta(2-2H,H-\frac{1}{2}}\right)^{\frac{1}{2}}$. The kernel K_H satisfies the condition

$$\frac{\partial K^H}{\partial u}(u,x) = d(H)(\frac{x}{u})^{\frac{1}{2}-H}(u-x)^{H-\frac{3}{2}}. \tag{1.9}$$

Theorem 1.1.1. *[81] Let $H \in (\frac{1}{2}, 1)$. Consider the process $(Y_t^{(q,H)})_{t\in[0,T]}$ with $q \geq 1$ given by*

$$Y_t^{(q,H)} = d(H)\int_0^t \cdots \int_0^t dW_{y_1}\cdots dW_{y_q} \times \left(\int_{y_1\vee\cdots\vee y_q} \partial_1 K^{H'}(u,y_1)\cdots\partial_1 K^{H'}(u,y_q)du\right), \quad t\in[0,1] \tag{1.10}$$

where $K_{H'}$ is the usual kernel of the fractional Brownian motion (1.7), $(W_t)_{t\in[0,T]}$ is a Wiener process and

$$H' = 1 + \frac{H-1}{q} \Longleftrightarrow (2H'-2)q = 2H-2. \tag{1.11}$$

Then, if $d(H)$ is such that $\mathbf{E}|Y_1^{(q,H)}|^2 = 1$ the process $(Y_t^{(q,H)})_{t\in[0,T]}$ has the same finite dimensional distributions as the Hermite process $(R_H^k(t))_{t\in[0,T]}$ given by (1.1).

Remark 1.1.2. *Another representation of $R_t^{(q,H)}$ can be obtained by using multiple stochastic integrals with respect to fractional Brownian motion; see [55] for further details. Let*

$B^{H'}$ be a fractional Brownian motion with Hurst parameter H' given by (1.11) and denote by $I_q^{B^H}$ the multiple integral of order q associated with this process. Then, for every $t \in [0,1]$,

$$R_t^{(q,H)} = c(H) I_q^{B^{H'}}(\mu_t), \tag{1.12}$$

where μ_t denotes the uniform measure on the diagonal D_t of $[0,t]^q$. The constant $c(H)$ is chosen so that the covariance function of $R^{(q)}$ is equal to $\frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big)$.

1.2 The Rosenblatt process

The Rosenblatt process is one of the most studied processes in the family of Hermite processes. It arises as a limit in the non-central limit theorem for normalized sums of long-range dependent random variables, and it belongs to the second Wiener chaos. It is a self-similar process with stationary increments and continuous non-differentiable paths. Due to these properties, it has attracted significant attention in probability theory and stochastic analysis, particularly in the study of long-range dependence phenomena. The Rosenblatt process was introduced by Rosenblatt [71] and later studied in detail by several authors, including Taqqu [79] and Tudor [80]. It admits a representation as a multiple Wiener–Itô integral with respect to Brownian motion, which provides a convenient framework for its analysis and for developing stochastic calculus with respect to it.

Definition 1.3.1. The Rosenblatt process $R = \{R_H^k(t), t \in [0,T]\}$ is a Hermite process with $k = 2$ and is an H-self-similar process with $(\frac{1}{2} < H < 1)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, has a zero mean and has stationary increments whose covariance function is equal to:

$$\mathbf{E}\big[R(t)R(s)\big] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

verifying:

- $R^H(0) = 0 \quad \text{a.s.},$

- $\mathbb{E}\big(R^H(t)\big)^2 = t^{2H}.$

We present some main properties of the Rosenblatt process.

1.2.1 Properties of the Rosenblatt process

Proposition 1.2.1. *[79]*

1. *The Rosenblatt process $\{R(t), t \in [0,T]\}$ has an almost sure continuous path version, since*
$$\mathbb{E}|R(t_1) - R(t_2)|^2 = |t_1 - t_2|^{2H}$$
with $2H > 1$ and thus Kolmogorov's criterion holds (see [14], Theorem 12.4).

2. *The Rosenblatt process $\{R(t), t \in [0,T]\}$ is H-self-similar with $1/2 < H < 1$, so that $\forall C > 0$:*
$$R_H(Ct) \overset{d}{=} C^H R_H(t). \tag{1.13}$$

3. *The increments of the process R are stationary. This implies that for every $h \geq 0$, the process $(R(t+h) - R(t))_{t\geq 0}$ has the same distribution as $(R(t))_{t\geq 0}$.*

4. *The Rosenblatt process is not differentiable in mean square since*
$$\lim_{t\to 0} \mathbb{E}\left(\frac{R(t)}{t}\right)^2 = \lim_{t\to 0} t^{2H-2} = \infty.$$
More precisely, it is non-differentiable almost surely. This follows from a self-similarity argument, identical to the one used in the study of fractional Brownian motion (see [53] and Proposition 1.7.1 in [13]).

5. *The increments of the Rosenblatt process have long-range dependence, that is*

$$\sum_{k=0}^{\infty} \mathbb{E}[(\Delta R(0)\Delta R(k))] = \infty,$$

where $\Delta R(k) = R(k+1) - R(k), \quad k \geq 0.$

6. *The Rosenblatt process is neither a semimartingale nor a Markov process.*

7. *It has zero quadratic variation.*

8. *It admits a Hölder continuous paths of order* $\delta < H$.

1.2.2 Semimartingale property

The asymptotic behavior of the p-variations of the Rosenblatt process will be examined in this subsection. We will demonstrate that the Rosenblatt process is never a semimartingale, but it can be approximated by a sequence of semimartingales.
Let us define, for every $\varepsilon > 0$,

$$R^{\varepsilon}(t) = d(H)\int_0^t\int_0^t \left(\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1)\frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2)\,du\right) dB(y_1)\,dB(y_2)$$

$$= \int_0^t \left(\int_0^u\int_0^u \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1)\frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2)\,dB(y_1)\,dB(y_2)\right) du$$

$$:= \int_0^t A_{\varepsilon}(u)\,du.$$

Since $A_{\varepsilon} \in L^2([0,T]\times\Omega)$ for every $\varepsilon > 0$ and it is adapted, it follows that the process R^{ε} is a semimartingale.

Proposition 1.2.2. *[80] For every* $t \in [0,T]$,

$$R^{\varepsilon}(t) \to R(t) \quad \text{in } L^2(\Omega).$$

Proof. We first write

$$R^{\varepsilon}(t) - R(t) = \int_0^t \int_0^t dB(y_1)\, dB(y_2)$$
$$\times \left(\int_{y_1 \vee y_2}^t \left[\frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2) - \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \right] du \right).$$

Using the isometry property of multiple Wiener integrals, we obtain

$$\mathbb{E}\left|R^{\varepsilon}(t) - R(t)\right|^2 = 2 \int_0^t \int_0^t dy_1\, dy_2 \int_{y_1 \vee y_2}^t \int_{y_1 \vee y_2}^t du\, dv$$
$$\times \left(\frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2) - \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \right)$$
$$\times \left(\frac{\partial K^{H'}}{\partial v}(v+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial v}(v+\varepsilon, y_2) - \frac{\partial K^{H'}}{\partial v}(v, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2) \right).$$

For every fixed u, y_1, y_2, the quantity

$$\left(\frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_1) \frac{\partial K^{H'}}{\partial u}(u+\varepsilon, y_2) - \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \right)$$

tends to zero as $\varepsilon \to 0$. Therefore, the result follows from the dominated convergence theorem. □

1.2.3 Stochastic representations of the Rosenblatt process

The Rosenblatt process admits several equivalent representations in the literature. In this part, we recall three classical representations introduced in [79].

1.2.3.1 Time representation of the Rosenblatt process

The time representation of the Rosenblatt process is :

$$R(t) = c(H,2) \int_{\mathbb{R}^2}' \left(\int_0^t (s-y_1)_+^{\frac{H}{2}-1} (s-y_2)_+^{\frac{H}{2}-1} ds \right) dB(y_1) dB(y_2), \tag{1.14}$$

with $x_+ = max(0, x)$ and $(B(y), y \in \mathbb{R})$ is a standard Brownian motion on $\mathbb{R}$.

The constant of normalizing $c(H, 2)$ is chosen to ensure that $\mathbb{E}(R_H^2(1)) = 1$, by

$$c(H, 2) = \frac{\sqrt{(H/2)(2H-1)}}{\beta(H/2, 1-H)}. \tag{1.15}$$

The prime $'$ indicates that the multiple Wiener–Itô integral is taken over $\mathbb{R}^2$ off the diagonal $\{(y_1, y_2) \in \mathbb{R}^2 : y_1 = y_2\}$, thereby excluding the set where the two integration variables coincide.

We now introduce another representation of the Rosenblatt process based on its spectral structure.

1.2.3.2 Spectral representation of the Rosenblatt process

The Rosenblatt process admits the following spectral representation:

$$R(t) \stackrel{d}{=} A_2(H) \int_{\mathbb{R}^2}^{''} \frac{\exp(it(\lambda_1 + \lambda_2)) - 1}{i(\lambda_1 + \lambda_2)} \frac{1}{|\lambda_1|^{H/2}} \frac{1}{|\lambda_2|^{H/2}} d\widetilde{B}(\lambda_1) d\widetilde{B}(\lambda_2), \tag{1.16}$$

where

$$A_2(H) = \left(\frac{H(2H-1)}{2[2\Gamma(1-H)\sin(H\pi/2)]^2} \right)^{1/2}, \tag{1.17}$$

to ensure that $\mathbb{E}(R^2(1)) = 1$.

And $\widetilde{B}$ denotes a complex Gaussian random measure on $\mathbb{R}$.

We finally present a representation of R as a stochastic integral with respect to a Brownian motion on the interval $[0, T]$. This representation is useful in several theoretical developments.

1.2.3.3 The Finite time interval representation of the Rosenblatt process

The representation is given by:

$$R(t) = A_3(H) \int_{[0,t]^2}^{'} \left(\int_{y_1 \vee y_2}^{t} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right) dB(y_1) dB(y_2), \tag{1.18}$$

with

$$A_3(H) = \frac{1}{H+1} \left(\frac{2(2H-1)}{H} \right)^{\frac{1}{2}}, \tag{1.19}$$

to ensure that $\mathbb{E}(R^2(1)) = 1$ and K is the self-similar kernel which is defined by:

$$K^{H'}(t,x) = d(H') x^{\frac{1}{2}-H'} \int_x^t (u-x)^{H'-\frac{3}{2}} u^{H'-\frac{1}{2}} du, \quad 0 < x < t,$$

where

$$d(H') = \left(\frac{H'(2H'-1)}{\beta(2-2H', H'-\frac{1}{2})} \right)^{\frac{1}{2}},$$

and

$$H' = \frac{H+1}{2},$$

with β being the beta function, i.e.

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt.$$

We have

$$\frac{\partial K^{H'}}{\partial u}(u,x) = d(H') (\frac{x}{u})^{\frac{1}{2}-H'} (u-x)^{H'-\frac{3}{2}}, \quad 0 < x < u.$$

1.2.4 On the Rosenblatt distribution

The Rosenblatt distribution is the simplest non-Gaussian law that appears in non-central limit theorems for long-range dependent random variables.

Definition 1.2.1. *Let $(R(t))_{t\geq 0}$ be the Rosenblatt process. The Rosenblatt distribution is defined as the probability law of the random variable $R(1)$. Its characteristic function is given by*

$$\phi(\theta) = \mathbb{E}\left[e^{i\theta R(1)}\right] = \exp\left(\frac{1}{2}\sum_{k=2}^{\infty}\frac{(2i\theta\sigma(D))^k}{k}c_k\right), \tag{1.20}$$

where

$$c_k = \int_0^1 ... \int_0^1 |x_1 - x_2|^{-D}|x_2 - x_3|^{-D}...|x_k - x_{k-1}|^{-D}|x_k - x_1|^{-D}dx_1...dx_k \tag{1.21}$$

and by Cauchy-Schwartz,

$$c_k \leq \left(\int_0^1\int_0^1 |x_1 - x_2|^{-2D}dx_1dx_2\right)^{\frac{k}{2}} = \left(\frac{1}{(1-2D)(1-D)}\right)^{\frac{k}{2}} = \left(2\sigma(D)^2\right)^{\frac{k}{2}}, \tag{1.22}$$

which ensures that the series (1.20) converges around the origin.

Here, $\sigma(D)$ is a normalizing constant so that $\forall D \in]0, \frac{1}{2}[$ is given by:

$$\sigma(D) = \left[\frac{1}{2}(1-2D)(1-D)\right]^{\frac{1}{2}}, \tag{1.23}$$

and

$$H = 1 - D. \tag{1.24}$$

1.2.5 Stochastic calculus with respect to the Rosenblatt process

Since the Rosenblatt process is not semimartingale, we can't apply Itô's theory. In general there are two methods to integrate stochastically with respect to this process: the first method is the pathwise type calculus, and the second method is the Malliavin calculus and the Skorohod integration theory.

1.2.5.1 Wiener integrals

Wiener integrals with respect to the Rosenblatt process can be constructed thanks to its covariance structure. For the definition of Wiener integrals with respect to general Hermite processes, we consult Maejima and Tudor [51], and for a broader context, we consult Kruk et al. [40]. Let's review the essential ideas and apply this construction to our situation.
Note that

$$R(t) = \int_0^T \int_0^T I(\mathbb{1}_{[0,t]})(y_1, y_2) dB(y_1) dB(y_2),$$

where the operator I is defined on the set of functions $f : [0,T] \to \mathbb{R}$ and takes values in the set of functions $g : [0,T]^2 \to \mathbb{R}^2$ and it is given by

$$I(f)(y_1, y_2) = A_3(H) \int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

Definition 1.2.2. *Let us denote by $\mathcal{E}$ the set of step functions. For $f \in \mathcal{E}$, we naturally define its Wiener integral with respect to R as :*

$$\int_0^T f(u) dR(u) = \sum_{i=0}^{n-1} a_i (R_{t_{i+1}} - R_{t_i}) = \int_0^T \int_0^T I(f)(y_1 y_2) dB(y_1) dB(y_2), \tag{1.25}$$

where

$$f(t) = \sum_{i=0}^{n-1} a_i \mathbb{1}_{(t_i, t_{i+1}]}(t), \qquad t_i \in [0,T]. \tag{1.26}$$

Let us denote $\mathcal{H}$ the set of functions f such that

$$\|f\|^2_{\mathcal{H}} = 2\int_0^T \int_0^T I(f)(y_1, y_2)^2 dy_1 dy_2 < \infty. \tag{1.27}$$

It can be seen that

$$\begin{aligned}
\|f\|^2_{\mathcal{H}} &= 2d(H)^2 \int_0^T \int_0^T \left(\int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2)\, du \right)^2 dy_1 dy_2 \\
&= 2d(H)^2 \int_0^T \int_0^T \int_{y_1 \vee y_2}^T \int_{y_1 \vee y_2}^T f(u) f(v) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) \\
&\qquad \times \frac{\partial K^{H'}}{\partial v}(v, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_2)\, du\, dv\, dy_1 dy_2 \\
&= 2d(H)^2 \int_0^T \int_0^T \left(\int_0^{u \wedge v} \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial v}(v, y_1)\, dy_1 \right)^2 du\, dv \\
&= H(2H-1) \int_0^T \int_0^T f(u) f(v) |u - v|^{2H-2}\, dv\, du.
\end{aligned} \tag{1.28}$$

As demonstrated in [51] or [40], the mapping

$$f \longrightarrow \int_0^T f(u) dR(u),$$

defines an isometry from $\mathcal{E}$ to $L^2(\Omega)$. Since $\mathcal{E}$ is dense in $\mathcal{H}$, it can be extended by continuity to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ (see [69] for more details). This extended operator is called the Wiener integral of $f \in \mathcal{H}$ with respect to R.

Remark 1.2.1. *According to Pipiras and Taqqu (see [69]):*

- *The elements of $\mathcal{H}$ may be not functions but distributions; it is therefore more practical to work with subspaces of $\mathcal{H}$ that are sets of functions. Such a subspace is*

$$|\mathcal{H}| = \{f : [0,T] \to \mathbb{R} | \int_0^T \int_0^T |f(u)||f(v)|u-v|^{2H-2} du dv < \infty\}.$$

Then $|\mathcal{H}|$ is a strict subspace of $\mathcal{H}$ and we actually have the inclusions

$$L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}.$$

- *The space $|\mathcal{H}|$ (and hence $\mathcal{H}$) is not complete with respect to the norm $\|.\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm*

$$\|f\|^2_{|\mathcal{H}|} = H(2H-1) \int_0^T \int_0^T |f(u)||f(v)||u-v|^{2H-2} du dv.$$

Proposition 1.2.3. *[80] Let $g, f \in \mathcal{H}$. Then, the Wiener integrals $\int_0^T f(u) dR(u)$ and $\int_0^T g(u) dR(u)$ are not necessarily independent when the functions f and g are orthogonal in $\mathcal{H}$; they are independent if and only if*

$$\left\langle f(.) \frac{\partial K^{H'}}{\partial u}(., y_1), g(.) \frac{\partial K^{H'}}{\partial u}(., y_2) \right\rangle_{\mathcal{H}'} = 0, \quad a.e. \text{ and } (y_1, y_2) \in [0,T]^2 \tag{1.29}$$

where $\mathcal{H}'$ is the space analogous to $\mathcal{H}$ corresponding to the Hurst parameter H'.

Remark 1.2.2. *The Non Central Limit Theorem established in [26, 77] admits an extension to Wiener integrals (see [51] for further details). More precisely, if $f \in \mathcal{H}$ satisfies suitable assumptions, then the sequence*

$$\frac{1}{n^H} \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) g(\xi_j)$$

converges weakly when $n \to \infty$, to the Wiener integral $\int_0^T f(u) dR(u)$ (g and ξ_j were introduced in the begining of Chapter.1).

1.2.5.2 Pathwise stochastic calculus: Russo-Vallois regularization

We can apply pathwise calculus to construct the stochastic integrals with respect to the Rosenblatt process because this process has a regular paths and zero quadratic variation. Here we use Russo and Vallois's approach [72]. We begin by presenting the main tools of the stochastic calculus via regularization.

Definition 1.2.3. *Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two continuous processes. For every $t \geq 0$, we define*

$$I^{-}(\varepsilon, Y, dX)(t) = \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\, ds,$$

$$I^{+}(\varepsilon, Y, dX)(t) = \int_0^t Y_s \frac{X_s - X_{(s-\varepsilon)^+}}{\varepsilon}\, ds,$$

and

$$I^{0}(\varepsilon, Y, dX)(t) = \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{(s-\varepsilon)^+}}{2\varepsilon}\, ds.$$

We also introduce

$$C_\varepsilon(X, Y)(t) = \int_0^t \frac{\left(X_{s+\varepsilon} - X_{(s-\varepsilon)^+}\right)\left(Y_{s+\varepsilon} - Y_{(s-\varepsilon)^+}\right)}{\varepsilon}\, ds.$$

Then, the forward, backward and symmetric integrals of Y with respect to X are given respectively by

$$\int_0^t Y\, d^{-}X = \lim_{\varepsilon \to 0^+} I^{-}(\varepsilon, Y, dX),$$

$$\int_0^t Y\, d^{+}X = \lim_{\varepsilon \to 0^+} I^{+}(\varepsilon, Y, dX),$$

and

$$\int_0^t Y\, d^{0}X = \lim_{\varepsilon \to 0^+} I^{0}(\varepsilon, Y, dX), \tag{20}$$

provided that the above limits exist uniformly in probability (ucp). The covariation process of X and Y is defined by

$$[X,Y]_t = \underset{\varepsilon\to 0^+}{\text{ucp-lim}}\, C_\varepsilon(X,Y)(t).$$

In the particular case where $X = Y$, we simply write $[X,X] = [X]$. If $[X]$ exists, then the process X is said to have finite quadratic variation. Moreover, when $[X] = 0$, the process X is called a zero quadratic variation process.

The Rosenblatt process is a zero quadratic variation process since

$$E(C_\varepsilon(R,R)(t)) = E\left(\int_0^t \frac{1}{\varepsilon}(X_{s+\varepsilon} - X_s)^2 ds\right) = t\varepsilon^{2H-1} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Consequently, the stochastic calculus via regularization can be applied directly to the Rosenblatt process.

The following result, proved in Proposition 4.2 of [73], provides an Itô-type formula for zero quadratic variation processes.

Proposition 1.2.4. *For every $f \in \mathcal{C}^2(\mathbb{R})$, the integrals (the forward, backward and symmetric integrals)*

$$\int_0^t f^{'}(R(s))d^- R(s), \int_0^t f^{'}(R(s))d^+ R(s) \quad \textit{and} \quad \int_0^t f^{'}(R(s))d^0 R(s) \tag{1.30}$$

exist and are equal, yielding the Itô formula

$$f(R(t)) = f(R(0)) + \int_0^t f^{'}(R(s))d^0 R(s). \tag{1.31}$$

1.2.5.3 Skorohod integral with respect to the Rosenblatt process

The Skorohod integral applies to processes that are not necessarily Gaussian or semi-martingales. We briefly recall some elements of Malliavin calculus associated with a

Wiener process $(W_t)_{t\in[0,T]}$. The presentation below follows [80]; see also [56] for a complete exposition. Let $\mathcal{S}$ denote the family of smooth random variables of the form

$$F = f(W_{t_1}, \ldots, W_{t_n}),$$

where $t_1, \ldots, t_n \in [0,T]$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$.

For such a random variable F, the Malliavin derivative of F is defined by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W_{t_1}, \ldots, W_{t_n}) \mathbf{1}_{[0,t_i]}(t), \qquad t \in [0,T].$$

The operator D is an unbounded closable operator and it can be extended to the closure of $\mathcal{S}$ (denoted by $\mathbb{D}^{k,p}$, for integers $k \geq 1$ and $p \geq 2$) with respect to the norm

$$\|F\|_{k,p}^p = E|F|^p + \sum_{j=1}^k E\left\|D^{(j)}F\right\|_{L^2([0,T]^j)}^p, \qquad F \in \mathcal{S}, \quad k \geq 1, \quad p \geq 2,$$

where the derivative $D^{(j)}$ is defined by iteration.

The Skorohod integral operator δ is introduced as the adjoint of the Malliavin derivative. Its domain consists of all stochastic processes $u \in L^2([0,T] \times \Omega)$ satisfying

$$\left| E\left(\int_0^T u_s D_s F \, ds \right) \right| \leq \mathcal{C}\|F\|_2.$$

Moreover, the duality relation

$$E(F\delta(u)) = E\left(\int_0^T D_s F \, u_s \, ds \right)$$

holds for every $F \in \mathcal{S}$ and $u \in \mathrm{Dom}(\delta)$.

We also denote by $\mathbb{L}^{k,p}$ the space $L^p([0,T]; \mathbb{D}^{k,p})$. Note that $\mathbb{L}^{k,p} \subset \mathrm{Dom}(\delta)$. We denote $\delta(u) = \int_0^T u_s \, \delta W_s$.

Finally, if $F \in \mathbb{D}^{1,2}$ and $u \in \mathbb{L}^{1,2}$, the following integration by parts formula holds:

$$F\delta(u) = \delta(Fu) + \int_0^T D_s F\, u_s\, ds.$$

Definition 1.2.4. *Let's consider a square integrable stochastic process* $(g(s))_{s\in[0,T]}$*; we define its Skorohod integral with respect to* R *by:*

$$\int_0^T g(s)dR(s) = \int_0^T \int_0^T I(g)(y,z)dB(y)dB(z),$$

with: $I(g)(y,z) = \displaystyle\int_{y\vee z}^T g(u)\frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u,y)\frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u,z)du.$

The process $(g(s))_{s\in[0,T]}$ *is Skorohod integrable with respect to* R *if the process* $I(g) \in Dom(\delta^2)$*, where* δ^2 *is the double Skorohod integral with respect to the Brownian motion* $(B(t))_{t\geq 0}$*.*

We conclude this chapter by recalling two results that will play a fundamental role in the study of stochastic differential equations driven by the Rosenblatt process. First, we present a continuity result for the indefinite Skorohod integral process, which shows that the indefinite integral preserves the same order of Hölder regularity as the Rosenblatt process. We then state Grönwall's lemma, a classical tool that will be used extensively in the analysis of the corresponding SDE.

Proposition 1.2.5. *[80]. Let* $\sigma \in \mathbb{L}^{2,p}$ *such that*

$$\sup_r \|\sigma_r\|_{2,p} < \infty.$$

Then the indefinite Skorohod integral process $\left(X_t = \int_0^t \sigma_s \delta R(s),\, t \in [0,T]\right)$ *admits a Hölder continuous version of order* $\delta < H$*.*

Remark 1.2.3. *For a fixed* $0 < \delta < H - \frac{1}{2}$, *we identify the process* $(\int_0^t \sigma_s \delta R(s),\ t \in [0,T])$ *with its version having the Hölder continuous trajectories with order* δ *for* σ *satisfying the assumptions of Proposition1.2.5.*

Lemma 1.2.1. *(Gronwall's lemma)[58] Let* h *a measurable positive function bounded on* $[0,T]$. *We assume that there exist a constant* $a \geq 0$, *and a continous function* $b(t) \geq 0$ *such that for all* $t \in [0,T]$, *we have*

$$h(t) \leq a + \int_0^t b(s)h(s)ds.$$

Then we have $h(t) \leq a.exp(\int_0^t b(s)ds)$ *for all* $t \in [0,T]$.

Chapter 2

Stochastic integration of non-adapted processes with respect to the Rosenblatt process

The question of defining stochastic integrals for non-adapted (anticipating) processes has been studied extensively over the past decades. One of the first significant contributions is due to Itô, who pointed out the difficulty of giving a meaning to $\int_0^t B(1)dB(s), 0 \leq t \leq 1$, where the integrand is not adapted to the filtration $\mathcal{F}_t$. His idea was to overcome this issue by enlarging the underlying filtration, by taking $\mathcal{G}_t$ to be the σ-field generated by $\mathcal{F}_t$ and $B(1)$, which makes the integrand $B(1)$ adapted to $\mathcal{G}_t$. However, this modification changes the nature of the integrator, which is no longer a Brownian motion in the usual sense, it admits a decomposition of the form

$$B(t) = \left(B(t) - \int_0^t \frac{B(1) - B(u)}{1-u}\, du \right) + \int_0^t \frac{B(1) - B(u)}{1-u}\, du \tag{2.1}$$

which allows one to define the integral with respect to a quasimartingale (i.e., an integrable adapted process whose conditional variation over all finite partitions is bounded). In this setting, one obtains

$$\int_0^t B(1)\, dB(s) = B(1)B(t), \quad 0 \leq t \leq 1. \tag{2.2}$$

A different point of view was later proposed by Ayed and Kuo[4], who instead of modifying the filtration, kept both the integrator and the filtration unchanged and focused on the structure of the integrand. Their approach consists in decomposing the integrand into the sum of an adapted component and an instantly independent one. The stochastic integral is then introduced through a Riemann-type sum where the adapted part is evaluated at the left endpoints of partitions, while the instantly independent part is evaluated at the right endpoints.
The aim of this chapter is to extend this methodology to the case where the integrator is the Rosenblatt process. In contrast with the Gaussian framework, the Rosenblatt process is a non-Gaussian self-similar process with long-range dependence, which makes the extension of these ideas more delicate. However, when the Hurst parameter satisfies $H > \frac{1}{2}$, its trajectories are regular enough to allow for a pathwise definition of the integral. Building on the above ideas, we develop a notion of stochastic integration for non-adapted processes with respect to the Rosenblatt process. Our approach is based on a suitable decomposition of the integrand, inspired by the method introduced by Ayed and Kuo [4]. We start by recalling the main ideas of this approach and mentioning some related results from the literature, and then we extend this framework to the Rosenblatt process in order to define the corresponding integral.

2.1 Stochastic integral for non adapted process: the Ayed-Kuo method

In this section, we present the main idea of the approach proposed by Ayed and Kuo for defining stochastic integrals of non-adapted processes. Their method is based on a simple but fundamental observation: many anticipating integrands can be decomposed into components with distinct probabilistic behaviors.

To illustrate this idea, we consider the anticipating integral

$$\int_0^t B(1)\, dB(s), \quad 0 \le t \le 1.$$

Using the decomposition

$$B(1) = \big(B(1) - B(s)\big) + B(s)$$

we write

$$\int_0^t B(1)\, dB(s) = \int_0^t B(s)\, dB(s) + \int_0^t (B(1) - B(s))\, dB(s). \tag{2.3}$$

The first term is now a classical Itô integral, since $B(s)$ is adapted. Hence,

$$\int_0^t B(s)\, dB(s) = \frac{1}{2}\left(B(t)^2 - t\right). \tag{2.4}$$

The second term has an anticipative nature. Still, it comes with a useful structure: for each fixed s, the increment $B(1)-B(s)$ is independent of the σ-algebra $\mathcal{F}_s$ That's the key point. Rather than trying to squeeze this term into the Itô framework, it's more natural to define it using Riemann sums where the integrand is taken at the right endpoint:

$$\int_0^t (B(1) - B(s))\, dB(s) = \lim_{\|\Delta\|\to 0} \sum_{i=1}^n (B(1) - B(s_i))(B(s_i) - B(s_{i-1})),$$

where $\Delta = \{s_0, s_1, s_2, \ldots, s_n\}$ is a partition of the interval $[0, t]$ with $s_0 = 0$ and $s_n = t$. We take the right endpoint s_i of the subinterval $[s_{i-1}, s_i]$ as the point where the integrand is evaluated. Then we establish the limit.

$$\sum_{i=1}^{n}(B(1)-B(s_i))(B(s_i)-B(s_{i-1})) = B(1)\sum_{i=1}^{n}(B(s_i)-B(s_{i-1})) - \sum_{i=1}^{n}B(s_i)(B(s_i)-B(s_{i-1}))$$

$$= B(1)B(t) - \left\{\sum_{i=1}^{n}(B(s_i)-B(s_{i-1}))^2 \right.$$

$$\left. + \sum_{i=1}^{n}B(s_{i-1})(B(s_i)-B(s_{i-1}))\right\}$$

$$\to B(1)B(t) - \left\{t + \frac{1}{2}(B(t)^2 - t)\right\}.$$

Consequently, we obtain the stochastic integral.

$$\int_0^t (B(1)-B(s))\,dB(s) = B(1)B(t) - \frac{1}{2}(B(t)^2+t), \quad 0 \le t \le 1. \tag{2.5}$$

From the equations (2.4) and (2.5), we deduce that

$$\int_0^t B(1)\,dB(s) = B(1)B(t) - t, \quad 0 \le t \le 1, \tag{2.6}$$

which is not the same as the right-hand side of the equation (2.2) provided by Itô. In particular, the integral defined by this construction has expectation zero; However, the one provided by Itô has a non-zero expectation.

This example highlights the main difficulty with anticipating integrals: the integrand may depend on future values of the driving process, so the usual Itô definition no longer applies. Rather than modifying the probabilistic framework, the Ayed-Kuo approach looks more closely at the structure of such integrands.

The key observation is that many anticipating processes can be decomposed into a sum of the products of two parts:

- The first part consists of adapted stochastic processes, depending only on past and present information,

- and the second part consists of instantly independent stochastic processes.

This leads to the following definition:

Definition 2.1.1. *(Ayed and Kuo [4], Definition 2.1) A stochastic process* $\{g(t), 0 \leq t \leq T\}$ *is said to be instantly independent with respect to a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if* $g(t)$ *and* $\mathcal{F}_t$ *are independent for each* t.

Lemma 2.1.1. *(Ayed and Kuo [4]) If a stochastic process* $\{g(t), 0 \leq t \leq T\}$ *is both adapted and instantly independent with respect to a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *, then* $g(t)$ *is a deterministic function.*

In this context, instantly independent stochastic processes can be regarded as the counterpart of adapted stochastic processes in the setting of the Itô integral.
The natural question is therefore how to define a stochastic integral involving the product of an adapted process and an instantly independent process. While the Itô integral relies on left-endpoint evaluations of the integrand, applying the same discretization to the instantly independent component fails to preserve its key properties. On the other hand, using right-endpoint evaluations for this part retains the desired structure. This observation motivates the definition introduced by Ayed and Kuo.

Definition 2.1.2. *(Ayed and Kuo [4], Definition 2.2) Let* $B(t)$ *be a Brownian motion, for an adapted stochastic process* $f(t)$ *with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *and an instantly independent stochastic process* $g(t)$ *with respect to the same filtration; we define the stochastic integral of* $f(t)g(t)$ *to be the limit :*

$$\int_0^T f(t)g(t)dB(t) = \lim_{||\Delta_n||\to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(B(t_i) - B(t_{i-1})) \tag{2.7}$$

provided that the limit exists in probability , where $\Delta_n = \{0 = t_0 < t_1 < ... < t_n = T\}$ *is the partition of interval* $[0, T]$.

To illustrate this approach, we now present an example. For additional examples, see Ayed et al. [4].

Example 2.1.1. *We consider the stochastic integral:* $\int_0^1 e^{B(1)}\, dB(t) \quad for \quad 0 \le t \le 1.$ *We decompose the integrand as*

$$e^{B(1)} = e^{B(s)} e^{B(1)-B(s)},$$

where the first factor is adapted and the second is instantly independent of the filtration. Let $\Delta_n = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = t\}$ *be a partition of the interval* $[0, t]$*. By definition:*

$$\begin{aligned}
\int_0^t e^{B(s)} e^{B(1)-B(s)}\, dB(s) &= \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} e^{B(s_{i-1})} e^{B(1)-B(s_i)} \big(B(s_i) - B(s_{i-1})\big) \\
&= e^{B(1)} \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} e^{-(B(s_i)-B(s_{i-1}))} \big(B(s_i) - B(s_{i-1})\big) \\
&= e^{B(1)} \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} \Big[1 - (B(s_i) - B(s_{i-1})) + \tfrac{1}{2}(B(s_i) - B(s_{i-1}))^2 \\
&\qquad + \; o\big((B(s_i) - B(s_{i-1}))^2\big)\Big] \big(B(s_i) - B(s_{i-1})\big) \\
&\to e^{B(1)}(B(t) - t), \quad \textit{in probability.}
\end{aligned}$$

Therefore, the stochastic integral is given by

$$\int_0^t e^{B(1)}\, dB(s) = e^{B(1)} \big(B(t) - t\big), \quad 0 \le t \le 1.$$

Moreover, we observe that

$$\int_0^t e^{B(1)}\, dB(s) = e^{B(1)} \big(B(t) - 1\big), \quad t > 1.$$

The above framework, has been successfully applied to various Gaussian processes. We recall in the next section some of the main results obtained in this setting, which will serve as a foundation for our extension.

2.2 Previous work on anticipating stochastic integrals for fractional Gaussian processes

In this section, we briefly review some known results on the stochastic integration of non-adapted processes in the Gaussian framework. In particular, we focus on constructions based on the approach introduced by Ayed et al., which allows one to define integrals of products of adapted and instantly independent processes. This methodology has been successfully applied to several Gaussian processes, including fractional Brownian motion [32] and its extensions such as sub-fractional [7] and mixed fractional Brownian motions [8]. We recall two definitions and one fundamental theorem illustrating how this framework can be adapted to these processes. These results highlight the main features of anticipating stochastic integration in the Gaussian setting and will serve as a basis for the extension developed later to the Rosenblatt process.
We begin by recalling the following definition in the setting of sub-fractional Brownian motion.

Definition 2.2.1. *[7] Let S^H, $H > \frac{1}{2}$, be a sub-fractional Brownian motion and let $\mathcal{F}_t$ be the σ-field generated by $\{S^H(t),\, t \geq 0\}$. For an adapted stochastic process $f(t)$ with respect to the filtration $\mathcal{F}_t$ and an instantly independent stochastic process $g(t)$ with respect to the same filtration, we define the stochastic integral of $f(t)g(t)$ to be the limit:*

$$\int_0^T f(t)g(t)\, dS^H(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})\, g(t_i) \left(S^H(t_i) - S^H(t_{i-1})\right), \tag{2.12}$$

provided that the limit in probability exists.

The following result provides a similar construction in the case of fractional Brownian motion.

Theorem 2.2.1. *[32] Let $\Delta = \{0 = t_0, t_1, t_2, \cdots, t_n = T\}$ be a partition of the interval $[0, T]$. On the subinterval $[t_{i-1}, t_i]$, we take the right endpoint t_i as the evaluation point* *for the integrand. For an adapted stochastic process $f(t)$ and an instantly independent stochastic process $g(t)$, we define the stochastic integral of $f(t)g(t)$ with respect to FBM to be the limit*

$$\int_0^T f(t)g(t)\,dB^H(t) = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \psi_H^{(1)}\big(f(t_{i-1})\big)\,\psi_H^{(2)}\big(g(t_i)\big)\,\big(B(t_i) - B(t_{i-1})\big).$$

We also recall the corresponding definition in the mixed fractional setting.

Definition 2.2.2. *[8] Let $M^H(t)$, $H > \frac{1}{2}$, be a mixed fractional Brownian motion and let $\mathcal{F}_t$ denote the σ-field generated by $\{M^H(t),\, t \geq 0\}$. For an adapted stochastic process $f(t)$ with respect to the filtration $\mathcal{F}_t$, and an instantly independent stochastic process $g(t)$ with respect to the same filtration, we define the stochastic integral of $f(t)g(t)$ as:*

$$\int_0^T f(t)g(t)\,dM^H(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})\,g(t_i)\,\big(M^H(t_i) - M^H(t_{i-1})\big), \tag{2.17}$$

provided that the convergence in probability exists.

The proofs and further properties of these constructions can be found in the cited references and will not be reproduced here. The results presented above illustrate how the Ayed–Kuo approach can be effectively applied in the Gaussian framework to define stochastic integrals of non-adapted processes. They provide a clear methodology based on the decomposition of the integrand and the use of suitable Riemann-type sums. However, these constructions rely strongly on the Gaussian nature of the underlying processes and their specific structural properties. Extending this approach to non-Gaussian processes is therefore not straightforward and requires additional considerations. In particular, the case of the Rosenblatt process presents new challenges due to its non-Gaussian behavior. The next section is devoted to the development of a stochastic integration framework in this setting.

2.3 New Anticipating Integral with respect to the Rosenblatt process

Motivated by the construction introduced in the previous section, we now consider the extension of the stochastic integration framework to the Rosenblatt process[1]. In this setting, we focus directly on the case of products of the form $f(t)g(t)$, where f is adapted and g is instantly independent. We now state the following proposition, which defines the corresponding stochastic integral, and we provide its proof below.

Proposition 2.3.1. *[10] Let $R(t)$, $t \geq 0$ be a Rosenblatt process and $\mathcal{F}_t$ the σ-field generated by $\{R(t),\, t \geq 0\}$ for an adapted stochastic process $f(t)$ with respect to the filtration $\mathcal{F}_t$ and an instantly independent stochastic process $g(t)$ with respect to the same filtration. We define the stochastic integral of $f(t)g(t)$ to be the limit:*

$$\int_0^T f(t)g(t)dR(t)$$
$$= \begin{cases} \lim\limits_{(\Delta_y,\Delta_z\to 0)} \sum\limits_{i=1}^{n}\sum\limits_{j=i+1}^{m} \phi_1^{H'}(f)(y_{j-1},z_{i-1})\phi_2^{H'}(g)(y_j)(B(y_j)-B(y_{j-1}))(B(z_i)-B(z_{i-1})) & \text{if } z<y, \\ \lim\limits_{(\Delta_y,\Delta_z\to 0)} \sum\limits_{i=1}^{n}\sum\limits_{j=i+1}^{m} \phi_3^{H'}(f)(z_{j-1},y_{i-1})\phi_4^{H'}(g)(z_j)(B(z_j)-B(z_{j-1}))(B(y_i)-B(y_{i-1})) & \text{if } z>y, \end{cases}$$

provided that the limit exists in probability.

Proof. We first recall that, for the computation of $\int_0^1 w(x)v(x)dx$ with $v(x) = \int_0^x v^{'}(y)dy$, both integration by parts and Fubini's theorem lead to the following identity:

$$\int_0^1 wvdx = \int v^{'}(x)\int_x^1 w(y)dydx.$$

[1]This result is the subject of a publication in Random Operators and Stochastic Equations, vol. 33, no. 3, pp. 271–281, 2025.

$$\int_0^T f(t)g(t)dR(t)$$

$$= \int_0^T \int_0^T I(f.g)(y,z)dB(y)dB(z)$$

$$= \int_0^T \int_0^T \left(\int_{y\vee z}^T (f.g)(u)\frac{\partial K^{H'}}{\partial u}(u,y)\frac{\partial K^{H'}}{\partial u}(u,z)du \right) dB(y)dB(z)$$

$$= \int_0^T \int_0^T \left(\int_y^T (f.g)(u)\frac{\partial K^{H'}}{\partial u}(u,y)\frac{\partial K^{H'}}{\partial u}(u,z)du\mathbb{1}_{z<y} \right.$$

$$+ \left. \int_z^T (f.g)(u)\frac{\partial K^{H'}}{\partial u}(u,y)\frac{\partial K^{H'}}{\partial u}(u,z)du\mathbb{1}_{y<z} \right) dB(y)dB(z)$$

$$= d(H')\int_0^T \int_0^T \left[\int_y^T (f.g)(u)(\frac{u}{y})^{H'-\frac{1}{2}}(u-y)^{H'-\frac{3}{2}}(\frac{u}{z})^{H'-\frac{1}{2}}(u-z)^{H'-\frac{3}{2}}du\mathbb{1}_{z<y} \right.$$

$$+ \left. \int_z^T (f.g)(u)(\frac{u}{y})^{H'-\frac{1}{2}}(u-y)^{H'-\frac{3}{2}}(\frac{u}{z})^{H'-\frac{1}{2}}(u-z)^{H'-\frac{3}{2}}du\mathbb{1}_{y<z} \right] dB(y)dB(z)$$

$$= d(H')\int_0^T \int_0^T \left[y^{\frac{1}{2}-H'}z^{\frac{1}{2}-H'}\int_y^T (f.g)(u)(u)^{H'-\frac{1}{2}}(u-y)^{H'-\frac{3}{2}}(u)^{H'-\frac{1}{2}}(u-z)^{H'-\frac{3}{2}}du\mathbb{1}_{z<y} \right.$$

$$+ \left. y^{\frac{1}{2}-H'}z^{\frac{1}{2}-H'}\int_z^T (f.g)(u)(u)^{H'-\frac{1}{2}}(u-y)^{H'-\frac{3}{2}}(u)^{H'-\frac{1}{2}}(u-z)^{H'-\frac{3}{2}}du\mathbb{1}_{y<z} \right] dB(y)dB(z)$$

$$= d(H')\int_0^T \int_0^T y^{\frac{1}{2}-H'}z^{\frac{1}{2}-H'}\left[\int_y^T f(u)g(u)(u)^{2H'-1}(u-y)^{H'-\frac{3}{2}}(u-z)^{H'-\frac{3}{2}}du\mathbb{1}_{z<y} \right.$$

$$+ \left. \int_z^T f(u)g(u)(u)^{2H'-1}(u-y)^{H'-\frac{3}{2}}(u-z)^{H'-\frac{3}{2}}du\mathbb{1}_{y<z} \right] dB(y)dB(z)$$

$$= d(H')\int_0^T \int_0^T y^{\frac{1}{2}-H'}z^{\frac{1}{2}-H'}\left[\int_y^T \left(g(u)(u)^{2H'-1} \right)' \int_u^T f(s)(s-y)^{H'-\frac{3}{2}}(s-z)^{H'-\frac{3}{2}}dsdu\mathbb{1}_{z<y} \right.$$

$$+ \left. \int_z^T \left(g(u)(u)^{2H'-1} \right)' \int_u^T f(s)(s-y)^{H'-\frac{3}{2}}(s-z)^{H'-\frac{3}{2}}dsdu\mathbb{1}_{y<z} \right] dB(y)dB(z)$$

$$
\begin{aligned}
= & \; d(H^{'})\int_0^T\int_0^T y^{\frac{1}{2}-H^{'}}z^{\frac{1}{2}-H^{'}}\Bigg(\Big[\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})\int_y^T\Big(g(u)(u)^{2H^{'}-1}\Big)^{'}\Big[-\frac{1}{\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})}\times \\
& \int_y^u f(s)(s-y)^{H^{'}-\frac{3}{2}}(s-z)^{H^{'}-\frac{3}{2}}ds+\frac{1}{\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})}\int_y^T f(s)(s-y)^{H^{'}-\frac{3}{2}}(s-z)^{H^{'}-\frac{3}{2}}ds\Big]du\mathbb{1}_{z<y}\Big] \\
+ & \Big[\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})\int_z^T\Big(g(u)(u)^{2H^{'}-1}\Big)^{'}\Big[-\frac{1}{\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})}\int_z^u f(s)(s-y)^{H^{'}-\frac{3}{2}} \\
& (s-z)^{H^{'}-\frac{3}{2}}ds+\frac{1}{\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})}\int_z^T f(s)(s-y)^{H^{'}-\frac{3}{2}}(s-z)^{H^{'}-\frac{3}{2}}ds\Big]du\mathbb{1}_{y<z}\Big]\Bigg)dB(y)dB(z)
\end{aligned}
$$

$$
\begin{aligned}
= & \; d(H^{'})\int_0^T\int_0^T y^{\frac{1}{2}-H^{'}}z^{\frac{1}{2}-H^{'}}\Bigg(\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})\int_y^T\Big(g(u)(u)^{2H^{'}-1}\Big)^{'}\Big[-(I_{u^-,tr}^{H^{'}-\frac{1}{2},H^{'}-\frac{1}{2}}f)(y,z) \\
+ & (I_{T^-,tr}^{H^{'}-\frac{1}{2},H^{'}-\frac{1}{2}}f)(y,z)\Big]du\mathbb{1}_{z<y}+\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})\int_z^T\Big(g(u)(u)^{2H^{'}-1}\Big)^{'}\times \\
& \Big[-(I_{u^-,tr}^{H^{'}-\frac{1}{2},H^{'}-\frac{1}{2}}f)(y,z)+(I_{T^-,tr}^{H^{'}-\frac{1}{2},H^{'}-\frac{1}{2}}f)(y,z)\Big]du\mathbb{1}_{y<z}\Bigg)dB(y)dB(z).
\end{aligned}
$$

Let

$$
\begin{aligned}
J = & \; \Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})\int_y^T\Big(g(u)(u)^{2H^{'}-1}\Big)^{'}\Big[-(I_{u^-,tr}^{H^{'}-\frac{1}{2},H^{'}-\frac{1}{2}}f)(y,z)+(I_{T^-,tr}^{H^{'}-\frac{1}{2},H^{'}-\frac{1}{2}}f)(y,z)\Big]du \\
= & \int_y^T f(u)g(u)(u)^{2H^{'}-1}(u-y)^{H^{'}-\frac{3}{2}}(u-z)^{H^{'}-\frac{3}{2}}du
\end{aligned}
$$

then,

$$
\begin{aligned}
J = & -\Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})\Big[g(y)y^{H^{'}-\frac{1}{2}}z^{H^{'}-\frac{1}{2}}(I_{T^-,tr}^{H^{'}-\frac{1}{2},H^{'}-\frac{1}{2}}f)(y,z)\Big] \\
- & \Gamma(H^{'}-\frac{1}{2})\Gamma(H^{'}-\frac{1}{2})\int_y^T g(u)u^{2H^{'}-1}f(u)(u-y)^{H^{'}-\frac{3}{2}}(u-z)^{H^{'}-\frac{3}{2}}du \\
= & \int_y^T f(u)g(u)(u)^{2H^{'}-1}(u-y)^{H^{'}-\frac{3}{2}}(u-z)^{H^{'}-\frac{3}{2}}du.
\end{aligned}
$$

It means that: $J=\frac{-\Gamma^2(H^{'}-\frac{1}{2})}{1+\Gamma^2(H^{'}-\frac{1}{2})}\Big[g(y)y^{H^{'}-\frac{1}{2}}z^{H^{'}-\frac{1}{2}}(I_{T^-,tr}^{H^{'}-\frac{1}{2},H^{'}-\frac{1}{2}}f)(y,z)\Big]$.

Let

$$\begin{aligned} J' &= \Gamma(H'-\frac{1}{2})\Gamma(H'-\frac{1}{2})\int_z^T \Big(g(u)(u)^{2H'-1}\Big)'\Big[-(I_{u^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(y,z)+(I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(y,z)\Big]du \\ &= \int_z^T f(u)g(u)(u)^{2H'-1}(u-y)^{H'-\frac{3}{2}}(u-z)^{H'-\frac{3}{2}}du \end{aligned}$$

then,

$$\begin{aligned} J' &= -\Gamma(H'-\frac{1}{2})\Gamma(H'-\frac{1}{2})\Big[g(z)y^{H'-\frac{1}{2}}z^{H'-\frac{1}{2}}(I_{T^-,tr}^{2H'-1}f)(z,y)\Big] \\ &- \Gamma(H'-\frac{1}{2})\Gamma(H'-\frac{1}{2})\int_z^T g(u)u^{2H'-1}f(u)(u-y)^{H'-\frac{3}{2}}(u-z)^{H'-\frac{3}{2}}du \\ &= \int_z^T f(u)g(u)(u)^{2H'-1}(u-y)^{H'-\frac{3}{2}}(u-z)^{H'-\frac{3}{2}}du. \end{aligned}$$

Which gives that: $J' = \frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}\Big[g(z)y^{H'-\frac{1}{2}}z^{H'-\frac{1}{2}}(I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(z,y)\Big]$. Then,

$$\begin{aligned} &\int_0^T f(t)g(t)dR(t) \\ &= d(H')\frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}\times\int_0^T\int_0^T\Big[g(y)(I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(y,z)\mathbb{1}_{z<y} \\ &+ g(z)(I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(y,z)\mathbb{1}_{y<z}\Big]dB(y)dB(z) \\ &= d(H')\frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}\Big(\int_0^T\int_0^T g(y)(I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(y,z)\mathbb{1}_{z<y}dB(y)dB(z) \\ &+ \int_0^T\int_0^T g(z)(I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(y,z)\mathbb{1}_{y<z}dB(y)dB(z)\Big) \end{aligned}$$

The first case : $z < y$

$$\int_0^T f(t)g(t)dR(t)$$

$$= d(H')\frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}\int_0^T\int_z^T g(y)(I_{T^-}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(y,z)dB(y)dB(z)$$

$$= d(H')\frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}\lim_{(\Delta_y,\Delta_z)\to 0}\sum_{i=1}^{n}\sum_{j=i+1}^{m}(I_{T^-,tr}^{H'-\frac{1}{2}}f)(y_{j-1},z_{i-1})g(y_j)(B(y_j)-B(y_{j-1}))(B(z_i)-B(z_{i-1}))$$

$$= \lim_{(\Delta_y,\Delta_z)\to 0}\sum_{i=1}^{n}\sum_{j=i+1}^{m}(I_T^{H'-\frac{1}{2}}f)(y_{j-1},z_{i-1})d(H')\frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}g(y_j)(B(y_j)-B(y_{j-1}))(B(z_i)-B(z_{i-1}))$$

$$= \lim_{(\Delta_y,\Delta_z\to 0)}\sum_{i=1}^{n}\sum_{j=i+1}^{m}\phi_1^{H'}(f)(y_{j-1},z_{i-1})\phi_2^{H'}(g)(y_j)(B(y_j)-B(y_{j-1}))(B(z_i)-B(z_{i-1}))$$

where,

$$\phi_1^{H'}(f)(y_{j-1},z_{i-1}) = (I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(y_{j-1},z_{i-1})$$

and

$$\phi_2^{H'}(g)(y_j) = d(H')\tfrac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}g(y_j).$$

The second case : $z > y$

$$
\begin{aligned}
&\int_0^T f(t)g(t)dR(t)\\
=&\, d(H')\frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}\int_0^T\int_y^T g(z)(I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(z,y)dB(z)dB(y)\\
=&\, d(H')\frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}\lim_{(\Delta_y,\Delta_z\to 0)}\sum_{i=1}^{n}\sum_{j=i+1}^{m}(I_T^{H'-\frac{1}{2}}f)(z_{j-1},y_{i-1})g(z_j)(B(z_j)-B(z_{j-1}))(B(y_i)-B(y_{i-1}))\\
=&\lim_{(\Delta_y,\Delta_z\to 0)}\sum_{i=1}^{n}\sum_{j=i+1}^{m}(I_T^{H'-\frac{1}{2}}f)(z_{j-1},y_{i-1})d(H')\frac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}g(z_j)(B(z_j)-B(z_{j-1}))(B(y_i)-B(y_{i-1}))\\
=&\lim_{(\Delta_y,\Delta_z\to 0)}\sum_{i=1}^{n}\sum_{j=i+1}^{m}\phi_3^{H'}(f)(z_{j-1},y_{i-1})\phi_4^{H'}(g)(z_j)(B(z_j)-B(z_{j-1}))(B(y_i)-B(y_{i-1}))
\end{aligned}
$$

where,

$$\phi_3^{H'}(f)(z_{j-1},y_{i-1}) = (I_{T^-,tr}^{H'-\frac{1}{2},H'-\frac{1}{2}}f)(z_{j-1},y_{i-1})$$

and

$$\phi_4^{H'}(g)(z_j) = d(H')\tfrac{-\Gamma^2(H'-\frac{1}{2})}{1+\Gamma^2(H'-\frac{1}{2})}g(z_j).$$

□

Chapter 3

Stochastic differential equation driven by the Rosenblatt process

The aim of this chapter is to study the following stochastic differential equation (SDE) [1]:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s)dR_s^H, \tag{3.1}$$

where $\{R_s^H,\, t \geq 0\}$ is a Rosenblatt process with $(\frac{1}{2} < H < 1)$. In the above stochastic differential equation, the integral $\int_0^t \sigma(s)dR_s$ should be interpreted as a Skorohod integral. Our main result is a general theorem about the existence and uniqueness of solution for the stochastic differential equation (3.1) under suitable conditions on the coefficients. The Gronwall inequality lemma is also used to provide the continuous dependency of solutions on the initial value. Euler-Maruyama numerical scheme is considered to obtain approximate solutions to (3.1). This chapter will be closed with Section 3.4 as an application in which an example of linear stochastic differential equation is solved. Moreover comparison between the exact solution and the approximation solution of this example is illustrated numerically, and we also solved a nonlinear SDE numerically. We use MATLAB for our simulations and support our results with graphs.

[1]This chapter is the subject of a publication in Acta Universitatis Sapientiae, Mathematica, vol. 18, no 1, p. 9, 2026.

3.1 Existence and uniqueness of the solution

In this section, we examine the existence and uniqueness of a solution for equation (3.1). To achieve this goal, we introduce a specific assumptions concerning the coefficients:
(**H1**): For any $t \in [0, T]$ and $x, y \in \mathbb{R}$
$|b(t, x)| \leq k(t)(1 + |x|)$,
(**H2**): $|b(t, x) - b(t, y)| \leq k(t)|x - y|$ with $k \in \mathbb{L}^1([0, T]; \mathbb{R})$.
Let $\mathcal{S}^2([0, T])$ denote the space of all adapted stochastic processes $(X_t)_{t \in [0,T]}$ with continuous paths such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^2\right] < \infty.$$

Equipped with the norm

$$\|X\|_{\mathcal{S}^2} := \left(\mathbb{E}\left[\sup_{t \in [0,T]} |X_t|^2\right]\right)^{1/2},$$

the space $\mathcal{S}^2([0, T])$ is a Banach space.

Theorem 3.1.1. *[11] Assume that b satisfies the assumption (**H1**) and (**H2**). If $\sigma(.) \in \mathbb{L}^2([0, T]; \mathbb{R})$, then equation* (3.1) *has a solution X. Furthermore, if for all $t \in [0, T]$, $b(t, .)$ is a non-increasing function, there exists a unique solution for* (3.1).

Proof. Let us begin by establishing the existence .
Existence:
We employ the iterative Picard method: we define $X_t^0 = x_0$ for all $t \in [0, T]$ and for all $n \in \mathbb{N}$ and $t \in [0, T]$:

$$X_t^{n+1} = x_0 + \int_0^t \sigma(s) dR_s^H + \int_0^t b(s, X_s^n) ds, \tag{3.2}$$

which is clearly defined and for all $t \in [0,T]$:

$$|X_t^{n+1}| \leq |x_0| + |\int_0^t \sigma(s) dR_s^H| + \sup_{r\in[0,T]} |X_r^n| \int_0^t k(s)ds + \int_0^t k(s)ds.$$

By Proposition 1.2.5; we have:

$$|X_t^{n+1}| \leq |x_0| + CT^H + \sup_{r\in[0,T]} |X_r^n| \int_0^t k(s)ds + \int_0^t k(s)ds,$$

where C is a constant that depends only on σ.
In contrast, we have the following, for all $n \in \mathbb{N}$ and $t \in [0,T]$:

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^{n+1} - X_t^n|\Big] &= \mathbb{E}\Big[\sup_{t\in[0,T]} |\int_0^t \big(b(s,X_s^n) - b(s,X_s^{n-1})\big)ds|\Big] \\
&\leq \mathbb{E}\Big[\sup_{t\in[0,T]} \int_0^t k(s)|X_s^n - X_s^{n-1}|ds\Big] \\
&\leq \frac{(\int_0^T k(s)ds)^n}{n!} \mathbb{E}\Big[\sup_{r\in[0,T]} |X_r^1 - X_r^0|\Big].
\end{aligned}$$

We deduce that the sequence $X_t^n (t \in [0,T])$ is a Cauchy sequence in the Banach space $\mathcal{S}^2([0,T])$. Hence it converges in $\mathcal{S}^2([0,T])$ to a process X, which is taken as the solution of the SDE (3.1).

Now we prove the uniqueness.

The uniqueness: We will divide the uniqueness proof into two cases.

The First case: if X is bounded, the proof proceeds as follows:

Let X and X' be two solutions of the equation (3.1), defined on the same probability space, adapted to the same filtration and with the same initial value $X_0 = X_0'$.

We have for all $t \geq 0$:

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s) dR_s^H,$$

and

$$X_t^{'} = X_0^{'} + \int_0^t b(s, X_s^{'})ds + \int_0^t \sigma(s)dR_s^H.$$

Let us note:

$$\begin{aligned}
h(t) &:= \mathbb{E}\left[\left(X_t - X_t^{'}\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^t (b(s, X_s) - b(s, X_s^{'}))ds\right)^2\right] \\
&\leq \mathbb{E}\left[\int_0^t 1^2 ds \int_0^t \left(b(s, X_s) - b(s, X_s^{'})\right)^2 ds\right] \\
&\leq \mathbb{E}\left[\int_0^T ds \int_0^t \left(b(s, X_s) - b(s, X_s^{'})\right)^2 ds\right] \\
&\leq T\mathbb{E}\left[\left(\int_0^t \left(b(s, X_s) - b(s, X_s^{'})\right)^2 ds\right)\right] \qquad \left(Cauchy \quad Schwarz \quad inequality\right) \\
&\leq T\mathbb{E}\left[\int_0^t K^2\left(X_s - X_s^{'}\right)^2 ds\right] \qquad \left(Lipschitz \quad condition\right) \\
&\leq TK^2\mathbb{E}\left[\int_0^t \left(X_s - X_s^{'}\right)^2 ds\right] \\
&\leq C\mathbb{E}\left[\int_0^t \left(X_s - X_s^{'}\right)^2 ds\right].
\end{aligned}$$

If we put $C = TK^2$; Thus we have shown that h satisfies the following condition for every $t \in [0, T]$:

$$h(t) \leq C \int_0^t h(s)ds.$$

The Gronwall lemma applies with $a = 0$ and $b = C$ (this shows that h satisfies the conditions of Grönwall's lemma), we obtain $h = 0$ which implies $X_t = X_t^{'}$ a.s.

The Second case: if we have X or $X^{'}$ one of them is not bounded, the proof will be as follows:

We consider two solutions X and $X^{'}$ of the equation (3.1), defined on the same probability space, adapted to the same filtration and with the same initial value $X_0 = X_0^{'}$.

For $M > 0$ fixed, we consider stopping time

$$\tau = \inf\left\{t \geq 0,\ |X_t| \geq M \quad or \quad \left|X_t^{'}\right| \geq M\right\}.$$

We have for all $t \geq 0$:

$$X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} b(s, X_s)ds + \int_0^{t\wedge\tau} \sigma(s)dR_s^H.$$

Since $X^{'}$ is also a solution, we have the analogous equation:

$$X_{t\wedge\tau}^{'} = X_0^{'} + \int_0^{t\wedge\tau} b(s, X_s^{'})ds + \int_0^{t\wedge\tau} \sigma(s)dR_s^H.$$

Let us note:

$$
\begin{aligned}
h(t) &:= \mathbb{E}\left[\left(X_{t\wedge\tau} - X'_{t\wedge\tau}\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^{t\wedge\tau} (b(s,X_s) - b(s,X'_s))ds\right)^2\right] \\
&\leq \mathbb{E}\left[\int_0^{t\wedge\tau} 1^2 ds \int_0^{t\wedge\tau}\left(b(s,X_s) - b(s,X'_s)\right)^2 ds\right] \\
&\leq \mathbb{E}\left[\int_0^{T} ds \int_0^{t\wedge\tau}\left(b(s,X_s) - b(s,X'_s)\right)^2 ds\right] \\
&\leq T\mathbb{E}\left[\left(\int_0^{t\wedge\tau}\left(b(s,X_s) - b(s,X'_s)\right)^2 ds\right)\right] \qquad \left(Cauchy\quad Schwarz\quad inequality\right) \\
&\leq T\mathbb{E}\left[\int_0^{t\wedge\tau} K^2\left(X_s - X'_s\right)^2 ds\right] \qquad \left(Lipschitz\quad condition\right) \\
&\leq TK^2\mathbb{E}\left[\int_0^{t\wedge\tau}\left(X_s - X'_s\right)^2 ds\right] \\
&\leq C\mathbb{E}\left[\int_0^{t\wedge\tau}\left(X_s - X'_s\right)^2 ds\right].
\end{aligned}
$$

If we set $C = TK^2$; then we have established that h verifies for every $t \in [0,T]$:

$$h(t) \leq C\int_0^t h(s)ds.$$

Moreover, by definition of τ, the function h is bounded by $4M^2$. The Grönwall lemma applies with $a = 0$ and $b = C$ (this shows that h satisfies the conditions of Grönwall's lemma), we obtain $h = 0$ which implies $X_{t\wedge\tau} = X'_{t\wedge\tau}$ a.s. Finally, by making M tends to infinity ($M \to +\infty$) we have ($\tau \to +\infty$) and therefore $X_t = X'_t$ a.s. $\square$

3.2 Stability of solutions

Non-Gaussian processes bring additional difficulties to stability analysis due to features such as heavy tails and long-range dependence. Studying stability in this setting is important to ensure that stochastic models remain reliable under more realistic sources of uncertainty. In this work, we consider the Rosenblatt process as a representative non-Gaussian noise. Although it is non-Gaussian and exhibits long memory, it has continuous paths, which makes it more tractable than processes with jumps. Moreover, it can be handled using well-established tools such as Malliavin calculus and Wiener–Itô multiple integrals. For these reasons, the Rosenblatt process provides a convenient framework for investigating the stability of stochastic differential equations driven by non-Gaussian noise. It allows us to capture essential effects such as memory and nonlinear dependence while avoiding the additional technical difficulties caused by discontinuities. Hence, it serves as a suitable model for deriving and analyzing stability results in a non-Gaussian context.

Definition 3.2.1. *A solution X_t^x of SDE (3.1) with initial value $X_0 = x$ is said to be stable in mean if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all y with $|x - y| < \delta$, we have*

$$\mathbb{E}(|X_t^x - X_t^y|) \leq \varepsilon.$$

where X_t^y is another solution of SDE (3.1) with initial value $X_0^y = y$.

Remark 3.2.1. *If $k \in \mathbb{L}^1([0,T];\mathbb{R})$, $\forall t > 0$, so the unique solution of (3.1) can be extended to $[0,+\infty)$. In particuler, if $k(t) \in \mathbb{L}^1([0,+\infty);\mathbb{R})$ we have the following result.*

Theorem 3.2.1. *[11] Let X_t^x and X_t^y be solutions of SDE (3.1) with initial value x and y, respectively. Assume the assumptions of Theorem 3.1.1 are satisfied, then the solution of SDE (3.1) is stable in mean.*

Proof. By assumptions, X_t^x and X_t^y are two solutions of (3.1) with initial value $X_0^x = x$

and $X_0^y = y$, respectively. We have

$$X_t^x = X_0^x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s)dR_s^H, \quad 0 \le t \le T, \; a.s. \tag{3.3}$$

$$X_t^y = X_0^y + \int_0^t b(s, X_s^y)ds + \int_0^t \sigma(s)dR_s^H, \quad 0 \le t \le T, \; a.s. \tag{3.4}$$

Subtracting the equation (3.3) from the equation (3.4), we obtain

$$X_t^x - X_t^y = x - y + \int_0^t b(s, X_s^x) - b(s, X_s^y)ds; \quad 0 \le t \le T, \; a.s.$$

By using expectation on both sides, we get

$$\begin{aligned}
\mathbb{E}|X_t^x - X_t^y| &\le \mathbb{E}|x - y| + \int_0^t \mathbb{E}|b(s, X_s^x) - b(s, X_s^y)|ds \\
&\le \delta + \int_0^t k(s)\mathbb{E}|X_s^x - X_s^y|ds \quad \Big(Lipschitz \quad condition\Big) \\
&\le \delta \exp\left(\int_0^t k(s)ds\right) \quad \Big(Gronwall \quad inequality\Big) \\
&\le \delta \exp\left(\int_0^T k(s)ds\right).
\end{aligned}$$

Let denote $\kappa = \exp\left(\int_0^T k(s)ds\right)$.

$\forall \varepsilon > 0$, $\exists \delta > 0$: such that $|x - y| < \delta$; let $\delta = \frac{\varepsilon}{\kappa}$; So we obtain for any $t \in [0, T]$

$$\mathbb{E}|X_t^x - X_t^y| \le \varepsilon.$$

This completes the proof of the theorem. □

3.3 Numerical simulation of SDE

In this section, we solve the equation (3.1) numerically using Euler Maryauma (EM) method; we start by computing the cumulative distribution function (CDF) of the Rosenblatt distribution.
For $M \geq 1$, define X_M and Y_M as

$$R^H = \sum_{n=1}^{M-1} \lambda_n(\varepsilon_n^2 - 1) + \sum_{n=M}^{\infty} \lambda_n(\varepsilon_n^2 - 1) := X_M + Y_M, \quad (\varepsilon_n)_{n\geq 1} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1),$$

the series converges a.s. in L^2, where the weights $\{\lambda_n\}_{n=1}^{\infty}$ are such that

$$\sum_{n=1}^{\infty} \lambda_n^k = \sigma^k(D)c_k, \qquad D = 1 - H, \qquad k = 2, 3, ...$$

To compute the CDF of $R^H = X_M + Y_M$ (see [82]), three steps are necessary:

- For $n = 1, 2, ...M$, estimate the eigenvalues λ_n of the Rosenblatt process for some $M > 0$.
- Compute the cumulative distribution function (CDF) of X_M and the probability density function (PDF) of Y_M separately in the decomposition (see[82]).
- The cumulative distribution function (CDF) of R^H is expressed as a convolution.

$$F_{R^H}(x) = \int_{-\infty}^{+\infty} F_{X_M}(x - y) f_{Y_M}(y) dy.$$

Utilizing standard numerical integration techniques, we calculate this convolution in MATLAB, where F_{X_M} represents the CDF of X_M and f_{Y_M} represents the PDF of Y_M; we take $D = 0.3$, $M = 50$ and $N = 5$; where M is the number of chi-squared distributions and

N is the order of Edgeworth expansion. For more details about the CDF of the Rosenblatt process see [82].

Using the inversion theorem we calculate the increments of the Rosenblatt process, then we apply Euler Maryauma method with the time step $\Delta t = 0.01$ on the interval $[0, 1]$ and 1000 realizations to find the numerical solution of the equation (3.1).

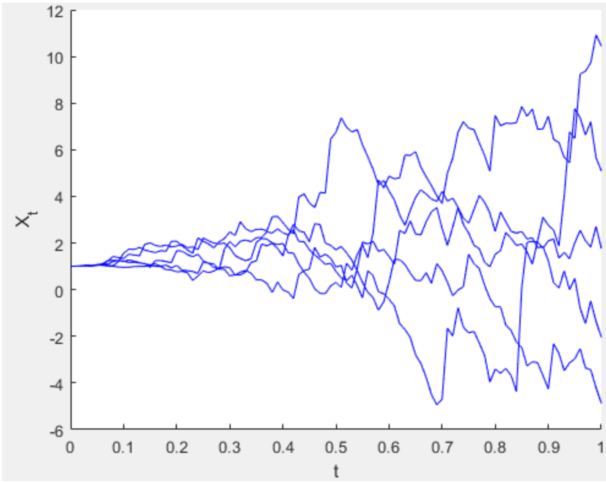

Figure 3.1: The numerical solution of the stochastic differential equation (3.1) driven by the Rosenblatt process.

3.4 Illustrative Example

In this section, we take an example of linear SDE, we solve it theoretically and numerically using Euler Maryauma method, then we calculate the error between two solutions for different steps. And we will also solve a nonlinear SDE numerically in another example.

Example 3.4.1. *We examine the linear stochastic differential equation given below:*

$$\begin{cases} dX_t = & -X_t\, dt + e^{-t} dR_t^H, \quad 0 \leq t \leq 1, \\ X_0 = & 1. \end{cases}$$

We have:

$$e^t dX_t = -e^t X_t dt + dR_t^H.$$

On the other hand, the integration by parts formula ensures that:

$$d(e^t X_t) = e^t dX_t + X_t e^t dt,$$

$$d(e^t X_t) = dR_t^H,$$

and therefore, the solution is written:

$$X_t = X_0 + e^{-t} R_t^H.$$

As we apply the Euler-Maruyama (EM) method to determine the numerical solution of this SDE, the following scheme is obtained:

$$X_{t_{i+1}} = X_{t_i} - X_{t_i}\Delta t + e^{-t_i}\Delta R_{t_i}^H.$$

For $i = 1, 2, ..., N_{time} - 1$, *where* $N_{time} = length(t)$, $X_0 = 1$ *and stepsize* $\Delta t = 0.01$.

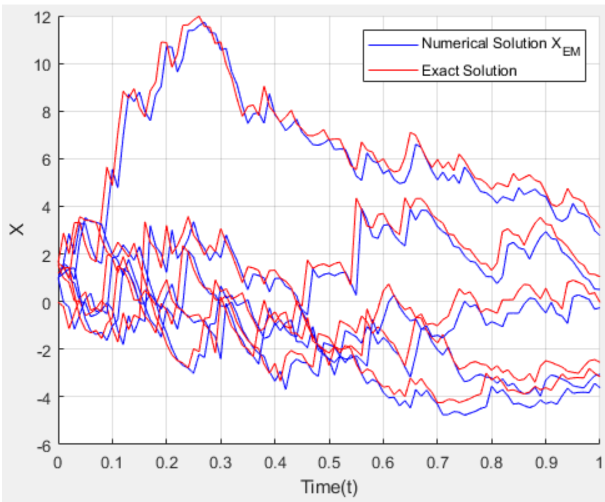

Figure 3.2: Exact solution (in red) and EM approximation (in blue) of 5 discretized sample paths for the example on the interval [0,1] with parameter $D = 0.3$, $M = 50$ and $N = 5$.

In Figure 3.2, the exact solution and the EM approximations are plotted on the same graph, the trajectory of the exact solution is plotted in red and the trajectory of the EM approximation is plotted in blue.

The error: is the the absolute value of the difference between the expected value of the approximate solution and the exact solution at time $t = 1$.
In this table we calculate the Error between the exact solution and the EM approximation for different steps for our example.

step	0.001	0.01	0.1
Error	0.0227	0.0435	0.12542

Table 3.1: the error between the exact solution and the numerical solution for different steps.

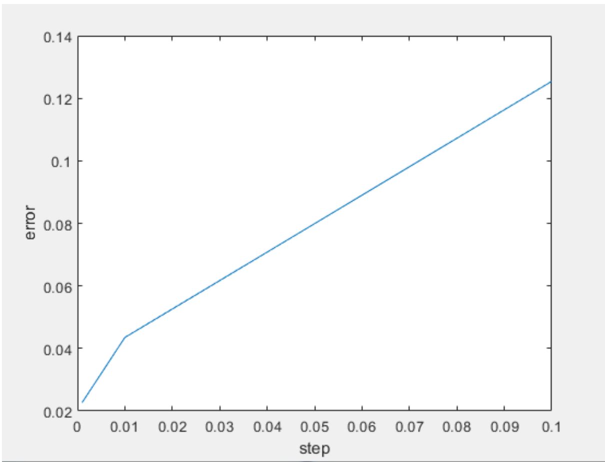

Figure 3.3: The graph of the error between the exact solution and the EM approximation for our example.

Observation: The table and the plot in Figure 3.3 depict the error between the exact solution and the EM approximation for our example when increasing the step. It is observed that, as the step decreases, the error decrease also, which justifies our approximation method (EM method) for our example. Finally, we say from our graphs that, if we minimize the stepsize Δt, we obtain more closed approximation to exact solution with the Euler-Maruyama method.

Example 3.4.2. *In this example, we solve numerically the following nonlinear stochastic differential equation using the Euler-Maruyama (EM) method:*

$$\begin{cases} dX_t = & \big(3/7X_t^{5/7} + bX_t^{6/7}\big)dt + X_t^{6/7}dR_t, \quad 0 \le t \le 1, \\ X(0) = & 1. \end{cases}$$

We obtain the following scheme by applying the EM method:

$$X_{t_{i+1}} = X_{t_i} + \big(3/7X_{t_i}^{5/7} + 7X_{t_i}^{6/7}\big)\Delta t + X_{t_i}^{6/7}\Delta R_t,$$

where $i = 0, 1, 2, ..., N_{time} - 1$, $b = 7$ *and step size* $\Delta t = 0.01$ *on the interval* $[0, 1]$.

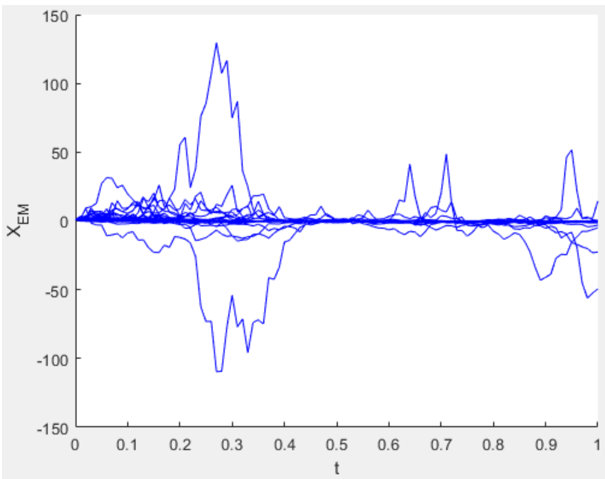

Figure 3.4: Numerical approximation of 100 sample paths for Example 3.4.2 on the interval $[0, 1]$ obtained with parameters $D = 0.3$, $M = 50$, and $N = 5$.

Figure 3.4 illustrates 100 simulated sample paths of the numerical solution of non linear SDE driven by the Rosenblatt process on the interval $[0, 1]$.

Chapter 4

The study of the no-arbitrage property in the Rosenblatt model

In this chapter, we study an important property of financial markets, namely the no-arbitrage property in a non-Gaussian framework, with particular emphasis on the Rosenblatt model [1].

A key tool in this analysis is the conditional full support (CFS) property, which plays an important role in stochastic analysis and mathematical finance. In particular, it provides information on the richness of the sample paths of a stochastic process and is closely related to the existence of consistent price systems and no-arbitrage properties in financial models. Establishing this property for the Rosenblatt process therefore contributes to a deeper understanding of its path behavior and underlying structure. The aim of this chapter is to investigate whether the Rosenblatt process satisfies the conditional full support property and to study the consequences of this result. To this end, we first recall the fundamental concepts related to the CFS property, which will serve as key tools throughout the chapter. We then review some previous results that will be used in our analysis. After that, we present our main contributions, where we establish the conditional full support property for the Rosenblatt process.

[1] This chapter is the subject of a paper submitted for publication in the journal Finance and Stochastics (Springer, Heidelberg), 2026.

Finally, we discuss an application to stochastic differential equations driven by the Rosenblatt bridge, where we verify the CFS property in this setting.

4.1 Fundamental Concepts and Findings for Conditional Full Support Property

In this part, we present conditional full support, a simple condition on asset prices that leads to a wide class of consistent price systems.

Notations

Let $T > 0$ be a fixed time horizon. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and saturatedness with respect to $\mathbb{P}$, as well as the conditions that $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ is trivial.

We introduce the following notation:

- $\mathbb{R}^d_{++} = (0, \infty)^d$.
- $C = C[u, v]$: the set of $\mathbb{R}^d$-valued continuous functions on $[u, v]$.
- $C_x[u, v]$: the subset of $C[u, v]$ consisting of functions f such that $f(u) = x$.
- $C^+[u, v]$: the set of $\mathbb{R}^d_{++}$-valued continuous functions on $[u, v]$.
- $C^+_x[u, v]$: the subset of $C^+[u, v]$ consisting of functions starting at $x \in \mathbb{R}^d_{++}$.
- All these spaces are endowed with the uniform norm topology.
- $\mathcal{B}(C)$: the σ-field of Borel subsets of C.
- $\mathcal{B}_\mu(C)$: the completion of $\mathcal{B}(C)$ with respect to a Gaussian probability measure μ.
- For a process X, we denote its support by $\mathrm{supp}(X)$.

The support of a $\mathbb{R}^d_{++}$-valued continuous process X on $[u, v]$ is defined as the smallest closed subset A of $C^+[u, v]$ such that $\mathbb{P}(X \in A) = 1$. We say that X has full support if $\text{supp}(X) = C^+_x[u, v]$, where $X_0 = x$.
The concepts of conditional full support and consistent price system are introduced in the next two definitions provided by Guasoni et al. [31].

Definition 4.1.1. *A continuous $\mathbb{R}^d_{++}$ -valued process $(X_t)_{t\in[0,T]}$ satisfies the conditional full support condition if, for all $t \in [0, T)$,*

$$(CFS) \qquad supp\ [\mathbb{P}(X|_{[t,T]}|\mathcal{F}_t)] = C^+_{X_t}[t, T] \qquad a.s.,$$

where $\mathbb{P}(X|_{[t,T]}|\mathcal{F}_t)$ denotes the $\mathcal{F}_t$ conditional distribution of the $C^+[t, T]$- valued random variable $X|_{[t,T]}$.

Remark 4.1.1. *The conditional full support property means that, conditionally on the present state, the process can approximate arbitrarily closely any continuous trajectory starting from that state, with positive probability.*

Definition 4.1.2. *Let $\varepsilon > 0$. An ε-consistent price system is a pair $(\tilde{X}, \tilde{\mathbb{P}})$ of a probability $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ and a $\tilde{\mathbb{P}}$-martingale $\tilde{X}$ (adapted to $\mathcal{F}_t$) such that*

$$\frac{1}{1+\varepsilon} \leq \frac{\tilde{X}^i_t}{X^i_t} \leq 1+\varepsilon, \quad a.s. \quad for \quad all \quad 1 \leq i \leq d \quad and \quad t \in [0, T].$$

According to Guasoni et al. [31] the existence of consistent price systems is implied by the CFS condition.

Theorem 4.1.1. *[31] Let $(X_t)_{t\in[0,T]}$ be an $\mathbb{R}^d_{++}$ continuous adapted process satisfying (CFS). Then X admits an ε-consistent pricing system for all $\varepsilon > 0$.*

Remark 4.1.2. *If $I \subset \mathbb{R}$ is an open interval and $f : \mathbb{R} \to I$ is a homeomorphism, then $g \mapsto f \circ g$ is a homeomorphism between $C_X([0,T])$ and $C_{f(X)}([0,T], I)$. Hence, for $f(X)$ understood as a process in I, we have*

$$f(X)\, has\, \mathbb{F} - CFS \Leftrightarrow X \;has\, \mathbb{F} - CFS.$$

Before presenting our main finding we start by summarizing the previous works on the CFS condition.

4.2 Previous results

Remember the CFS results for the form's processes.

$$Q_t = L_t + \int_0^t h_s dW_s, \qquad t \in [0,T],$$

where L is a continuous process, the integrator W is a Brownian motion, and the integrand h satisfies some varying assumptions.
Pakkanan [59], studied three cases in his article each of which requires a separate treatment:

- First case: L and h are (jointly) independent of W.
- Second case: L and h depend progressively on W and some additional continuous process that does not anticipate W; and $L = \int_0^t l_s ds$ is absolutely continuous.
- Third case : a (general) continuous process X with CFS takes place of W; processes L and h are (jointly) independent of X; and h is of finite variation (extension of the first case).

We focus on the first case, for more details about the second and the third case see Pakkanan [59].

4.2.1 Independent integrands and Brownian integrators

Theorem 4.2.1. *(Pakkanan, 2009 [59]) Let us define*

$$Q_t = L_t + \int_0^t h_s dW_s, \qquad t \in [0, T].$$

Suppose that:

- $(L_t)_{t\in[0,T]}$ *is a continuous process,*
- $(h_t)_{t\in[0,T]}$ *is a progressively measurable process s.t.* $\int_0^T h_s^2 ds < \infty$ *a.s.,*
- $(W_t)_{t\in[0,T]}$ *is a standard Brownian motion independent of* L *and* h*.*

If we have

$$\lambda(t \in [0, T] : h_t = 0) = 0 \quad a.s. \quad (\lambda : Lebesgue \quad measure),$$

then Q *possesses CFS.*

4.2.2 Independent integrands and fractional Brownian motion integrators

Dani et al. [25] study the CFS property for stochastic integral in which the fractional Brownian motion is the integrator, its main result is given in the following theorem:

Theorem 4.2.2. *Dani et al. [25] Let*

$$S_t = L_t + \int_0^t h_s dB_s^H, \qquad t \in [0, T],$$

Suppose that:

- $(L_t)_{t\in[0,T]}$ *is a continuous adapted process,*

- $(h_t)_{t\in[0,T]}$ *is elementary predictable s.t.* $\int_0^T h_s^2 ds < \infty \quad a.s.$,

- $(B_t^H)_{t\in[0,T]}$ *is a fractional Brownian motion independent of* L *and* h.

If we have

$$\lambda(t \in [0,T] : h_t = 0) = 0 \quad a.s. \quad (\lambda : Lebesgue \quad measure),$$

then S *has CFS.*

Models based on the Rosenblatt process have recently drawn attention due to their ability to capture long-range dependence and non-Gaussian behavior properties observed in empirical financial data. However, like fractional Brownian motion, the Rosenblatt process does not possess the semimartingale property, and in a frictionless market framework, this can again lead to arbitrage opportunities, making such models unsuitable for pricing and portfolio optimization. In this paper, we demonstrate that the introduction of arbitrarily small transaction costs changes the situation fundamentally. In particular, we prove that the Rosenblatt process satisfies the conditional full support property, a key result that ensures the existence of consistent price systems. This result provides a rigorous foundation for studying optimal investment strategies and derivative pricing in markets driven by the Rosenblatt process, a non-Gaussian process exhibiting long-range dependence.

4.3 Main Result

The main purpose of this section is to investigate the CFS property by considering the no-arbitrage problems for stochastic integrals when the integrator is the Rosenblatt process. We extend the work of Guasoni et al. [31].

We start by recalling the definition of the topological support and its characterization for Gaussian processes, which will serve as a basis for establishing the full support property of our process.

Definition 4.3.1. *Let $X = (X_t)_{t\in[0,T]}$ be a stochastic process with continuous paths, taking values in a topological space E (typically $E = C^+[u, v]$ endowed with the uniform norm. The topological support of the process X denoted by* $\mathrm{supp}(\mathcal{L}(X))$ *is defined as the support of its law $\mathcal{L}(X)$, i.e., the smallest closed subset $S \subset E$ such that $\mathbb{P}(X \in S) = 1$. Equivalently, it can be characterized as:*

$$\mathrm{supp}(\mathcal{L}(X)) = \{f \in E : \mathbb{P}(X \in U) > 0 \textit{ for every open neighborhood } U \textit{ of } f\}$$

.

Interpretation:

- A path $f \in E$ belongs to the support of the law of X if the process can get arbitrarily close to f with positive probability.
- In path space (e.g., $C^+[u, v]$), this means:

$$\mathbb{P}\big(\|X - f\|_\infty < \varepsilon\big) > 0, \quad \forall\varepsilon > 0.$$

In more practical terms, a path $f \in E$ belongs to the support of the law of X if every open neighborhood of f has strictly positive probability of containing a sample path of the process X. Equivalently, the support consists of all trajectories that the process can approximate arbitrarily closely with positive probability.

We recall the following classical result, which characterizes the support of a Gaussian measure in terms of its associated reproducing kernel Hilbert space. This result will be an essential tool in the proof of the Conditional Full Support property.

Theorem 4.3.1. *[38] Let μ be a Gaussian measure on $(C, \mathcal{B}_\mu(C))$ with mean zero and continuous covariance Γ. Denote by $\mathcal{H}(\Gamma)$ the reproducing kernel Hilbert space (RKHS) associated with Γ. Then*

$$supp(\mu) = \overline{\mathcal{H}(\Gamma)},$$

where $\overline{\mathcal{H}(\Gamma)}$ is the closure of $\mathcal{H}(\Gamma)$ in C.

In the following theorem we present our main result.

Theorem 4.3.2. *[12] Let us define*

$$Y_t = L_t + \int_0^t h_s dR_s^H, \qquad t \in [0,T].$$

Suppose that:

- $(L_t)_{t\in[0,T]}$ *is a continuous adapted process,*
- $(h_t)_{t\in[0,T]}$ *is predictable process s.t.* $\int_0^T h_s^2 ds < \infty \quad a.s.$,
- $(R_t^H)_{t\in[0,T]}$ *is a Rosenblatt process independent of* L *and* h.

If we have

$$\lambda(t \in [0,T] : h_t = 0) = 0 \quad a.s. \quad (\lambda : Lebesgue \quad measure),$$

then Y *has CFS.*

Proof. We first emphasize that the conditional full support (CFS) property does not depend on the Gaussian nature of the process. Rather, it is a purely topological/pathwise property, related to the ability of the process to approximate arbitrary trajectories in the underlying path space with positive probability. In this sense, the CFS property is determined by the support of the law of the process and not by its specific distributional structure. However, in the case of the Rosenblatt process, establishing this property is more involved due to its representation in the second Wiener chaos, which makes the analysis significantly more delicate compared to the Gaussian framework. Let

$$Z_t = \int_0^t h_s dR_s^H.$$

Using the finite-time representation of the Rosenblatt process, we obtain:

$$Z_t = A_3(H) \int_{[v,t]^2} h(x_1,x_2) \left(\int_{x_1 \vee x_2}^t \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u,x_1) \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u,x_2) du \right) dB_{x_1} dB_{x_2}.$$

Let us fix $v \in [0, T]$. It is sufficient to demonstrate that the conditional law $\mathbb{P}(Z|_{[v,T]}|\mathcal{F}_v)$ has full support on $C_{Z_v}([v, T], \mathbb{R})$ almost surely, by taking into account the restriction of Y on an interval $[v, T]$. Furthermore, it suffices to demonstrate this property on an interval where h is continuous and constant with respect to time. As a result, we can take T small enough that h is $\mathcal{F}_v$ measurable and has the form $h_{t_1,t_2} = \xi$ on $[v, T]$, with $\xi \neq 0$. Thus, we must demonstrate that Z has a full support on $C_0([v, T], \mathbb{R})$.

The reproducing kernel Hilbert space (RKHS) of a non Gaussian process Z indexed in $[0, T]$ with covariance $\Gamma(t_1, t_2)$ is the Hilbert space spanned by the functions of the form $\Gamma(., t_2)$ with the scalar product defined by $\langle \Gamma(., t_1), \Gamma(., t_2)\rangle = \Gamma(t_1, t_2)$. Since the Rosenblatt process shares the same covariance structure as the fractional Brownian motion, its reproducing kernel Hilbert space (RKHS) can be represented as follows:

$$\mathbb{H} := \{f \in C_0([v,T],\mathbb{R}) : f(t) = A_3(H)\int_v^t\int_v^t h(x_1,x_2)\widetilde{K}^{H'}(t,x_1,x_2)g(x_1)g(x_2)dx_1dx_2,\ for\, some\, g \in L^2([v,T])a.s.,\}$$

with

$$\widetilde{K}^{H'}(t, x_1, x_2) = \int_{x_1 \vee x_2}^t \frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_1)\frac{\partial K^{\frac{H+1}{2}}}{\partial u}(u, x_2)\, du.$$

To characterize the topological support of the process Z, we exploit its representation as a second-order Wiener–Itô integral. Let $B = (B_t)_{t\in[v,T]}$ denote a standard Brownian motion. A fundamental result in Gaussian analysis identifies the topological support of the Wiener measure on the Banach space $(C_0[v, T], \|\cdot\|_\infty)$ with the closure of its Cameron–Martin space. We define the Cameron–Martin space associated with the Brownian motion B as the Hilbert space:

$$\mathcal{H}_B := \left\{ f \in C_0([v, T], \mathbb{R}) : f(t) = \int_v^t g(s)\, ds,\ \text{with } g \in L^2([v, T]) \right\}$$

equipped with the inner product $\langle f_1, f_2\rangle_{\mathcal{H}_B} = \int_v^T g_1(s)g_2(s)\, ds$. It is well established that the law $\mathcal{L}(B)$ satisfies:

$$\mathrm{Supp}(\mathcal{L}(B)) = \overline{\mathcal{H}_B}^{\|\cdot\|_\infty}.$$

Given that the process Z can be expressed as a continuous functional of the Brownian motion, specifically $Z = \Phi(B)$, where $\Phi(B)_0 = 0$, so the support of its law $\mathcal{L}(Z)$ is determined by the image of $\mathcal{H}_B$ under the deterministic mapping Φ. Specifically, if we define the "skeleton" of the process for any $f \in \mathcal{H}_B$ as:

$$\Phi(f)(t) = A_3(H) \int_v^t \int_v^t h_{x_1,x_2} \left(\int_{x_1 \vee x_2}^t \frac{\partial K^{H'}}{\partial u}(u, x_1) \frac{\partial K^{H'}}{\partial u}(u, x_2)\, du \right) g(x_1) g(x_2)\, dx_1 dx_2,$$

then the topological support of Z in $C_0([v,T],\mathbb{R})$ is given by the closure of the set of all such trajectories:

$$\mathrm{Supp}(\mathcal{L}(Z)) = \overline{\{\Phi(f) : f \in \mathcal{H}_B\}}^{\|\cdot\|_\infty} = \overline{\mathbb{H}}^{\|\cdot\|_\infty}.$$

This reduction shows that the problem of determining the support of the process Z can be expressed in terms of deterministic functions obtained from the Cameron–Martin space of Brownian motion through a quadratic transformation. Therefore, to prove that Z has full support in $C_0([v,T],\mathbb{R})$ it suffices to show that $\mathbb{H}$ is norm-dense in $C_0([v,T],\mathbb{R})$.

To this end, we begin by recalling the definition of the second-order fractional integral see [21].

$$(I^{\alpha_1,\alpha_2}_{a+,a+} f)(t_1,t_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^{t_1} \int_a^{t_2} f(x_1,x_2)(t_1 - x_1)^{\alpha_1 - 1}(t_2 - x_2)^{\alpha_2 - 1} dx_1 dx_2,\ a \le t_1 \le b \quad a \le t_2 \le b.$$

Since the RKHS $\mathbb{H}$ consists of functions of a single variable t, we consider the diagonal restriction of the second-order fractional integral:

$$(I^{\alpha_1,\alpha_2}_{a+,a+} f)(t,t) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^{t} \int_a^{t} f(x_1,x_2)(t - x_1)^{\alpha_1 - 1}(t - x_2)^{\alpha_2 - 1} dx_1 dx_2, \quad a \le t \le b.$$

This yields a function depending only on t, making it compatible with the RKHS $\mathbb{H}$, and to present the kernel operator $\widetilde{K}^{H'}$,

$$(\widetilde{K}^{H'}f)(t) := \int_0^t \int_0^t \widetilde{K}^{H'}(t, x_1, x_2) f(x_1, x_2) dx_1 dx_2, \quad f \in L^2[0,T], \quad t \in [0,T].$$

$$(\widetilde{K}^{H'}fh)(t) := \xi \int_v^t \int_v^t \widetilde{K}^{H'}(t, x_1, x_2) f(x_1, x_2) dx_1 dx_2, \quad f \in L^2[0,T], \quad t \in [0,T].$$

We have :

$$\widetilde{K}^{H'}(fh) = I^{1,1}_{0^+,0^+}(x_1^{H'-\frac{1}{2}} x_2^{H'-\frac{1}{2}} I^{H'-\frac{1}{2},H'-\frac{1}{2}}_{0^+,0^+}(x_1^{\frac{1}{2}-H'} x_2^{\frac{1}{2}-H'}(fh)(x_1,x_2))). \tag{4.1}$$

Going forward, we presume that every function on a subset of $\mathbb{R}$ is extended to the entire real line by making them equal to zero outside of their defined domain.

We now continue the proof of our fundamental result, which is divided into two cases.

First case: For $v \neq 0$ the argument must be divided into two steps. We first state and prove Lemma 4.3.1, then state and prove Lemma 4.3.2. These two lemmas together lead to the proof of our theorem 4.3.2 .

Lemma 4.3.1. *if $f \in C_0[v,T]$, then $L_1 fh \in C_0[v,T]$, where*

$$(L_1(fh))(t) = (I^{H'-\frac{1}{2},H'-\frac{1}{2}}_{0^+,0^+}(x_1^{\frac{1}{2}-H'} x_2^{\frac{1}{2}-H'}(fh)(x_1,x_2)))(t).$$

Moreover, $L_1 : C_0[v,T] \to C_0[v,T]$ is continuous and has a dense range in $C_0[v,T]$ with respect to the uniform norm, i.e., $\overline{\mathrm{Im}(L_1)}^{\|\cdot\|_\infty} = C_0[v,T]$.

Proof. Evidently, $(L_1 f)(0) = 0$ and $L_1 f$ is a continuous function. The operator is continuous by the following estimate.

$$\|L_1 f - L_1 \phi\|_\infty \leq v_1^{\frac{1}{2}-H'} v_2^{\frac{1}{2}-H'} \int_0^T \int_0^T (T-x_1)^{H'-\frac{3}{2}} (T-x_2)^{H'-\frac{3}{2}} dx_1 dx_2 \|f-\phi\|_\infty.$$

Remember the identity for a, $b > 0$,

$$\int_0^t (t-u)^{a-1} u^{b-1} du = \mathbf{B}(a,b) t^{a+b-1},$$

where $\mathbf{B}(a,b) \neq 0$ and $\mathbf{B}(a,b) = \frac{\Gamma(a).\Gamma(b)}{\Gamma(a+b)}$. For a fixed $\alpha > 0$ defining :

$$\phi(x_1, x_2) = \mathbb{1}_{[v,T]^2} \frac{(x_1 - v)^\alpha (x_2 - v)^\alpha}{\xi x_1^{\frac{1}{2}-H'} x_2^{\frac{1}{2}-H'}}.$$

For $t \in [v, T]$ we get

$$\begin{aligned}
(L_1(\phi h))(t) &:= \frac{\xi}{\Gamma(H' - \frac{1}{2})\Gamma(H' - \frac{1}{2})} \int_v^t \int_v^t x_1^{\frac{1}{2}-H'} x_2^{\frac{1}{2}-H'} \phi(x_1, x_2)(t - x_1)^{H'-\frac{3}{2}}(t - x_2)^{H'-\frac{3}{2}} dx_1 dx_2 \\
&= \frac{1}{\Gamma(H' - \frac{1}{2})\Gamma(H' - \frac{1}{2})} \int_v^t \int_v^t (x_1 - v)^\alpha (x_2 - v)^\alpha (t - x_1)^{H'-\frac{3}{2}}(t - x_2)^{H'-\frac{3}{2}} dx_1 dx_2 \\
&\leq \int_0^{t-v} \int_0^{t-v} (u_1)^\alpha (u_2)^\alpha (t - v - u_1)^{H'-\frac{3}{2}}(t - v - u_2)^{H'-\frac{3}{2}} du_1 du_2 \\
&\leq \mathbf{B}\left(H' - \frac{1}{2}, \alpha + 1\right)^2 (t - v)^{2\alpha + 2H' - 1}.
\end{aligned}$$

By varying α, we obtain that $(t - v)^m \in Im(L_1)$ for $m \geq 1$ and the Stone–Weierstrass theorem ensures that $Im(L_1)$ is dense in $C_0[v, T]$. $\square$

Lemma 4.3.2. *if $f \in C_0[v, T]$, then $L_2 f h \in C_0[v, T]$, where*

$$(L_2(fh))(t) = (I^{1,1}_{0^+,0^+}(x_1^{H'-\frac{1}{2}} x_2^{H'-\frac{1}{2}}(fh)(x_1, x_2)))(t)$$

and, $L_2 : C_0[v, T] \to C_0[v, T]$ is continuous and has a dense range.

Proof. The same reasoning as in the previous lemma applies, however this time we employ the estimation

$$\|L_1 f - L_1\phi\|_\infty \le \int_0^T \int_0^T x_1^{H'-\frac{1}{2}} x_2^{H'-\frac{1}{2}} dx_1 dx_2 \|f - \phi\|_\infty,$$

and the function

$$\phi(s) = \mathbb{1}_{[v,T]^2} \frac{(x_1 - v)^\alpha (x_2 - v)^\alpha}{\xi x_1^{H'-\frac{1}{2}} x_2^{H'-\frac{1}{2}}}.$$

Since the restriction of $\widetilde{K}^{H'}$ to $C_0[v,T]$ is exactly $L_2 \circ L_1$, we may conclude that $\widetilde{K}^{H'}$: $C_0[v,T] \to C_0[v,T]$ has a dense range and, a fortiori, $\mathbb{H}$ is norm-dense in $C_0[v,T]$.

□

Second case: We might use the same computations as in [30], in the scenario where $v = 0$.

Proof. Let $f(x_1, x_2) = \frac{x_1^\alpha x_2^\alpha}{\xi}$, *where* $\xi \neq 0$ *is a constant.*

$$\begin{aligned} I^{1,1}_{0^+,0^+}\left(x_1^{H'-\frac{1}{2}} x_2^{H'-\frac{1}{2}} I^{H'-\frac{1}{2},H'-\frac{1}{2}}_{0^+,0^+}\left(x_1^{\frac{1}{2}-H'} x_2^{\frac{1}{2}-H'} fh\right)\right) &:= I^{1,1}_{0^+,0^+}\left(\frac{\left[\Gamma(\alpha+\frac{3}{2}-H')\right]^2}{\left[\Gamma(\alpha+1)\right]^2} x_1^{\alpha+H'-\frac{1}{2}} x_2^{\alpha+H'-\frac{1}{2}}\right) \\ &= \left(\frac{\Gamma(\alpha+\frac{3}{2}-H')}{\Gamma(\alpha+1)}\right)^2 \left(\frac{\Gamma(\alpha+H'+\frac{1}{2})}{\Gamma(\alpha+H'+\frac{3}{2})}\right)^2 x_1^{\alpha+H'+\frac{1}{2}} x_2^{\alpha+H'+\frac{1}{2}}. \end{aligned}$$

Since $\alpha > -1$ can be chosen arbitrarily, the image of $\widetilde{K}_{H'}$ contains functions of the form

$$x_1^\beta x_2^\beta, \qquad \beta > H' + \tfrac{1}{2}.$$

If these functions are inserted into the representation of the RKHS elements associated with the process Z, and using the separable structure of the kernel, the resulting RKHS elements reduce to functions of the time variable of the form t^γ, where γ ranges over an interval of the form $(\gamma_0, +\infty)$ for some $\gamma_0 > 0$.

Moreover, the finite linear combinations of such functions generate an algebra of polynomial-type functions on $[0,T]$ which vanish at the origin. This algebra separates points in $[0,T]$ and vanishes at 0. By the Stone–Weierstrass theorem, this algebra is dense in the space

$$C_0[0,T] = \{f \in C[0,T] : f(0) = 0\}.$$

Therefore, the RKHS of the process Z is dense in $C_0[0,T]$. □

□

It follows that, the process Y possesses the CFS property, then it admits consistent price systems for arbitrarily small transaction costs. We deduce that this price process doesn't admit an arbitrage opportunities.

4.4 Application

In insider-trading models, a central question is how private information affects asset-price dynamics and the resulting investment opportunities. Typically, an insider possesses information about a future characteristic of the asset, such as its terminal value, future payoff, earnings, or liquidation value. Through the insider's trading decisions, this information is progressively incorporated into market prices, leading to dynamics that differ from those observed under the standard market filtration.
Monique Jeanblanc [36] investigated a simple scenario of insider trading. They considered a Black-Scholes model $[dS_t = S_t(\mu dt + \sigma dB_t)]$, where B_t is a Brownian Motion. The insider is assumed to know, at time $(t = 0)$, the future value of the stock price at the terminal date $(T = 1)$. This additional information enlarges the information available to the insider and creates arbitrage opportunities that are not accessible to ordinary market participants. The authors showed that such anticipative information can be exploited to improve the insider's trading performance, even when trading is restricted to a time horizon strictly smaller than the terminal date.

This result illustrates a fundamental idea in insider-trading theory: knowledge of the future value of an asset may significantly alter the structure of the market and generate arbitrage opportunities. In particular, the information concerning the terminal stock price (S_1) provides the insider with a decisive advantage, which becomes fully reflected in the market outcome at the terminal time.
Our goal is to establish the absence of arbitrage for the case ($0 \leq t < 1$) without explicitly deriving the dynamics of wealth or the risk-neutral probability in the same model studied by Monique Jeanblanc [36], namely the Black- Sholes model, but in our example the integrator is the Rosenblatt Bridge.
The Rosenblatt Bridge is used in insider trading models to capture non-Gaussian, long-memory, and non-linear effects in the information flow from insiders to the market. It generalizes the Brownian Bridge by allowing for richer statistical structures, making it a powerful tool for modeling realistic insider-driven price dynamics where the terminal value is known in advance. Moreover, the Rosenblatt Bridge is particularly well-suited for modeling situations where long-range memory effects are present for instance, when market participants react to information slowly or when the insider releases information gradually across a long time horizon. These features are naturally captured by the Rosenblatt bridge due to its long-memory and non-Markovian structure.

Before presenting the application of CFS, we begin by recalling essential facts about the Rosenblatt Bridge.
The Rosenblatt Bridge $(r_t, 0 \leq t \leq 1)$ is defined as the conditioned process $(R_t, t \leq 1 | R_1 = 0)$. Note that $R_t = (R_t - tR_1) + tR_1$ where, from the non Gaussian property, the process $(R_t - tR_1, t \leq 1)$ and the random variable R_1 are independent. Hence $(r_t, 0 \leq t \leq 1) \overset{Law}{=} (R_t - tR_1, 0 \leq t \leq 1)$. The Rosenblatt Bridge process is a non Gaussian process, with zero mean and it satisfies $r_0 = r_1 = 0$ where $(R_t, t \leq T)$ is a Rosenblatt process starting from 0.

Example of application: We consider the following example. Let

$$d^\circ S_t = S_t(\mu dt + \sigma d^\circ r_t)$$

where μ and σ are constants, r_t is the Rosenblatt Bridge. The price of a risky asset is defined by S_t. Assume that the interest rate r on the riskless asset is constant. We use the symmetric integral because the Itô formula used later is formulated in this framework. The Rosenblatt Bridge r_t is a solution of the following stochastic equation.

$$\begin{cases} d^\circ r_t = & -\frac{r_t}{1-t}dt + d^\circ R_t; \quad 0 \leq t < 1, \\ r_0 = & 0. \end{cases}$$

The equation above can be solved as

$$r_t = (1-t)\int_0^t \frac{1}{1-s} d^\circ R_s.$$

We check whether S has the CFS property. Using Itô formula in the Rosenblatt case (see Equation (1.31)) we obtain:

$$\log S_t = \log S_0 + \mu t + \sigma(1-t)\int_0^t \frac{1}{1-s} d^\circ R_s, \quad 0 \leq t < 1$$

$$= \underbrace{\log S_0 + \mu t}_{=:L_t} + \int_0^t \underbrace{\sigma(1-t)\frac{1}{1-s}}_{=:h_s} d^\circ R_s, \quad 0 \leq t < 1.$$

We observe that

- $(L_t) = (\log S_0 + \mu t)$ is a continuous process,
- $(h_s) = (\sigma(1-t)\frac{1}{1-s})$ is a predictable process s.t. $\int_0^T h_s^2 ds < \infty \quad a.s.$,
- (R_t) is a Rosenblatt process independent of L and h,

which evidently meets the conditions stated in Theorem 4.3.2, then $\log S_t$ has CFS property. According to Remark 4.1.2 the process S has CFS for $0 \leq t < 1$. Therefore, by Theorem 4.1.1, S admits an ε-consistent pricing system for all $\varepsilon > 0$. Consequently, the model is free of arbitrage opportunities.

Conclusion and Perspectives

Conclusion

The aim of this thesis was to contribute to the development of stochastic analysis with respect to the Rosenblatt process by investigating three fundamental aspects: stochastic integration, stochastic differential equations, and the conditional full support property.

First, we developed a stochastic integration framework with respect to the Rosenblatt process that goes beyond the classical adapted setting. In particular, we constructed integrals for a class of non-adapted processes by decomposing the integrands into suitable components involving adapted and instantly independent parts. This allows one to handle situations involving anticipative information, while preserving a rigorous stochastic framework driven by a non-Gaussian process.

Second, we studied stochastic differential equations driven by the Rosenblatt process, focusing on a class of models with constant diffusion coefficient. We established well-posedness of the equations by proving existence, uniqueness, and stability of solutions. The theoretical results were complemented by explicit examples and numerical simulations that illustrate the behavior of the solutions. Overall, this shows that, despite the intrinsic complexity of the Rosenblatt process, it is possible to construct and analyse well-defined stochastic dynamics driven by this type of noise.

Finally, we established the conditional full support (CFS) property for processes when the integrator is the Rosenblatt process. This study shows that the CFS property does not depend on the non-Gaussian nature of the underlying process, but is instead fundamentally a topological property; however, proving it in the Rosenblatt setting is significantly more challenging due to its second-order Wiener chaos representation and long-range dependence structure. As a consequence, the CFS property implies the existence of an ε-consistent pricing system for every $\varepsilon > 0$, and thus the market is free of arbitrage. Our results not only extend classical notions of financial no-arbitrage to a more complex stochastic setting but also provide a robust framework for studying stochastic differential equations driven by the Rosenblatt Bridge, for which we proved the preservation of the CFS property. These results show that the Rosenblatt process, despite its non-Gaussian nature, provides a suitable framework for modelling financial markets under uncertainty, while remaining consistent with the no-arbitrage principle.

Perspectives

Although several fundamental aspects of stochastic systems driven by the Rosenblatt process have been addressed in this thesis, many challenging questions remain open. The results obtained here naturally lead to further investigations in stochastic analysis and mathematical finance. We conclude by highlighting some research directions that appear particularly interesting for future work:

- **Anticipating stochastic calculus:**
  Developing an anticipating Itô-type formula in the non-adapted setting with respect to the Rosenblatt process.

- **Non-adapted stochastic differential equations:**
  Extending the study of stochastic differential equations driven by the Rosenblatt process to the case of anticipating or non-adapted coefficients, particularly in situations involving future information.

- **Extension to broader non-Gaussian frameworks:**
  Extending the integration framework and the study of stochastic dynamics to broader classes of processes, including higher-order Hermite processes and other non-Gaussian models exhibiting long-range dependence.

- **CFS in the non-adapted (anticipating) setting:**

  Does CFS still hold when the integrand is non-adapted? This is very interesting because:

  CFS property is mainly related to the pathwise structure of the process, whereas non-adaptedness concerns the filtration and the underlying information structure.

- **Applications in stochastic modeling and finance:**

  Developing stochastic models incorporating non-Gaussian features and long-range dependence, and investigating their implications in areas such as mathematical finance, particularly in connection with no-arbitrage conditions.

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