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On the flow of a stochastic differential equation

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Dedication

All praise to **Allah**; today we fold the day's tiredness and the errand summing up between the covers of this humble work.

I dedicate my work to:

My great teacher and messenger, **Mohammed-peace and grace from Allah be upon him**-who taught us the purpose of life.

My parents, who have been my source of inspiration and gave us strength when we thought of giving up, continually provide their moral, spiritual, emotional, and financial support. God save them.

My grandmother. God bless her.

My brothers **Zaid** and **Sohaib**.

To my friends **Tayeb Hamlat**, **Moulay Larbi Mahdi**, **Khelifa Berkane** and their families.

Last but not least, I am dedicating this to my aunts and my maternal aunts.

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Contents

Abstract

In this thesis, we focus on a prominent financial model, known as the natural model. Specifically, this model is represented by a stochastic differential equation called the natural equation, which plays an important role in our research. Firstly, we investigate the existence and uniqueness of solutions to the natural equation in the one-dimensional case under a condition weaker than the Lipschitz condition, termed the non-Lipschitz condition. Additionally, we study the mean-square stability of the solutions to the natural equation. Secondly, we study the bicontinuity and homeomorphism properties of stochastic flows of the solution to the one-dimensional natural equation with non-Lipschitz coefficients with respect to the initial value based on Kolmogorov's criterion.

Keywords: Stochastic flows, non-Lipschitz coefficients, stochastic differential equations, existence and uniqueness, financial market, bicontinuity, homeomorphism, Bihari inequality, Kolmogorov's theorem

Résumé

Dans cette thèse, nous nous concentrons sur un modèle financier important, connu sous le nom de modèle naturel. Plus précisément, ce modèle est décrit par une équation différentielle stochastique appelée équation naturelle, qui joue un rôle important dans notre étude. Dans un premier temps, nous étudions l'existence et l'unicité des solutions de l'équation naturelle dans le cas unidimensionnel sous une condition plus faible que la condition de Lipschitz, appelée condition de non-Lipschitz. De plus, nous étudions la stabilité en moyenne quadratique des solutions de l'équation naturelle. Dans un second temps, nous examinons les propriétés de bicontinuité et d'homéomorphisme des flots stochastiques de la solution de l'équation naturelle unidimensionnelle à coefficients non lipschitziens, par rapport à la condition initiale, en nous basant sur le critère de Kolmogorov.

Mots clés : Flots stochastiques, coefficients non lipschitziens, équations différentielles stochastiques, existence et unicité, marché financier, bicontinuité, homéomorphisme, inégalité de Bihari, théorème de Kolmogorov.

الملخص

في هذه الاطروحة، نركز على نموذج مالي بارز يُعرف بالنموذج الطبيعي. وعلى وجه التحديد، يتم تمثيل هذا النموذج بمعادلة تفاضلية عشوائية تسمى المعادلة الطبيعية، والتي تلعب دورًا مهما في بحثنا. أولاً، نقوم بدراسة وجود وتفرد الحلول للمعادلة الطبيعية في حالة أحادية البعد في ظل شرط أضعف من شرط ليبشيتز، والتي تسمى بشرط غير ليبشيتز. بالإضافة إلى ذلك، ندرس استقرار متوسط مربع حلول المعادلة الطبيعية. ثانياً، ندرس خصائص الاستمرارية الثنائية والهومومورفية للتدفقات العشوائية لحل المعادلة الطبيعية أحادية البعد ذات المعاملات غير ليبشيتز فيما يتعلق بالقيمة الأولية بناءً على معيار كولموغوروف.

الكلمات المفتاحية: التدفقات العشوائية، المعاملات غير الليبشيتزية، المعادلات التفاضلية العشوائية، الوجود و التفرد، السوق المالية، الاستمرارية الثنائية، الهومومورفية، متراجحة بيهاري، مبرهنة كولموغوروف.

List of works

Publications:

1. Mohammed Chikouche, E. T. E., Benziadi, F., Caraballo, T., and Kandouci, A. (2025). On the existence and uniqueness of solutions to natural equations with non-Lipschitz conditions. *Random Operators and Stochastic Equations*, 33(2), 143-155. https://doi.org/10.1515/rose-2025-2008

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Communications:

1. The continuity of the solution of stochastic differential equation in one dimensional case, 2^{nd} International Seminar on Industrial Engineering and Applied Mathematics (ISIEAM'2022), Skikda, Algeria, October 23^{rd} and 24^{th}, 2022.

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6. On the stochastic differential equation under Non-Lipschitz conditions, National Seminar on Modern Mathematics & Applications (MMA'24), Annaba, Algeria, October 6^{th} to 8^{th}, 2024.

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8. Advanced Mathematical Model in the Financial Market, 4^{th} National Conference of Mathematics and Applications (CNMA'24), Mila, Algeria, December 7^{th} and 8^{th}, 2024.

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Presentations in Laboratory

1. Thesis progress state, September 27^{th}, 2022.

2. Mathematical models in finance: The natural model, Mathematics Week, March 12^{td} to 16^{th}, 2023.

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6. Thesis progress state, September 28^{th}, 2024.

7. Thesis progress state, September 20^{th}, 2025.

Research events

1. Participation in autumn school, Deep Learning Methods for Stochastic Dynamics, Laboratory of Stochastic Models, Statistics and Applications, Saida, Algeria, October 29^{th} to November 2^{nd}, 2023.

2. Participation in Student Week, University Dr Moulay Tahar, Saida, Algeria, May 19^{th} to 23^{rd}, 2024.

3. Participation in the School of Artificial Intelligence for Scientific Research in Applied Mathematics, Laboratory of Stochastic Models, Statistics and Applications, Saida, Algeria, June 17^{th} to 19^{th}, 2025.

4. Participation in Student Week, University Dr Moulay Tahar, Saida, Algeria, May 18^{th} to 22^{nd}, 2025.

5. Participation in the CIMPA'PAA summer school, Probability and Applied Analysis, Feza Gursey Center for Physics and Mathematics, Bogazici University, Istanbul, Türkiye, June 29^{th} to July 12^{th}, 2025.

Introduction

Stochastic differential equations (SDEs for short) constitute a powerful framework for modeling dynamical systems subject to random fluctuations. They have been successfully applied in many scientific fields, including biology, physics, chemistry, economics, and finance. In biology, SDEs are used to describe population dynamics and the spread of infectious diseases. In physics, they provide a framework for modeling the random motion of particles, while in chemistry they help account for uncertainties arising in chemical reactions. In economics and finance, SDEs play a central role in the modeling of financial markets, particularly in option pricing, risk management, and portfolio optimization. Owing to their ability to describe uncertainty in evolving systems, SDEs have become an indispensable tool in both theoretical and applied research (see for instance, [46], [21], [3], [42], [43]).

The theory of stochastic flows for stochastic differential equations (SDEs) has received considerable attention and has been the subject of extensive research over the last three decades, with particular emphasis on the qualitative and geometric properties of their solutions. In the classical framework, it is well known that solutions of SDEs driven by a Wiener process, Brownian motion, and continuous semimartingales define a stochastic flow of diffeomorphisms under suitable regularity assumptions on the coefficients. This fundamental result reflects the regular dependence of solutions on both time and initial conditions and plays a central role in stochastic analysis. Under strong assumptions for the coefficients. Numerous authors have contributed to this theory. We mention some of them. In 1973, Neveu [41] proved the theorem of continuity of the solution according to the initial condition for classical SDEs governed by dB_t and dt terms. In 1981, Yamada and Ogura [65] showed the existence of global flows of homeomorphisms for one-dimensional

SDEs. In the same year, Emmery [16] treated the general case of weak injectivity but without the time reversal technique. Bismut [11] studied stochastic flows in the case where the initial condition is itself a stochastic process. Using an extension of Itô's formula, he proved that if the initial condition is a semimartingale, then the corresponding solution is also a semimartingale. In 1984, Kunita [30] investigated the stochastic flow of diffeomorphisms for stochastic differential equations. Stochastic flow of diffeomorphisms of a set D onto itself was defined by Taniguchi [57] in 1989 by establishing necessary conditions for the solution to a stochastic differential equation on an open set $D \subset \mathbb{R}^n$. In 2002, Airault and Ren [2] provided an explicit modulus of Hölder continuity of stochastic flows of the canonic Brownian motion on the group of diffeomorphisms of the circle. In 2004, Protter [45] investigated the continuity and differentiability of solutions to SDEs governed by semimartingales with local Lipschitz coefficients.

A natural question arises as to whether stochastic flows of solutions to stochastic differential equations (SDEs) with non-Lipschitz coefficients can be established. The key step for stochastic flows for SDEs with Lipschitz coefficients is to get the bicontinuity of $(t, x) \longmapsto X_t^x$ by using Kolmogorov's criterion. In the case of Lipschitz coefficients, such results can be obtained by means of the classical Gronwall inequality. However, in the non-Lipschitz case, the situation becomes more delicate since the direct application of Gronwall's inequality is no longer possible.

Recently, there has been growing interest in the study of stochastic flows for stochastic differential equations (SDEs) with non-Lipschitz coefficients. For this, let us mention a few papers. The stochastic flows of a class of stochastic differential equations with non-Lipschitz coefficients were demonstrated by Fang and Zhang [18] in 2003. In the same year, Ren and Zhang [50] proved the bicontinuity and homeomorphism property of solutions of stochastic differential equations with non-Lipschitz coefficients and driven by infinitely many Brownian motions. In 2005, Zhang [66] investigated the homeomorphism property of solutions for stochastic differential equations with non-Lipschitz coefficients and in the multi-dimensional case. In 2006, Ren and Zhang [51] studied the continuity modulus of stochastic homeomorphism flows for SDEs with non-Lipschitz coefficients. In 2008, Qiao and Zhang [48] made a significant contribution to the field of stochastic flows by proving the continuity and homeomorphism properties for the solutions of multi-dimensional stochastic differential equations with jumps and non-Lipschitz coefficients. The bicontinuous modifications of the solutions for multivalued stochastic differential equations with non-Lipschitz coefficients were shown by Xu [55] in 2009. In 2012, Qiao [47] demonstrated the homeomorphism flows for stochastic differential equations with non-Lipschitz coefficients and driven by Lévy processes. In 2019, Xu et al. [61] examined stochastic homeomorphism flows for stochastic differential equations (SDEs) with non-Lipschitz coefficients and singular time in the multi-dimensional case. In 2021, Liu et al. [33] demonstrated the homeomorphism property of solutions to multi-dimensional stochastic differential equations governed by G-Brownian motion under non-Lipschitz coefficients.

In order to establish the properties of stochastic flows of the solutions for SDEs with non-Lipschitz coefficients, we first investigate that the solutions of the stochastic differential equations under non-Lipschitz coefficients exist and are unique. Indeed, the uniqueness of solutions ensures that the mapping $(t,x) \longmapsto X_t^x$ is well defined for each initial condition. Once the existence and uniqueness are obtained, we then study the regularity properties of the flow of the solutions, such as the bicontinuity with respect to both time and space variables, and establish the homeomorphism for all almost ω.

The study of existence and uniqueness of solutions has attracted considerable interest, particularly within the framework of stochastic differential equations (SDEs). This area of inquiry has attracted significant interest. In the past few years, there has been a growing and significant interest in the study of the existence and uniqueness of stochastic differential equations (SDEs) with non-Lipschitz conditions. In 2003, Cao and He [12] contributed significantly to the field by proving two theorems that address the existence and uniqueness of solutions of SDEs with non-Lipschitz coefficients. In addition to establishing these theorems, their work includes the non-contact property and the strong comparison theorem for solutions. In the same year, Fang and Zhang [19] investigated a class of stochastic differential equations with non-Lipschitz coefficients. They obtained a unique strong solution to these equations. Furthermore, their work extended to establishing a large deviation principle of Freidlin-Wentzell type. In 2006, Zhang and Zhu [67] applied successive approximations to establish the existence and uniqueness of solutions. Their focus was on SDEs with non-Lipschitz coefficients driven by the multi-parameter Brownian motion. In 2013, Wang et al. [58] studied the existence and uniqueness of initial value problems for stochastic differential equations driven by both Wiener processes and Poisson processes under local non-Lipschitz conditions. In 2014, the existence and uniqueness of solutions for neutral stochastic differential equations with Markovian switching and jumps (NSDEwMSJs) under non-Lipschitz conditions were demonstrated by Wei Mao and Xuerong Mao [35]. In 2015, Xu et al. [62] utilized the successive approximation method to investigate the existence and uniqueness of solutions to stochastic differential equations (SDEs) driven by the Lévy noise. Their focus was on non-Lipschitz conditions, which are less than classical Lipschitz conditions. This research extended to prove the stability of solutions for the same equation. In 2016, Pei and Xu [44] proved the existence and uniqueness theorem for solutions to non-Lipschitz stochastic differential equations (SDEs). Their study centred on SDEs driven by the fractional Brownian motion (fBm), with a specific focus on the Hurst parameter $H \in (\frac{1}{2}, 1)$. In 2025, Haddadi and Akdim [22] proved the existence and uniqueness of solutions to SDEs driven by optional semimartingales under non-Lipschitz conditions. Furthermore, they presented the stability conditions and the theorem that the solutions are stochastic stable in the sense of mean-square.

Regarding the published works on the stochastic flows field. In 2016, Benziadi and Kandouci [6] proved the continuity of the solution of the natural equation in the one-dimensional case with Lipschitz coefficients, based on Kolmogorov's theorem. In 2017, Benziadi and Kandouci [7] investigated the homeomorphism property of stochastic flows for the natural equation in the multi-dimensional case. In 2020, Benziadi [5] demonstrated the homeomorphism property of stochastic flows for the natural equation in the one-dimensional case and with Lipschitz coefficients. In 2021, Khatir et al. [28] made a substantial advance to the study of stochastic flows by proving the differentiability of the solution for the natural equation in the multi-dimensional case. In 2023, Khatir et al. [29] examined the same property with respect to the initial data of the solution for the natural equation but in the one-dimensional case. In 2026, Mohammed Chikouche et al. [40] investigated the bicontinuity and homeomorphism properties of stochastic flows of the solution for the natural equation in the one-dimensional case with non-Lipschitz coefficients.

Contribution of the thesis

In this thesis, we focus on a fundamental model in credit risk modeling, known as the natural model. This model is expressed by a stochastic differential equation called the "natural equation" which is considered one of the best ways to describe the evolution of the market after the default time. The following points summarize the primary aim of this thesis.

(i) Under non-Lipschitz conditions, we show the existence and uniqueness of the solution to the natural equation based on the successive approximation method. In addition, we show the continuous dependence of solutions on the initial value by means of the same equation.

(ii) Based on Kolmogorov's continuity criterion and Bihari's inequality, we study the bicontinuity and homeomorphism properties of stochastic flows of the solution for the natural equation driven by continuous local martingales in the one-dimensional case under weaker coefficients than the Lipschitz one.

Outline of thesis

This thesis is organized as follows:

Chapter one "Preliminaries": In this chapter, we recall some important inequalities and essential tools on stochastic calculus, such as stochastic processes, martingales, continuous local martingales, continuous semimartingales, etc., as well as the definition and main properties of flow in both deterministic and stochastic cases, and we give some results about the continuity and homeomorphism presented by H. Kunita [30].

Chapter two "Existence, uniqueness, and stability of solutions to natural equations under non-Lipschitz conditions": In this chapter, we present our natural model, and then we establish the existence and uniqueness of the solution to the natural equation in the one-dimensional case under non-Lipschitz conditions. Finally, we establish the mean-square stability of solutions of the natural equation with respect to the initial value that is the subject of our published research paper [39].

Chapter three "Stochastic flows for one-dimensional natural equations with non-Lipschitz coefficients": In this last chapter, we give the presentation of our problem and main hypothesis, and then we show that the solution for the one-dimensional natural equation with non-Lipschitz coefficients has a bicontinuous modification in (t, x). Finally, we establish the homeomorphism property of the solution for the same equation by following the approach introduced by Yamada and Ogura [65]. This chapter is the subject of our submitted paper [40].

Chapter 1

Preliminaries

In this chapter, we introduce the fundamental concepts that will serve as the foundation for the subsequent developments of this work. We begin by presenting several fundamental inequalities and mathematical tools that will be used throughout the subsequent chapters. We then discuss generalities on stochastic calculus. Finally, we focus on the study of stochastic flows of solutions generated by stochastic differential equations (SDEs), where we present the notion of stochastic flows and their properties, given their significance in both theoretical and applied contexts.

1.1 Fundamental inequalities and mathematical tools

In this section, we present the basic inequalities and some mathematical tools which are fundamental in proving our main results in the next chapters.

Theorem 1.1.1 (Burkholder-Davis-Gundy inequality). [32] *Let $T > 0$ and let $(N_t)_{0\leq t\leq T}$ be a continuous local martingale such that $N_0 = 0$. For every $0 < p < \infty$, there exist universal constants c_p and C_p such that*

$$c_p\mathbb{E}\left[\langle N\rangle_T^{p/2}\right] \leq \mathbb{E}\left[(N_t^*)^p\right] \leq C_p\mathbb{E}\left[\langle N\rangle_T^{p/2}\right], \tag{1.1}$$

where $N_t^* = \sup\limits_{0\leq t\leq T} |N_t|$.

Proof. We first prove the right-hand side inequality (1.1), by stopping, we can reduce to the case where N is a bounded martingale. Let $q \geq 2$. We apply Itô's formula to the function $|x|^q$:

$$|N_t|^q = \int_0^t q|N_s|^{q-1}\,\mathrm{sgn}(N_s)\,dN_s + \frac{1}{2}\int_0^t q(q-1)|N_s|^{q-2}\,d\langle N\rangle_s. \tag{1.2}$$

Consequently, by Doob's stopping theorem 1.2.1, it follows that for all bounded stopping time τ,

$$\mathbb{E}\left[|N_\tau|^q \mid \mathcal{F}_0\right] \le \frac{1}{2}q(q-1)\,\mathbb{E}\left[\int_0^\tau |N_t|^{q-2}\,d\langle N\rangle_t \ \mid\ \mathcal{F}_0\right].$$

Thus, we deduce from the Lenglart's domination inequality (1.8) that for all $\hat{r} \in (0,1)$

$$\mathbb{E}\left[\sup_{0\le t\le T}|N_t|^q\right]^{\hat{r}} \le \frac{2-\hat{r}}{1-\hat{r}}\left(\frac{1}{2}q(q-1)\right)^{\hat{r}}\mathbb{E}\left[\left(\int_0^T |N_s|^{q-2}\,d\langle N\rangle_s\right)^{\hat{r}}\right].$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T |N_s|^{q-2}\,d\langle N\rangle_s\right)^{\hat{r}}\right] &\le \mathbb{E}\left[\left(\sup_{0\le t\le T}|N_t|\right)^{\hat{r}(q-2)}\left(\int_0^T d\langle N\rangle_s\right)^{\hat{r}}\right]\\ &\le \mathbb{E}\left[\left(\sup_{0\le t\le T}|N_t|\right)^{\hat{r}q}\right]^{1-\frac{2}{q}}\mathbb{E}\left[\langle N\rangle_T^{\frac{\hat{r}q}{2}}\right]^{\frac{2}{q}}.\end{aligned}$$

Thus, we obtain

$$\mathbb{E}\Big[\sup_{0\le t\le T}|N_t|^q\Big]^{\hat{r}} \le \frac{2-\hat{r}}{1-\hat{r}}\left(\frac{1}{2}q(q-1)\right)^{\hat{r}}\mathbb{E}\Big[\big(\sup_{0\le t\le T}|N_t|\big)^{\hat{r}q}\Big]^{\left(1-\frac{2}{q}\right)}\mathbb{E}\Big[\langle N\rangle_T^{\frac{\hat{r}q}{2}}\Big]^{\frac{2}{q}}.$$

Choosing $p = \hat{r}q$, we obtain

$$\mathbb{E}\Big[\sup_{0\le t\le T}|N_t|^p\Big] \le C_p\,\mathbb{E}\Big[\langle N\rangle_T^{\frac{p}{2}}\Big].$$

We now go on to the proof of the inequality (1.1) on the left side for $p \ge 4$. We apply Itô's formula to the function $F(x) = x^2$, we get

$$N_t^2 = \langle N\rangle_t + 2\int_0^t N_s\,dN_s.$$

Consequently, using the convexity inequality (The convexity of $x \mapsto |x|^p$ gives $|x+y|^p \leq a_p\big(|x|^p + |y|^p\big)$), we obtain

$$\mathbb{E}\Big[\langle N\rangle_T^{\frac{p}{2}}\Big] \leq a_p\left(\mathbb{E}\Big[\sup_{0\leq t\leq T}|N_t|^p\Big] + \mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t N_s\,dN_s\Big|^{\frac{p}{2}}\Big]\right).$$

Using the preceding argumentation, we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t N_s\,dN_s\Big|^{\frac{p}{2}}\Big] &\leq \mathbb{E}\Big[\Big(\int_0^T N_s^2\,d\langle N\rangle_s\Big)^{\frac{p}{4}}\Big] \\
&\leq \mathbb{E}\Big[\Big(\sup_{0\leq t\leq T}|N_t|\Big)^{\frac{p}{2}}\langle N\rangle_T^{\frac{p}{4}}\Big] \\
&\leq \mathbb{E}\Big[\sup_{0\leq t\leq T}|N_t|^p\Big]^{\frac{1}{2}}\,\mathbb{E}\Big[\langle N\rangle_T^{\frac{p}{2}}\Big]^{\frac{1}{2}}.
\end{aligned}$$

Finally, we get

$$\mathbb{E}\Big[\langle N\rangle_T^{\frac{p}{2}}\Big] \leq a_p\left(\mathbb{E}\Big[\sup_{0\leq t\leq T}|N_t|^p\Big] + \mathbb{E}\Big[\sup_{0\leq t\leq T}|N_t|^p\Big]^{\frac{1}{2}}\,\mathbb{E}\Big[\langle N\rangle_T^{\frac{p}{2}}\Big]^{\frac{1}{2}}\right).$$

Set $x = \mathbb{E}\Big[\langle N\rangle_T^{\frac{p}{2}}\Big]^{\frac{1}{2}}$ and $y = \mathbb{E}\Big[\sup_{0\leq t\leq T}|N_t|^p\Big]^{\frac{1}{2}}$, so

$$x^2 - a_p xy - a_p y^2 \leq 0.$$

This inequality has the following form $x^2 \leq a_p(y^2 + xy)$, which easily implies, $c_p\,x^2 \leq y^2$, thanks to the inequality $2xy \leq \frac{1}{\zeta}x^2 + \zeta y^2$, with a suitably chosen ζ.

This completes the proof of Theorem 1.1.1. □

The following Young's inequality will be used repeatedly throughout our work.

Theorem 1.1.2 (Young's inequality)**.** [39] *For $a, b \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ one has*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1.3}$$

and the equality holds if and only if $a^p = b^q$.
For the case $p = q = 2$, *we find*

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

Proof. Since $\frac{1}{p} + \frac{1}{q} = 1$, we can write

$$p = \frac{m+n}{m}, \quad q = \frac{m+n}{n},$$

where m, n are positive integers. Let

$$a = x^{1/p}, \quad b = y^{1/q}.$$

Therefore,

$$\frac{a^p}{p} + \frac{b^q}{q} = \frac{x}{p} + \frac{y}{q} = \frac{m}{m+n}x + \frac{n}{m+n}y = \frac{mx+ny}{m+n}.$$

From the AM-GM inequality, we deduce that

$$\frac{mx+ny}{m+n} \geq (x^m y^n)^{\frac{1}{m+n}} = x^{\frac{m}{m+n}} y^{\frac{n}{m+n}} = x^{1/p} y^{1/q} = ab.$$

Consequently,

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

This completes the proof of Theorem 1.1.2. □

We recall the classical Gronwall inequality.

Lemma 1.1.1 (Gronwall's inequality)**.** [6] *Let* $\alpha \geq 0$, $\beta(t) \geq 0$, *and let* $\psi(t)$ *be a real continuous function on* $[a, b]$. *If*

$$\psi(t) \leq \alpha + \int_a^t \beta(s)\psi(s)ds, \quad \textit{for all } t \in [a, b],$$

then

$$\psi(t) \leq \alpha \exp \int_a^t \beta(s)ds, \quad \textit{for all } t \in [a, b]. \tag{1.4}$$

Proof. We define $H : [a, b] \to \mathbb{R}$ by

$$H(u) \longmapsto \left(\int_a^u \psi(s)\beta(s)\, ds\right) \exp\left(-\int_a^u \beta(s)\, ds\right).$$

Since β and ψ are continuous functions, and H is continuously derivable on $[a, b]$. Moreover, for all $u \in [a, b]$,

$$H'(u) = \psi(u)\beta(u) \exp\left(-\int_a^u \beta(u)\, ds\right) - \beta(u)\left(\int_a^u \psi(s)\beta(s)\, ds\right) \exp\left(-\int_a^u \beta(s)\, ds\right).$$

Furthermore,

$$H'(u) = \beta(u) \exp\left(-\int_a^u \beta(s)\, ds\right)\left(\psi(u) - \int_a^u \psi(s)\beta(s)\, ds\right).$$

but we have,

$$\psi(u) \leq \alpha + \int_a^u \beta(s)\psi(s)\, ds, \quad \text{for all } u \in [a, b].$$

Hence,

$$H'(u) \leq \alpha\, \beta(u) \exp\left(-\int_a^u \beta(s)\, ds\right), \quad \text{for all } u \in [a, b].$$

Let $t \in [a, b]$. Integrating this inequality for i from a to t, we obtain

$$H(t) - H(a) \leq \alpha \int_a^t \beta(u) \exp\left(-\int_a^u \beta(s)\, ds\right) du.$$

Since $H(a) = 0$, this gives

$$\begin{aligned}\left(\int_a^t \beta(s)\psi(s)\, ds\right) \exp\left(-\int_a^t \beta(s)\, ds\right) &\leq \alpha \left[-\exp\left(-\int_a^u \beta(s)\, ds\right)\right]_a^t \\ &\leq -\alpha \exp\left(-\int_a^t \beta(s)\, ds\right) + \alpha \exp(0).\end{aligned}$$

Then,

$$\left(\int_a^t \beta(s)\psi(s)\, ds\right) \leq -\alpha + \alpha \exp\left(\int_a^t \beta(s)\, ds\right).$$

Finally,

$$\psi(t) \leq \alpha \exp\left(\int_a^t \beta(s)\, ds\right).$$

This completes the proof of Lemma 1.1.1. □

The following Bihari inequality is the main tool for handling non-Lipschitz conditions.

Lemma 1.1.2 (Bihari's inequality)**.** [10] *Let $T > 0$ and $z_0 \geq 0$, and let $z(t), q(t)$ be two continuous functions on $[0,T]$. Let $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave continuous and non-decreasing function such that $\lambda(v) > 0$ for $v > 0$. If*

$$z(t) \leq z_0 + \int_0^t q(s)\lambda(z(s))\mathrm{d}s, \quad \textit{for all } 0 \leq t \leq T, \tag{1.5}$$

then for $t \in [0,T]$,

$$z(t) \leq F^{-1}\left(F(z_0) + \int_0^t q(s)ds\right), \tag{1.6}$$

so that

$$F(z_0) + \int_0^t q(s)ds \in \mathrm{Dom}\left(F^{-1}\right),$$

where $F(v) = \int_1^v \frac{ds}{\lambda(s)}, v \geq 0$ and F^{-1} is the inverse function of F.
Furthermore, if $z_0 = 0$ and $\int_{0^+} \frac{ds}{\lambda(s)} = \infty$, it follows

$$z(t) = 0, \quad \textit{for all } 0 \leq t \leq T.$$

Proof. Set

$$h(t) = z(0) + \int_0^t q(s)\, \lambda(z(s))\, ds.$$

By the non-decreasing property of λ and (1.5), we have

$$\frac{q(t)\lambda(z(t))}{\lambda(h(t))} \leq q(t).$$

Furthermore,

$$\frac{d}{dt}F(h(t)) = \frac{h'(t)}{\lambda(h(t))} = \frac{q(t)\lambda(z(t))}{\lambda(h(t))} \leq q(t).$$

Now, by integrating both sides, we obtain

$$F(h(t)) \leq F(h(0)) + \int_0^t q(s)\, ds.$$

Thus, it follows that

$$z(t) \leq h(t) \leq F^{-1}\left(F(z(0)) + \int_0^t q(s)\, ds\right).$$

This completes the proof of Lemma 1.1.2 □

The following Jensen inequality will be used frequently throughout this work when taking expectations.

Theorem 1.1.3 (Jensen's inequality)**.** [14] *Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a convex function and let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, P)$. If $\lambda(X)$ is integrable, then*

$$\lambda\big(\mathbb{E}[X]\big) \leq \mathbb{E}\big[\lambda(X)\big], \ \ a.s. \tag{1.7}$$

Proof. We suppose that λ is twice differentiable with λ'' continuous. Let $\mu = E[X]$. Expand λ in a Taylor series about μ and get

$$\lambda(x) = \lambda(\mu) + \lambda'(\mu)(x-\mu) + \frac{1}{2}\lambda''(\zeta)(\zeta-\mu)^2,$$

with ζ in between x and μ.
Since λ is convex, $\lambda'' \geq 0$, then

$$\lambda(x) \geq \lambda(\mu) + \lambda'(\mu)(x-\mu).$$

Replacing x by the random variable X, and taking the expectation, we obtain

$$\begin{aligned} E[\lambda(X)] \geq E\big[\lambda(\mu) + \lambda'(\mu)(X-\mu)\big] &= \lambda(\mu) + \lambda'(\mu)(E[X]-\mu) \\ &= \lambda(\mu) = \lambda(E[X]). \end{aligned}$$

This completes the proof of Theorem 1.1.3. □

Theorem 1.1.4 (Lenglart's inequality)**.** [14] *Let X and G be non-negative adapted right-continuous processes, and let G be in addition non-decreasing and predictable such that $\mathbb{E}[X_T \mid \mathcal{F}_0] \leq \mathbb{E}[G_T \mid \mathcal{F}_0] < \infty$ for any bounded stopping time T. Then, for all $p \in (0,1)$,*

$$\mathbb{E}\left[\left(\sup_{t\geq 0} X_t\right)^p \,\Big|\, \mathcal{F}_0\right] \leq c_p\, \mathbb{E}\left[\left(\sup_{t\geq 0} G_t\right)^p \,\Big|\, \mathcal{F}_0\right], \tag{1.8}$$

where $c_p = \frac{2-p}{1-p}$.

The following lemma gives the quadratic variation of a stochastic integral with respect to a continuous local martingale.

Lemma 1.1.3. [40] *Let Y be a continuous local martingale, and let $\langle Y \rangle$ denote its quadratic variation. If X_s is a progressively measurable process, then the quadratic variation of the stochastic integral with respect to Y_s is given by*

$$\left\langle \int_0^t X_s dY_s \right\rangle_t = \int_0^t |X_s|^2 ds. \tag{1.9}$$

Proof.

$$\int_0^t X_s dY_s = \int_0^t X_s dB_{\langle Y \rangle_s^{-1}}.$$

Making

$$\langle Y \rangle_s^{-1} = r \Rightarrow s = \langle Y \rangle_r, \quad ds = \frac{d\langle Y \rangle_r}{dr} dr.$$

Using the change of variables formula as presented in (Øksendal page 119 [53]), we derive

$$\int_0^t X_s dY_s = \int_0^t X_s dB_{\langle Y \rangle_s^{-1}} = \int_0^{\langle Y \rangle_t^{-1}} X_{\langle Y \rangle_r} \sqrt{\frac{d\langle Y \rangle_r}{dr}} dB_r.$$

Now, the quadratic variation of this term is given by

$$\left\langle \int_0^{\langle Y \rangle_t^{-1}} X_{\langle Y \rangle_r} \sqrt{\frac{d\langle Y \rangle_r}{dr}} dB_r \right\rangle = \int_0^{\langle Y \rangle_t^{-1}} |X_{\langle Y \rangle_r}|^2 \frac{d\langle Y \rangle_r}{dr} dr.$$

By changing variables in the last integral by taking $\langle Y\rangle_r = \sigma$, we have

$$r = \langle Y\rangle_\sigma^{-1} \Rightarrow \sigma = \langle Y\rangle_r\,,$$

then,

$$\left\langle \int_0^{\langle Y\rangle_t^{-1}} X_{\langle Y\rangle_r}\sqrt{\frac{d\,\langle Y\rangle_r}{dr}}dB_r \right\rangle = \int_0^{\langle Y\rangle_t^{-1}} |X_{\langle Y\rangle_r}|^2\frac{d\,\langle Y\rangle_r}{dr}dr = \int_0^t |X_\sigma|^2 d\sigma,$$

(the reason why t is when $r = \langle Y\rangle_t^{-1}$ then $\sigma = t$). So,

$$\left\langle \int_0^t X_s dY_s \right\rangle_t = \int_0^t |X_s|^2 ds.$$

This completes the proof of Lemma 1.1.3. □

The following technical lemma will be used throughout the subsequent chapters.

Lemma 1.1.4. [40] *Let N and Y be two continuous local martingales. Then for any measurable function h, we have*

$$\int_s^t |h(u)|\,|d\langle N,Y\rangle_u| \le \int_s^t |h(u)|\sqrt{d\langle N\rangle_u}\sqrt{d\langle Y\rangle_u}. \tag{1.10}$$

Proof. For any $\lambda \in \mathbb{R}$, we consider the local martingale

$$M_t = N_t + \lambda Y_t.$$

Since the quadratic variation $\langle M\rangle$ is an increasing process, it follows that

$$d\langle N + \lambda Y\rangle_u \ge 0.$$

By bilinearity of the quadratic covariation, we have

$$d\langle N\rangle_u + 2\lambda\, d\langle N,Y\rangle_u + \lambda^2\, d\langle Y\rangle_u \ge 0.$$

This is a quadratic polynomial in λ which is always non-negative. Therefore, its discriminant is non-positive

$$\Delta' = \big(d\langle N,Y\rangle_u\big)^2 - d\langle N\rangle_u\, d\langle Y\rangle_u \le 0.$$

Hence,

$$\left|d\langle N, Y\rangle_u\right| \leq \sqrt{d\langle N\rangle_u}\,\sqrt{d\langle Y\rangle_u}.$$

Multiplying by $|h(u)|$ and integrating over $[s,t]$, we obtain

$$\int_s^t |h(u)|\left|d\langle N, Y\rangle_u\right| \leq \int_s^t |h(u)|\,\sqrt{d\langle N\rangle_u}\,\sqrt{d\langle Y\rangle_u}.$$

This completes the proof of Lemma 1.1.4. □

1.2 Basic concepts of stochastic calculus

In this section, we recall the main notions and results of stochastic calculus. We first present some basic concepts related to stochastic processes, including conditional expectation, martingales, local martingales, and semi-martingales. We then introduce stochastic integration with respect to local martingales and semimartingales and discuss the fundamental Itô's formula. The section concludes with an overview of stochastic differential equations which form the basis of our study.

1.2.1 Notions on stochastic processes

A stochastic process is any process describing the evolution in time of a random phenomenon. Denote by T the set of time. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is a set, $\mathcal{F}$ is a σ-field contained in the set of parts of Ω and $\mathbb{P}$ is a probability measure.

Definition 1.2.1 (Stochastic process)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let T be an arbitrary set (called the index set). Any collection of random variables $X = \{X_t, t \in T\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a stochastic process with index set T. Therefore, the stochastic process X can be written as a function*

$$\begin{aligned} X:\ T \times \Omega &\longrightarrow \mathbb{R}, \\ (t,\omega) &\longrightarrow X_t(\omega), \quad \omega \in \Omega. \end{aligned} \tag{1.11}$$

Definition 1.2.2 (Filtration)**.** *A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t\in T}$, of sub-σ-fields of $\mathcal{F}$ i.e. $\mathcal{F}_s \subset \mathcal{F}_t$, for all $s < t$.*

Definition 1.2.3 (Adapted process)**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T}, \mathbb{P})$ be a filtered probability space. A stochastic process $(X_t)_{t\in T}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in T}$ if, for all $t \in T$, X_t is $\mathcal{F}_t$-measurable.*

Definition 1.2.4. *A stochastic process X_t on $\mathcal{J} = [0, T]$ is*

i) integrable if $\sup\limits_{t\in\mathcal{J}} \mathbb{E}\left[|X_t|\right] < \infty$,

ii) square integrable if $\sup\limits_{t\in\mathcal{J}} \mathbb{E}\left[X_t^2\right] < \infty$,

iii) uniformly integrable if $\lim\limits_{n\to\infty} \sup\limits_{t\in\mathcal{J}} \mathbb{E}\left[|X_t|\, \mathbb{1}_{|X_t|>n}\right] = 0$.

Definition 1.2.5. *A stochastic process $(\widehat{X}_t)_{t\geq 0}$ is called a modification of the process $(X_t)_{t\geq 0}$ if for every $t \geq 0$, $\mathbb{P}\left(X_t = \widehat{X}_t\right) = 1$.*

We can observe that if $(\widehat{X}_t)_{t\geq 0}$ is a modification of $(X_t)_{t\geq 0}$ then $(\widehat{X}_t)_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$.

Definition 1.2.6. *Let X_t and X'_t be two stochastic processes. We say that X_t and X'_t are indistinguishable if and only if:*

$$\mathbb{P}\big(\{\omega \in \Omega : X_t(\omega) = X'_t(\omega)\ \forall t \geq 0\}\big) = 1,$$

i.e., there is a negligible part $\mathcal{N}$ such as $\forall \omega \in \mathcal{N}$ and $\forall t \geq 0$, we have $X_t(\omega) = X'_t(\omega)$. Hence, there is an equivalence relation:

$$X \,\mathcal{R}\, X' \iff X \text{ and } X' \text{ are indistinguishable.}$$

Definition 1.2.7. *A stochastic process $\{X_t, t \geq 0\}$ is said to be continuous if $\mathbb{P}(\omega \in \Omega :$ $t \to X_t(\omega)$ is continuous$) = 1$, i.e., its sample paths are continuous a.s.*

Definition 1.2.8. *A stochastic process is said to be cádlág (resp. cáglád) if every sample path is right-continuous with left-hand limits (resp. left-continuous with right-hand limits).*

1.2.2 Conditional Expectation and Martingale

A martingale plays a fundamental role in the theory of stochastic calculus. To understand the definition, we need to define conditional expectation.

Definition 1.2.9. *A conditional expectation of a random variable X on $(\Omega, \mathcal{F}, I\!P)$ with respect to a σ-field $\mathcal{G} \subset \mathcal{F}$ is a random variable denoted by $\mathbb{E}[X \mid \mathcal{G}]$ that satisfies the following conditions:*

1. *$\mathbb{E}[X \mid \mathcal{G}]$ is measurable with respect to $\mathcal{G}$,*
2. *$\int_A X d\mathbb{P} = \int_A \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P}$ for all $A \in \mathcal{G}$.*

$\mathbb{E}[X \mid \mathcal{G}]$ called the conditional expectation of X given $\mathcal{G}$.

Proposition 1.2.1 (Properties of conditional expectation)**.** [14] *Suppose that X and X' be two random variables on $(\Omega, \mathcal{F}, I\!P)$. Suppose also that $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ are σ-fields and $a, b \in I\!R$ is a constant. The following are simple properties of the conditional expectation:*

1. *$\mathbb{E}[aX + bX' \mid \mathcal{G}] = a\mathbb{E}[X \mid \mathcal{G}] + b\mathbb{E}[X' \mid \mathcal{G}]$.*
2. *$\mathbb{E}(\mathbb{E}[X \mid \mathcal{G}]) = \mathbb{E}[X]$.*
3. *If X is independent of $\mathcal{G}$ then, $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$.*
4. *If X is $\mathcal{G}$-measurable then, $\mathbb{E}[X \mid \mathcal{G}] = X$.*
5. *$\mathbb{E}[X \mid \mathcal{H}] = \mathbb{E}(\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H})$.*
6. *If X' is $\mathcal{G}$-measurable and $\mathbb{E}[XX'] < \infty$ then, $\mathbb{E}[XX' \mid \mathcal{G}] = X'\mathbb{E}[X \mid \mathcal{G}]$.*

Definition 1.2.10 (Martingale)**.** *A martingale is a stochastic process $X = (X_t)_{t \geq 0}$ such that:*

(i) *X is adapted, i.e. X_t is $\mathcal{F}_t$-measurable for each $t \geq 0$.*

(ii) *X is integrable, i.e. $\mathbb{E}\big[|X_t|\big] < \infty$, for all $t \geq 0$.*

(iii) *$\mathbb{E}\big[X_t \,\big|\, \mathcal{F}_s\big] = X_s$, for all $0 \leq s \leq t$.*

If property (iii) holds with $\geq$ (resp. $\leq$) instead of $=$, then X is called a submartingale (resp. supermartingale).

Definition 1.2.11 (Stopping time). *A random variable $T : \Omega \to [a, b]$ is called a stopping time with respect to the filtration $\{\mathcal{F}_t; a \leq t \leq b\}$ if $\{\omega; T(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in [a, b]$.*

Theorem 1.2.1 (Doob's Stopping Theorem). [14] *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(X_t)_{t\geq 0}$ be a stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ whose paths are right-continuous and locally bounded. Then the following assertions are equivalent:*

- *$(X_t)_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$;*
- *For every almost surely bounded stopping time T with respect to $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathbb{E}(|X_T|) < +\infty$, we have $\mathbb{E}(X_T) = \mathbb{E}(X_0)$.*

1.2.3 Local martingale

We relax the integrability condition in the definition of a martingale to handle more general processes, which naturally leads to the following notion.

Definition 1.2.12 (Local martingale). *An adapted process $N = (N_t)_{t\geq 0}$ with continuous sample paths and such that $N_0 = 0$ a.s. is called a continuous local martingale if there exists a sequence $(T_n)_{n\geq 0}$ of stopping times (called a localizing sequence) with $T_n \to \infty$ almost-surely as $n \to \infty$, such that for each $n \in \mathbb{N}$, the stopped process X^{T_n} is a uniformly integrable martingale.*

Remark 1.2.1. *Any continuous bounded local martingale is a true martingale.*

Theorem 1.2.2 (Quadratic variation). [32] *Let $N = (N_t)_{t\geq 0}$ be a continuous local martingale. Then there exists a unique (up to indistinguishability) continuous adapted increasing process $(\langle N\rangle_t)_{t\geq 0}$ such that $N_t^2 - \langle N\rangle_t$ is a continuous local martingale. Furthermore, for every fixed $t > 0$, if $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is an increasing sequence of subdivisions of $[0, t]$ with mesh tending to 0, we have*

$$\langle N\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{p_n} \left(N_{t_i^n} - N_{t_{i-1}^n} \right)^2,$$

in probability.

Definition 1.2.13. *The process* $\langle N\rangle$ *is called the quadratic variation of* N.

Definition 1.2.14 (Covariation)**.** *If* N *and* Y *are two continuous local martingales, the bracket* $\langle N, Y\rangle$ *is the finite variation process defined by setting, for every* $t \geq 0$,

$$\langle N, Y\rangle_t = \frac{1}{2}\left(\langle N+Y, N+Y\rangle_t - \langle N, N\rangle_t - \langle Y, Y\rangle_t\right).$$

In particular, $\langle N, N\rangle = \langle N\rangle$.

Proposition 1.2.2. [32] *For any continuous local martingale* N *and* Y*:*

(i) $\langle N, Y\rangle$ *is the unique (up to indistinguishability) finite variation process such that* $N_t Y_t - \langle N, Y\rangle_t$ *is a continuous local martingale.*

(ii) The mapping $(N, Y) \longmapsto \langle N, Y\rangle$ *is bilinear and symmetric.*

(iii) if $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ *is an increasing sequence of subdivisions of* $[0, t]$ *with mesh tending to* 0, *we have*

$$\langle N, Y\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{p_n} \big(N_{t_i^n} - N_{t_{i-1}^n}\big)\big(Y_{t_i^n} - Y_{t_{i-1}^n}\big),$$

in probability.

(iv) For every stopping time T,

$$\langle N^T, Y^T\rangle_t = \langle N^T, Y\rangle_t = \langle N, Y\rangle_{t\wedge T}.$$

(v) If N, Y *are two martingales (with continuous sample paths) bounded in* $\boldsymbol{L}^2$, *then* $N_t Y_t - \langle N, Y\rangle_t$ *is a uniformly integrable martingale. Consequently,* $\langle N, Y\rangle_\infty$ *is well defined as the almost sure limit of* $\langle N, Y\rangle_t$ *as* $t \to \infty$, *is integrable, and satisfies*

$$\mathbb{E}[N_\infty Y_\infty] = \mathbb{E}[N_0 Y_0] + \mathbb{E}[\langle N, Y\rangle_\infty].$$

Definition 1.2.15. *Two continuous local martingales* N *and* Y *are said to be orthogonal if* $\langle N, Y\rangle = 0$, *it holds if and only if* NY *is a continuous local martingale.*

Proposition 1.2.3 (Kunita-Watanabe)**.** [32] *Let N and Y be two continuous local martingales and let H and K be two measurable processes. Then almost surely*

$$\int_0^\infty |H_s|\,|K_s|\,|\mathrm{d}\langle N,Y\rangle_s| \le \left(\int_0^\infty H_s^2\;\mathrm{d}\langle N\rangle_s\right)^{\frac{1}{2}}\left(\int_0^\infty K_s^2\;\mathrm{d}\langle Y\rangle_s\right)^{\frac{1}{2}}.$$

1.2.4 Semimartingale

Definition 1.2.16 (Semimartingale)**.** *A process $X=(X_t)_{t\geq 0}$ is a continuous semimartingale if it can be written in the form*

$$X_t = N_t + A_t,$$

where N is a continuous local martingale and A is a finite variation process.

Definition 1.2.17 (Quadratic variation)**.** *Let $X = N+A$ and $X' = N'+A'$ be the canonical decompositions of two continuous semimartingales X and X'. The bracket $\langle X, X'\rangle$ is the finite variation process defined by*

$$\langle X, X'\rangle_t = \langle N, N'\rangle_t.$$

In particular, we have $\langle X\rangle_t = \langle N\rangle_t$.

Proposition 1.2.4. [32] *Let $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ be an increasing sequence of subdivisions of $[0,t]$ whose mesh tends to 0. Then,*

$$\lim_{n\to\infty}\sum_{i=1}^{p_n}\big(X_{t_i^n} - X_{t_{i-1}^n}\big)\big(X'_{t_i^n} - X'_{t_{i-1}^n}\big) = \langle X, X'\rangle_t,$$

in probability.

1.2.5 Stochastic integration w.r.t. local martingales

We have now defined the stochastic integral for continuous local martingales.

Definition 1.2.18. *Let N be a continuous local martingale, and let $L^2_{loc}(N)$ (resp. $L^2(N)$) the set of all progressive processes H such that*

$$\int_0^t H_s^2\,d\langle N\rangle_s < \infty, \text{ for all } t \geq 0, \text{ a.s.}$$

Theorem 1.2.3. [32] *Let N be a continuous local martingale.*

(i) For every $H \in L^2_{loc}(N)$, there is a unique continuous local martingale $H \cdot N$ with $(H \cdot N)_0 = 0$ such that, for every continuous local martingale Y,

$$\langle H \cdot N, Y\rangle = H \cdot \langle N, Y\rangle.$$

(ii) If T is a stopping time, we have

$$(\mathbb{1}_{[0,T]} H) \cdot N = (H \cdot N)^T = H \cdot N^T.$$

(iii) If $H \in L^2_{loc}(N)$ and K is a progressive process, we have $K \in L^2_{loc}(H \cdot N)$ if and only if $HK \in L^2_{loc}(N)$, and then

$$K \cdot (H \cdot N) = (KH) \cdot N.$$

1.2.6 Stochastic integration w.r.t. semimartingales

We now turn to the extension of stochastic integrals to continuous semimartingales.

Definition 1.2.19 (Locally bounded previsible process)**.** *A previsible process H is locally bounded if for all $t \geq 0$, we have*

$$\sup_{s \leq t} |H_s| < \infty, \ \ a.s.$$

In particular, any adapted continuous process is locally bounded. If H is progressive and locally bounded and A is a finite variation process, then for all $t \geq 0$, we have

$$\int_0^t |H_s|\,|dA_s| < \infty, \ \ a.s.$$

Definition 1.2.20 (Stochastic integral)**.** *Let $X = N + A$ be a continuous semimartingale with its canonical decomposition and H a locally bounded previsible process. Then the stochastic integral $H \cdot X$ is the continuous semimartingale defined by*

$$H \cdot X = H \cdot N + H \cdot A,$$

in other words, we write

$$(H \cdot X)_t = \int_0^t H_s\, dX_s.$$

Proposition 1.2.5 (Tanaka's Formula)**.** [32] *Let X be a continuous semimartingale. For any real number a, there exists an increasing continuous process ℓ^a called the local time of X in a such that,*

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\, dX_s + \ell_t^a(X), \tag{1.12}$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbb{1}_{\{X_s > a\}}\, dX_s + \frac{1}{2}\ell_t^a(X), \tag{1.13}$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t \mathbb{1}_{\{X_s \le a\}}\, dX_s + \frac{1}{2}\ell_t^a(X). \tag{1.14}$$

Furthermore, for every stopping time T, we have $\ell_t^a\big(X^T\big) = \ell_{t\wedge T}^a(X)$.

1.2.7 Itô formula

Itô's formula lies at the heart of stochastic calculus, and it is the tool to calculate the stochastic integrals without approximating. We present first the integration by parts.

Theorem 1.2.4 (Integration by parts)**.** [32] *Let X, X' be continuous semimartingales. Then almost surely,*

$$X_t X'_t = X_0 X'_0 + \int_0^t X_s\, dX'_s + \int_0^t X'_s\, dX_s + \langle X, X' \rangle_t.$$

The last term is called the Itô correction.

Remark 1.2.2. *In particular, if $X' = X$,*

$$X_t^2 = X_0^2 + 2\int_0^t X_s\, dX_s + \langle X \rangle_t. \tag{1.15}$$

When $X = N$ is a continuous local martingale, by the definition of the quadratic variation (Theorem 1.2.2)*, the process $N_t^2 - \langle N \rangle_t$ is a continuous local martingale. The previous formula (1.15) shows that this continuous local martingale is*

$$N_0^2 + 2\int_0^t N_s\, dN_s.$$

Theorem 1.2.5 (Itô's formula)**.** [32] *Let X be a continuous semimartingale, and let f be $\mathcal{C}^2$ on $\mathbb{R}$. Then $f(X)$ is a continuous semimartingale as well and it holds that*

$$f\left(X_t\right)=f\left(X_0\right)+\int_0^t f'(X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^t f''(X_s)\mathrm{d}\langle X\rangle_s,\ \textit{for every } t\geq 0,\ \textit{a.s.} \tag{1.16}$$

Remark 1.2.3. *The following multivariate extension of the Itô formula* (1.16) *holds true. Let $\left(X^1,\ldots,X^d\right)$ be continuous semimartingales, and let $f:\mathbb{R}^d\to\mathbb{R}$ be $\mathcal{C}^2$ Then, for every $t\geq 0$,*

$$f\left(X_t\right)=f\left(X_0\right)+\sum_{i=1}^{d}\int_0^t\frac{\partial f}{\partial x_i}(X_s)\mathrm{d}X_s^i+\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t\frac{\partial^2 f}{\partial x_i\partial x_j}(X_s)\mathrm{d}\left\langle X^i,X^j\right\rangle_s. \tag{1.17}$$

1.2.8 Stochastic differential equations

Stochastic differential equations constitute a powerful mathematical framework for modeling and analysing systems subject to random fluctuations. They have become an essential tool in many fields, including finance, physics, biology, and engineering, where uncertainty and noise play a fundamental role. In particular, stochastic differential equations are widely used to describe phenomena such as fluctuating financial markets, physical systems influenced by random perturbations, biological processes, and various engineering applications.

The aim of stochastic differential equations is to study differential equations that are influenced by random effects or noise. More precisely, Consider an ordinary differential equation of the form

$$\dot{x}(t)=b(x(t)). \tag{1.18}$$

If we take random perturbations of the system (1.18) into account, we add a noise term, which is typically of the form σdB_t. This leads to a stochastic differential equation of the form

$$dx_t=b(x_t)\,dt+\sigma dB_t,$$

where $(B_t)_{t\geq 0}$ is Brownian motion and σ is a constant corresponding to the intensity of the noise.

Or, equivalently, in the integral form which has a precise mathematical meaning,

$$x_t = x_0 + \int_0^t b(x_s)\, ds + \sigma B_t. \tag{1.19}$$

By allowing σ to depend on the state of the system (1.19) at time t, we have

$$x_t = x_0 + \int_0^t b(x_s)\, ds + \int_0^t \sigma(x_s) dB_s.$$

Definition 1.2.21 (Stochastic differential equation)**.** *Let d and m be positive integers, and let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be locally bounded and measurable. A solution to the stochastic differential equation $E(\sigma, b)$ given by*

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t,$$

consists of:

(i) a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ obeying the usual conditions,

(ii) an m-dimensional Brownian motion B with $B_0 = 0$,

(iii) an $(\mathcal{F}_t)$-adapted continuous process X with values in $\mathbb{R}^d$ with continuous sample paths, such that

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds.$$

Remark 1.2.4. *If $X_0 = x \in \mathbb{R}^d$, then we say that X is a solution of $E_x(\sigma, b)$. Moreover, X is said to be a strong solution if it is adapted with respect to the canonical filtration of B.*

Definition 1.2.22 (Existence and uniqueness for SDEs)**.** *For the stochastic differential equation $E(\sigma, b)$, we say that there is*

- *Weak existence if, for every $x \in \mathbb{R}^d$, there exists a solution of $E_x(\sigma, b)$.*

- *Weak uniqueness if, for every $x \in \mathbb{R}^d$, all solutions of $E_x(\sigma, b)$ have the same law.*
- *Pathwise uniqueness if, whenever the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and the $(\mathcal{F}_t)$-Brownian motion B are fixed, any two solutions X and X' such that $X_0 = X'_0$ a.s. are indistinguishable.*

The central result in the study of stochastic differential equations concerns the existence and uniqueness of solutions under suitable assumptions. Before stating this theorem, we introduce the following Lipschitz conditions.

Definition 1.2.23 (Lipschitz coefficients)**.** *The coefficients $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d, \sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Lipschitz in variable x if there exists a constant $K > 0$ such that, for all $t \geq 0$ and $x, y \in \mathbb{R}^d$,*

$$|b(t,x) - b(t,y)| \leq K|x-y|,$$

and

$$|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|.$$

Remark 1.2.5. *To ensure the existence of the solution, it is usually assumed that the coefficients satisfy the linear growth condition*

$$|\sigma(t,x)| \leq K(1+|x|), \qquad |b(t,x)| \leq K(1+|x|),$$

for some constant $K > 0$ and $x \in \mathbb{R}^d$.

Theorem 1.2.6. [32] *Assume that b and σ are Lipschitz in x. Then there is pathwise uniqueness for the $E(\sigma, b)$ and for every choice of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and every $(\mathcal{F}_t)$-Brownian motion B, for every $x \in \mathbb{R}^d$, there exists a unique strong solution to $E_x(\sigma, b)$.*

Remark 1.2.6. *The proof of Theorem* 1.2.6 *is based on the Picard approximation method and some inequalities, such as the Burkholder-Davis-Gundy inequality and Gronwall's inequality* (see [32]).

Remark 1.2.7. *In the case of Lipschitz coefficients, the existence and uniqueness theorem can be obtained by Gronwall's inequality. The situation changes completely, however, in the non-Lipschitz case since the use of Gronwall's inequality meets an obvious difficulty. So, we use another inequality which is the generalization of the Gronwall-Bellman type inequality, so-called Bihari's inequality.*

1.3 Stochastic flows

In this section, we introduce the notion of flows in both the deterministic and stochastic cases. We then investigate some fundamental properties of stochastic flows with particular emphasis on the continuity and homeomorphism properties of the solution for SDEs with respect to the initial condition. The results concerning the continuity and homeomorphism of stochastic flows on $\mathbb{R}^d$ that we will present in this section are borrowed from [30].

1.3.1 Definition of flows in the deterministic case

In mathematics, the concept of flow originates from the study of ordinary differential equations and continuous dynamical systems. Let $f(x) = \big(f^1(x), \dots, f^d(x)\big)$ be a Lipschitz continuous $\mathbb{R}^d$-valued function (or vector field on $\mathbb{R}^d$), and let $X_t(x)$ be the solution of the ordinary differential equation starting from $x \in \mathbb{R}^d$ at time 0:

$$\frac{dX_t}{dt} = f(X_t), \quad X_0 = x.$$

This solution satisfies the following properties:

1. For each t, $X_t : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a homeomorphism.
2. $X_s \circ X_t = X_{s+t}$ for each s, t.
3. $(t, x) \longmapsto X_t(x)$ gives a continuous map from $\mathbb{R} \times \mathbb{R}^d$ onto $\mathbb{R}^d$.

This family $\{X_t\}$ is called a flow of homeomorphisms generated by the vector field f.

1.3.2 Definition of flows in the stochastic case and their properties

In stochastic analysis, stochastic flows of homeomorphisms (or simply flows) play an important role in the study of stochastic differential equations. They describe the evolution of random dynamical systems and provide information on how solutions depend on the initial conditions. In many situations, they can be viewed as the random counterpart of the flows generated by ordinary differential equations. More specifically, let $X_{s,t}(x,\omega), s,t \in [0,T], x \in \mathbf{R}^d$ be a continuous $\mathbb{R}^d$-valued random field defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for almost all ω, $X_{s,t}(\omega) \equiv X_{s,t}(.,\omega)$ defines a continuous map from $\mathbf{R}^d$ into itself for any s,t such that $s < t$. The first represents the initial time of the flow, and the second represents the state of the flow. The family $\{X_{s,t}(\omega) : s,t \in [0,T]\}$ defines a flow of homeomorphisms, i.e., it satisfies the following properties:

1. For any $x \in \mathbb{R}^d, X_{s,t}(\omega)$ is continuous.

2. The map $X_{s,t}(\omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a homeomorphism for any s,t.

3. $X_{s,s}(\omega) = Id_{\mathbb{R}^d}$ for any s.

4. $X_{s,u}(\omega) = X_{t,u}(\omega) \circ X_{s,t}(\omega)$ holds for all s,t,u, where $\circ$ denotes the composition of maps.

5. $X_{s,t}(x,\omega)$ is k-times differentiable with respect to x for all $s,t \in \mathbb{R}^d$ and the derivatives are continuous in (s,t,x)

Furthermore, if $X_{s,t}(\omega)$ satisfies properties (2) and (5), then it is called a stochastic flow of C^k–diffeomorphisms.

1.3.3 Continuity of the solution w.r.t the initial data

To establish the continuity properties of the flow, we shall use Kolmogorov's continuity criterion. Although the proofs are somewhat technical, they mainly rely on classical inequalities such as Hölder's inequality, Burkholder-Davis-Gundy inequality, and Gronwall's inequality, which together allow us to obtain suitable estimates on the solutions and deduce their continuity properties.

Let $f_0(t,x),\dots,f_m(t,x), t\in[0,T],\ x\in\mathbb{R}^d$, be $\mathbb{R}^d$-valued functions continuous in (t,x) and Lipschitz continuous in x. Let $B_t^k=(B_t^1,\dots,B_t^m), k=1,\dots,m$ be an m-dimensional Brownian motion. Consider the following stochastic differential equation on $\mathbb{R}^d$:

$$dX_t = f_0(t,X_t)\,dt + \sum_{k=1}^{m} f_k(t,X_t)\,dB_t^k. \tag{1.20}$$

Remark 1.3.1. *The existence and uniqueness of the solution of the Equation* (1.20) *were proved by H. Kunita, assuming that the coefficients $f_0,\dots,f_m$ are Lipschitz continuous* (see [30])*.*

Henceforth, let $X_{s,t}(x)$ the solution of the stochastic differential equation with globally Lipschitz continuous coefficients starting from (s,x)

$$X_{s,t}(x) = x + \sum_{k=0}^{m}\int_s^t f_k\big(r,X_{s,r}(x)\big)\,dB_r^k. \tag{1.21}$$

We begin by recalling the well-known Kolmogorov's continuity criterion for stochastic processes in $\mathbb{R}^d$.

Theorem 1.3.1. [30] *Let $X_u(\omega)$ be a real-valued random field with parameter $u=(u_1,\dots,u_d)\in\Lambda=[0,1]^d$. Suppose that there exist constants $p>0$, $\alpha_i>d$, $i=1,\dots,d$, and $C_p>0$ such that*

$$\mathbb{E}[|X_u-X_v|^p] \le C_p\sum_{i=1}^{d}|u_i-v_i|^{\alpha_i}, \qquad \textit{for all } u,v\in\Lambda. \tag{1.22}$$

Then X_u has a continuous modification $\widehat{X}_u$.

The goal is to show that the solution $X_{s,t}(x)$ admits a continuous modification. Our approach relies on the following theorem.

Theorem 1.3.2. [30] *For any $p\ge 2$, there exists a positive constant C_1^p such that*

$$\mathbb{E}\big[|X_{s,t}(x)-X_{s',t'}(x')|^p\big] \le C_1^p\left[|x-x'|^p+\big(1+|x|^p+|x'|^p\big)\big(|t-t'|^{p/2}+|s-s'|^{p/2}\big)\right]. \tag{1.23}$$

This estimate holds for all (s, t, x) and (s', t', x') such that $s < t$ and $s' < t'$.

Proof. Without loss of generality, assume that $s < s' < t < t'$. Then,

$$X_{s',t'}(x') = x' + \sum_{k=0}^{m} \int_{s'}^{t} f_k\big(r, X_{s',r}(x')\big)\, dB_r^k + \sum_{k=0}^{m} \int_{t}^{t'} f_k\big(r, X_{s',r}(x')\big)\, dB_r^k,$$

$$X_{s,t}(x) = X_{s,s'}(x) + \sum_{k=0}^{m} \int_{s'}^{t} f_k\big(r, X_{s,r}(x)\big)\, dB_r^k.$$

Therefore,

$$|X_{s,t}(x) - X_{s',t'}(x')|^p \le (2m+3)^p \Bigg\{ \sum_{k=0}^{m} \left| \int_{t}^{t'} f_k\big(r, X_{s',r}(x')\big)\, dB_r^k \right|^p + |X_{s,s'}(x) - x'|^p$$
$$+ \sum_{k=0}^{m} \left| \int_{s'}^{t} \Big(f_k\big(r, X_{s,r}(x)\big) - f_k\big(r, X_{s',r}(x')\big) \Big)\, dB_r^k \right|^p \Bigg\}.$$

Consequently, it suffices to establish the following estimates:

$$\mathbb{E}\left[\left| \int_{t}^{t'} f_k\big(r, X_{s',r}(x')\big)\, dB_r^k \right|^p \right] \le C_2^p\, |t - t'|^{p/2} \big(1 + |x'|^p\big), \tag{1.24}$$

$$\mathbb{E}\left[|X_{s,s'}(x) - x'|^p \right] \le C_3^p \left[|x - x'|^p + |s - s'|^{p/2} (1 + |x|^p) \right], \tag{1.25}$$

$$\mathbb{E}\left[\left| \int_{s'}^{t} \Big(f_k(r, X_{s,r}(x)) - f_k(r, X_{s',r}(x')) \Big)\, dB_r^k \right|^p \right] \le C_4^p \left[|x - x'|^p + |s - s'|^{p/2} (1 + |x|^p) \right]. \tag{1.26}$$

For the proof of the estimates (1.24), (1.25) and (1.26) see [30]. □

Theorem 1.3.3. [30] *There exist modifications of the solution $X_{s,t}(x)$ and of the stochastic integrals $\int_s^t f_k\big(r, X_{s,r}(x)\big)\, dB_r^k$ with $k = 0, \dots, m$ are continuous in (s, t, x), and the Equation* (1.21) *holds for every s, t, x a.s. Furthermore, the solution $X_{s,t}(x)$ is (α, β)-Hölder continuous in (s, t, x), where α is an arbitrary number less than $\frac{1}{2}$ and β is an arbitrary number less than 1.*

Proof. Assuming that (1.23) holds, then by Kolmogorov's theorem 1.3.1, $X_{s,t}(x)$ has a modification which is locally (α, β)-Hölder continuous in (s,t,x), where $\alpha < p^{-1}(\frac{p}{2} - d)$ and $\beta < 2p^{-1}(\frac{p}{2} - d)$.

We show now the continuity of the stochastic integral $\int_s^t f_k\big(r, X_{s,r}(x)\big)\, dB_r^k$. The case $k = 0$ being immediate, we only consider the case $k \geq 1$. Assume that $s < s' < t < t'$. Then

$$\begin{aligned}\int_s^t f_k\big(r, X_{s,r}(x)\big)\, dB_r^k - \int_{s'}^{t'} f_k\big(r, X_{s',r}(x')\big)\, dB_r^k = &\int_s^{s'} f_k\big(r, X_{s,r}(x)\big)\, dB_r^k \\ &+ \int_{s'}^t \Big(f_k\big(r, X_{s,r}(x)\big) - f_k\big(r, X_{s',r}(x')\big)\Big)\, dB_r^k \\ &- \int_t^{t'} f_k\big(r, X_{s',r}(x')\big)\, dB_r^k.\end{aligned} \tag{1.27}$$

We recall that the L^p-estimates for the first and third terms are given in (1.24) and the L^p-estimate of the second term is given by (1.26). Hence, the L^p-norm of the left-hand side of (1.27) is dominated by a term of the same type as the right-hand side of (1.23). Therefore, the stochastic integral $\int_s^t f_k\big(r, X_{s,r}(x)\big)\, dB_r^k$, $k = 1, \ldots, m$ have the same type of continuity as that of $X_{s,t}(x)$.

This completes the proof of Theorem 1.3.3 □

1.3.4 Stochastic flow of homeomorphisms for SDEs

The main objective is to show that for almost every ω, the map $X_{s,t}(\cdot, \omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a homeomorphism. We begin by stating the main theorem and then prove that the map is bijective (one-to-one and onto) and that the inverse map is continuous.

Theorem 1.3.4. [30] *Let $X_{s,t}(x)$ admit a modification which is continuous in three variables (s,t,x) a.s. Then, for any $s < t$, $X_{s,t}(\cdot, \omega)$ defines a continuous map $\mathbb{R}^d \longrightarrow \mathbb{R}^d$ for almost every ω, $X_{s,t}(\cdot, \omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ defines a stochastic flow of homeomorphisms.*

To prove Theorem 1.3.4 we need to show first that the map $X_{s,t}(\cdot, \omega)$ is bijective.

• **Injectivity (one-to-one property)**: We first prove the one-to-one property of the map $X_{s,t}(\cdot,\omega)$.

Theorem 1.3.5 (Injectivity)**.** [30] *The map $X_{s,t}:\mathbb{R}^d\to\mathbb{R}^d$ is one-to-one for any $s<t$ a.s.*

To go forward in the proof of Theorem 1.3.5, let us first establish some helpful lemmas.

Lemma 1.3.1. [30] *Let p be any real number and $\varepsilon>0$. Then there is a positive constant C_5^p independent of ε such that*

$$\mathbb{E}\Big[\big(\varepsilon+|X_{s,t}(x)-X_{s,t}(y)|^2\big)^p\Big]\le C_5^p\left(\varepsilon+|x-y|^2\right)^p, \tag{1.28}$$

holds for any $s<t$ and $x,y\in\mathbb{R}^d$.

Remark 1.3.2. *Let ε tend to zero in Lemma* 1.3.1*. Then, we have*

$$\mathbb{E}|X_{s,t}(x)-X_{s,t}(y)|^{2p}\le C_5^p|x-y|^{2p}. \tag{1.29}$$

From this inequality in the case $p<0$ we get that if $x\neq y$, than $X_{s,t}(x)\neq X_{s,t}(y)$ a.s. for any $s<t$. Furthermore, H. Kunita calls this property "weak injectivity" (see [30]).

To prove that the map $X_{s,t}(\cdot,\omega)$ is one-to-one a.s., we need the following lemma.

Lemma 1.3.2. [30] *Define*

$$\zeta_{s,t}(x,y)=\frac{1}{|X_{s,t}(x)-X_{s,t}(y)|},\ \text{for all } s<t.$$

Then for all $p>2$ there exists a constant C_6^p such that for any $\eta>0$

$$\mathbb{E}\big[|\zeta_{s,t}(x,y)-\zeta_{s',t'}(x',y')|^p\big]\le C_6^p\,\eta^{-2p}\Big[|x-x'|^p+|y-y'|^p+\big(1+|x|^p+|x'|^p+|y|^p+|y'|^p\big)\times\big(|t-t'|^{p/2}+|s-s'|^{p/2}\big)\Big]. \tag{1.30}$$

holds for any x,y,x',y' such that $|x-x'|\ge\eta$, $|y-y'|\ge\eta$.

By using the previous lemmas, we can now demonstrate Theorem 1.3.5.

Proof of Theorem 1.3.5. Choose p as large as such that $\frac{p}{2} > 2(d+1)$ in Lemma 1.3.2. By Kolmogorov's theorem 1.3.1, the random field $\zeta_{s,t}(x,y)$ is continuous in the variables (s,t,x,y) in the domain $\{(s,t,x,y) \mid s<t, \; |x-y| \geq \eta\}$. Since η is arbitrary, it follows that it is also continuous in the domain $\mathcal{H} = \{(s,t,x,y) \mid s<t, \; x \neq y\}$. Therefore, if $x \neq y$, then $X_{s,t}(x) \neq X_{s,t}(y)$ a.s. for any $s<t$. Consequently, the map $X_{s,t} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is one-to-one for any $0 < s < t < T$ a.s.
This completes the proof of Theorem 1.3.5. $\square$

• **Surjectivity (onto property)**: We now prove the onto property of the map $X_{s,t}(\cdot,\omega)$.

Theorem 1.3.6 (Surjectivity)**.** [30] *The map $X_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ is onto for any $s<t$ a.s.*

Before proving Theorem 1.3.6, we need the following useful lemma.

Lemma 1.3.3. [30] *Let $\widehat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^d$, set*

$$\widehat{x} = \frac{x}{|x|^2}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Define for every $s<t$

$$\beta_{s,t}(\widehat{x}) = \begin{cases} \dfrac{1}{1+\left|X_{s,t}(x)\right|}, & \text{if } \widehat{x} \in \mathbb{R}^d, \\ 0, & \text{if } \widehat{x} = 0. \end{cases}$$

Moreover, for any positive p there exists a constant C_7^p such that

$$\mathbb{E}\big[\left|\beta_{s,t}(\widehat{x}) - \beta_{s',t'}(\widehat{y})\right|^p\big] \leq C_7^p \Big(|\widehat{x}-\widehat{y}|^p + |t-t'|^{p/2} + |s-s'|^{p/2}\Big).$$

By using the previous lemma, we can now prove Theorem 1.3.6.

Proof of Theorem 1.3.6. Choose $p > 2(d+3)$. An application of Kolmogorov's theorem 1.3.1 yields $\beta_{s,t}(\widehat{x})$ is continuous at $\widehat{x} = 0$. Then, $X_{s,t}(\cdot,\omega)$ can be extended to a continuous map from $\widehat{\mathbb{R}}^d$ into itself for any $s<t$ a.s. This extension $\widehat{X}_{s,t}(x,\omega)$ is continuous in (s,t,x)

a.s. Let ω be fixed, the map $\widehat{X}_{s,t}(\cdot,\omega) : \widehat{\mathbb{R}}^d \longrightarrow \widehat{\mathbb{R}}^d$ is homotopically equivalent to the identity map $\widehat{X}_{s,s}(\cdot,\omega)$. Consequently, it is an onto map by a classical homotopic theory theorem. Since $\widehat{X}_{s,t}(\infty,\omega) = \infty$, the restriction of $\widehat{X}_{s,t}(\cdot,\omega)$ in $\mathbb{R}^d$ is also an onto map.
This completes the proof of Theorem 1.3.6. $\square$

We combine the previous results to prove the main theorem 1.3.4.

Proof of Theorem 1.3.4. Since the continuous map $X_{s,t}(\cdot,\omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is one-to-one and onto according to Theorem 1.3.5 and Theorem 1.3.6 (respectively). Then the inverse map $X^{-1}_{s,t}(\cdot,\omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is also one-to-one and onto. We now show that it is continuous. Indeed, since the map $\widehat{X}_{s,t}(\cdot,\omega)$ is a one-to-one and continuous from the compact space $\widehat{\mathbb{R}}^d$ into itself. Then, the inverse map $\widehat{X}^{-1}_{s,t}(\cdot,\omega) : \widehat{\mathbb{R}}^d \longrightarrow \widehat{\mathbb{R}}^d$ is continuous.
As a result, for almost all ω, the map $X_{s,t}(\cdot,\omega) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a homeomorphism for all $s < t$.
This completes the proof of Theorem 1.3.4. $\square$

1.4 Concluding remarks

In this chapter, we introduced the main concepts and tools that will be used throughout this thesis. The first part was devoted to recalling some fundamental inequalities and mathematical tools that are essential for the next chapters. In the second part, we presented the basic notions of stochastic calculus, which provide the theoretical framework of our study. In the last part, we discussed the notion of flows in the deterministic case and stochastic case and their main properties. These preliminary results lay the groundwork for the developments that follow.

Chapter 2

Existence, uniqueness and stability of solutions to natural equations under non-Lipschitz conditions

The objective of this chapter is to investigate the existence and uniqueness of solutions to the natural equation (2.4) with non-Lipschitz conditions, which is a much weaker condition than the Lipschitz one, by using a successive approximation method. Furthermore, we examine the mean-square stability of solutions on the initial value. This work contributes to the theory of stochastic differential equations (SDEs), showing that the solutions exist and are unique under weaker conditions.

This chapter is organized as follows. In Section 2.1, we describe the natural model. In Section 2.2, under specific assumptions we imposed, we prove the existence and uniqueness theorem of the solution to the natural equation (2.4) under non-Lipschitz conditions. In Section 2.3, we discuss the mean-square stability of solutions for the natural equation (2.4). We conclude this chapter with concluding remarks.

2.1 Description of the natural model

In this section, we are interested in a famous model in finance, especially in the realm of credit risk modeling. This model is known as the natural model. It is a one-default model that gives the conditional law of a random time with respect to a reference filtration which represents market information and is widely applied in financial risk modeling and in price valuation of financial products like Credit Default Swaps (CDS).

The credit risk is the risk associated with losses incurred by the lender from the failure of a borrower to make timely and full payments of interest or principal, leading to financial losses for the lender. To deal with this risk, two main models are commonly used: the structural models and the intensity models. The structure models is based on modeling the evolution of the total value of a firm's assets. The default time is considered a predictable event, and it is modeled as the first passage of a stochastic process with a barrier. On the other hand, in the intensity models the default time is considered an unpredictable event and is modeled as the first jump time of a homogeneous Poisson process. Furthermore, the two models have been studied before the default time but it is important to study what happens after the default time and it is essential to analyze the impact of default on the counterparty and the rest of the market. These models did not address these problems adequately.

The effectiveness of a one-default model depends on how the conditional laws of τ can be computed with respect to the filtration $\mathbb{F}$. The most commonly used examples of random times include the independent time, honest time, Cox time, pseudo-stopping time, initial time, etc. (for example, [8], [9], [15], [25], [26]). A new class of random times has been presented by M. Jeanblanc and S. Song (see [27]). Specifically, there exists a random variable τ and a probability measure $\mathbb{Q}$ within an extension of the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, such that the family of conditional expectations $dX_t^u = \mathbb{Q}\left[\tau \leq u/\mathcal{F}_t\right], 0 < u, t < \infty$, satisfies the following one-dimensional stochastic differential equation:

$$(\natural_u) : \begin{cases} dX_t = X_t\left(-\frac{e^{-\Lambda_t}}{1-Z_t}dN_t + f(X_t - (1-Z_t))dY_t\right), & t \in [u, \infty), \\ X_u = x, \end{cases} \tag{2.1}$$

where

x is the initial condition that is $\mathcal{F}_u$-measurable random variable,

Λ is a given continuous increasing process null at the origin,

Y is a continuous local martingale,

N is a continuous non-negative local martingale,

Z is an Azéma supermartingale such that $Z_t = N_t e^{-\Lambda_t}$ satisfies $0 < Z_t < 1$, $t > 0$,

f is a Lipschitz function on $\mathbb{R}$ satisfying $f(0) = 0$.

Equation (2.1), called the natural equation, is cited in references ([5], [6], [27], [29], [39]). It is worth noting that the financial models are defined by stochastic differential equations, which are one of the most efficient ways to represent the evolution of a financial market. Moreover, the natural equation (2.1) admits a rich system of parameters (Z, Y, f). The parameter Z defines the default intensity. The parameters Y and f describe the evolution of the market after the default time τ. This system of parameters establishes a favorable framework for deducing market behavior and refining the calibration of financial data.

Then, the integral form of Equation (2.1) can be expressed as follows

$$X_t = x + \int_u^t X_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_s} \right) dN_s + \int_u^t X_s f(X_s - (1 - Z_s)) dY_s, \quad s \in [u, t], \tag{2.2}$$

with the initial value $X_u = x$.

This result is from [27], which enables a comparison between a submartingale $1 - Z$ and a local martingale.

Lemma 2.1.1. [27] *Let $u \in \mathbb{R}_+^*$ be fixed. Let X be a $(\mathbb{P}, \mathbb{F})$ local martingale on $[u, \infty)$ such that $X_u \leq 1 - Z_u$. Furthermore, $X_t \leq 1 - Z_t$ for $t \in [u, \infty)$ if and only if the local time at zero $\ell^0(X - (1 - Z))$ of $X - (1 - Z)$ on $[u, \infty)$ is identically null. Here the local time is taken right continuous in the space variable a, i.e., the map $a \to \ell_t^a(X - (1 - Z))$ is right continuous.*

Proof. (a) Sufficient condition. Let $\Delta = X - (1 - Z)$. If $\ell^0(\Delta) \equiv 0$, for any $\mathbb{F}$-stopping time $T \geq u$ such that any quantity in the following computation is integrable and that local martingales stopped at time T are uniformly integrable martingales, by applying Tanaka's formula 1.2.5, we obtain

$$\mathbb{E}_{\mathbb{P}} \left[(\Delta_T)^+ \right] = \mathbb{E}_{\mathbb{P}} \left[(\Delta_u)^+ \right] + \mathbb{E}_{\mathbb{P}} \left[\int_u^T \mathbb{1}_{\{\Delta_s > 0\}} \mathrm{d} \left(\Delta_s \right) \right] + \frac{1}{2} \mathbb{E}_{\mathbb{P}} \left[\ell_T^0(\Delta) \right] \leq 0,$$

since Δ is a $(\mathbb{P}, \mathbb{F})$ supermartingale and its martingale part is a uniformly integrable martingale, by the choice of T. This proves that the condition is sufficient.

(b) Necessary condition. If $\Delta \leq 0$ on $t \in [u, \infty)$, we calculate the predictable bracket $\langle \Delta \rangle$ by applying Itô's formula

$$d\Delta_t = -\Delta_t \frac{e^{-\Lambda_t}}{1 - Z_t} dN_t + X_t f(\Delta_t) \, dY_t - Z_t d\Lambda_t.$$

Moreover,

$$\begin{aligned} d\langle \Delta \rangle_t &= \Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d\langle N \rangle_t + X_t^2 f(\Delta_t)^2 \, d\langle Y \rangle_t - 2\Delta_t \frac{e^{-\Lambda_t}}{1 - Z_t} X_t f(\Delta_t) \, d\langle N, Y \rangle_t \\ &\leq 2\Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d\langle N \rangle_t + 2X_t^2 f(\Delta_t)^2 \, d\langle Y \rangle_t \\ &\leq 2\Delta_t^2 \left(\frac{e^{-\Lambda_t}}{1 - Z_t} \right)^2 d\langle N \rangle_t + 2X_t^2 K^2 \Delta_t^2 d\langle Y \rangle_t. \end{aligned}$$

It follows that, for $0 < \epsilon$ and $0 < t < \infty$, one has

$$\int_0^t \mathbb{1}_{\{0 < \Delta_s < \epsilon\}} \frac{1}{\Delta_s^2} \; \mathrm{d}\langle \Delta \rangle_s < \infty.$$

Tanaka's formula 1.2.5 implies that $\ell^0(\Delta) \equiv 0$.

This completes the proof of Lemma 2.1.1. □

2.1.1 The multi-dimensional natural equation

In 2017, F. Benziadi and A. Kandouci (see [7]) introduced the natural equation in higher dimensions. Let $(\Lambda_1, \ldots, \Lambda_d)$ be d-dimensional continuous increasing process null at the origin, and let N be a d-dimensional continuous non-negative local martingale such that $0 < N_t e^{-\Lambda_t} < 1, t > 0$. Let F be a continuous Lipschitz mapping from $\mathbb{R}^d$ into itself and $Y(t, \omega) = (Y_1(t, \omega), \ldots, Y_{\mathrm{r}}(t, \omega))$ denote r-dimensional continuous local martingale defined on a probability space $\left(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$. We consider the multi-dimensional version of the natural equation defined as follows:

$$
(\natural_u) : \begin{cases} dX_1(t) = X_1(t)\left(-\dfrac{e^{-\Lambda_1(t)}}{1-Z_1(t)}\, dN_1(t) + F_{11}\, dY_1 + \cdots + F_{1d}\, dY_r\right), \\ \vdots \\ dX_d(t) = X_d(t)\left(-\dfrac{e^{-\Lambda_d(t)}}{1-Z_d(t)}\, dN_d(t) + F_{r1}\, dY_1 + \cdots + F_{rd}\, dY_r\right). \end{cases}
$$

Thus,

$$
(\natural_u) : \begin{cases} dX_t = X_t\left(-\frac{e^{-\Lambda_t}}{1-Z_t} dN_t + F(X_t - (1-Z_t))dY_t\right), \quad t \in [u, \infty), \\ X_u = x, \end{cases} \tag{2.3}
$$

where

$$
X(t) = (X_1(t), \ldots, X_d(t))^{\mathbf{T}}, -\frac{e^{-\Lambda_t}}{1-Z_t} = \left(-\frac{e^{-\Lambda_1}(t)}{1-Z_1(t)}, \ldots, -\frac{e^{-\Lambda_d}(t)}{1-Z_d(t)}\right)^{\mathbf{T}},
$$

$$
dN(t) = (dN_1(t), \ldots, dN_d(t))^{\mathbf{T}}, dY(t) = (dY_1(t), \ldots, dY_r(t))^{\mathbf{T}},
$$

$$
F = \begin{pmatrix} F_{11} & \cdot & \cdot & F_{1d} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ F_{r1} & \cdot & \cdot & F_{rd} \end{pmatrix}, \text{ where } \mathbf{T} \text{ denotes the transpose of the vector.}
$$

We can express the Equation (2.3) in the following integral form:

$$
X_t^u = x + \int_u^t X_s \left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right) dN_s + \int_u^t X_s \sum_{i=1}^{d} \sum_{j=1}^{r} F_s^{ij}\big(X_s - (1-Z_s)\big)\, dY_s^j, \quad s \in [u, t].
$$

where $x = (x_1, \ldots, x_d)^{\mathbf{T}}$ is the initial condition and $F_j^i(x)$ is the i-th component of the vector function $F_j(x)$.

2.2 Existence and uniqueness of solution to the natural equation

In this section, we present our main result, where we prove the existence and uniqueness of the solution to the natural equation (2.4) with non-Lipschitz conditions using an iteration of the Picard type. Let us first present the problem.

2.2.1 Problem and assumptions

Let $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, our focus is to introduce a technical modification to the natural equation(2.1). It is unknown to us if the quantity $\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)$ is finite or not. So, we present the following stopping time: $\tau_n = \inf\left\{t : 1 - Z_t < \frac{1}{n}\right\}$. Hence, we assume the process $\tilde{X}$ rather than X in the above Equation (2.1).
The modified equation can be expressed as follows:

$$d\tilde{X}_t = \tilde{X}_t \left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right) dN_t + \tilde{X}_t f(\tilde{X}_t - (1 - Z_t))dY_t, \quad u \leq t \leq T, \tag{2.4}$$

such that $\tilde{X}_t = X_t$, for all $t \leq \tau_n$ and $n \in \mathbb{N}$.

Equation (2.4) can thus be expressed in its integral form as follows:

$$\tilde{X}_t = x + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \tilde{X}_s f(\tilde{X}_s - (1 - Z_s))dY_s, \quad s \in [u, t], \tag{2.5}$$

with the initial value $\tilde{X}_u = x$.
We aim to establish the existence and uniqueness of the solution to the natural equation (2.4) in the one-dimensional case. To achieve this, let us suppose the following:

Hypotheses (H)

(H1) We assume that the stochastic integral $\int_u^t \frac{e^{-\Lambda_s}}{1-Z_s} dN_s$, $u \leq t < \infty$, exists and defines a continuous local martingale. (see [27] for a similar assumption)

(H2) Let $\lambda(\cdot)$ be a concave non-decreasing continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\lambda(0) = 0$, $\lambda(v) > 0$ for $v > 0$ and $\int_{0+} \frac{dv}{\lambda(v)} = \infty$. For all $x, y \in \mathbb{R}$, f is continuous, bounded and satisfying

$$|f(x) - f(y)|^2 \leq \lambda(|x - y|^2). \tag{2.6}$$

(H3) We assume the existence of a positive constant K such that for all $x \in \mathbb{R}$, f satisfies the linear growth condition

$$|f(x)|^2 \leq K(1 + |x|^2). \tag{2.7}$$

Remark 2.2.1. *The hypothesis* **(H2)** *is the so-called non-Lipschitz condition. In particular, if $\lambda(v) = Kv$, then the non-Lipschitz condition reduces to the Lipschitz condition. Therefore, the non-Lipschitz condition is weaker than the Lipschitz condition.*

Remark 2.2.2. *Let $\tilde{K} > 0$, and let $\rho \in (0, 1)$ be sufficiently small. To show the generality of our results, we illustrate them using a specific choice of the function $\lambda(\cdot)$ defined as follows:*

$$\lambda_1(v) = \tilde{K}v, \quad v \geqslant 0,$$

$$\lambda_2(v) = \begin{cases} v \log\left(v^{-1}\right), & 0 \leqslant v \leqslant \rho, \\ \rho \log\left(\rho^{-1}\right) + \lambda_2'(\rho-)(v - \rho), & v > \rho, \end{cases}$$

$$\lambda_3(v) = \begin{cases} v \log\left(v^{-1}\right) \log\log\left(v^{-1}\right), & 0 \leqslant v \leqslant \rho, \\ \rho \log\left(\rho^{-1}\right) \log\log\left(\rho^{-1}\right) + \lambda_3'(\rho-)(v - \rho), & v > \rho, \end{cases}$$

where λ' denotes the derivative of function λ. All of them are concave non-decreasing functions that satisfy $\int_{0+} \frac{\mathrm{d}v}{\lambda_i(v)} = +\infty$ $(i = 1, 2, 3)$. Therefore, we see that the Lipschitz condition is a special case of the above conditions.

Remark 2.2.3. *The existence and uniqueness of the solution to the natural equation with Lipschitz coefficients were proved by M. Jeanblanc and S. Song* (see [27]).

2.2.2 Main result

The existence and uniqueness theorem of the solution for the natural equation (2.4) with the initial value $\tilde{X}_u = x$ is now presented under the above hypotheses.

Theorem 2.2.1. [39] *Suppose that* **(H1)**, **(H2)** *and* **(H3)** *hold, then there exists a unique solution to the natural equation* (2.4).

Based on the Picard-type iteration method, we construct an approximation sequence $\tilde{X}^m$ to obtain the existence of the solution to the natural equation (2.4). Let $\tilde{X}^0 = x$. For each $m = 1, 2, \ldots$, we define:

$$
\begin{aligned}
\tilde{X}_t^0 &= x \\
\tilde{X}_t^1 &= x + \int_u^t x \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \int_u^t x f(x - (1 - Z_s)) dY_s \\
\tilde{X}_t^2 &= x + \int_u^t \tilde{X}_s^1 \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \int_u^t \tilde{X}_s^1 f(\tilde{X}_s^1 - (1 - Z_s)) dY_s \\
\tilde{X}_t^3 &= x + \int_u^t \tilde{X}_s^2 \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \int_u^t \tilde{X}_s^2 f(\tilde{X}_s^2 - (1 - Z_s)) dY_s \\
\cdots &= \cdots \\
\tilde{X}_t^m &= x + \int_u^t \tilde{X}_s^{m-1} \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \int_u^t \tilde{X}_s^{m-1} f(\tilde{X}_s^{m-1} - (1 - Z_s)) dY_s. \qquad (2.8)
\end{aligned}
$$

To go forward in the proof of Theorem 2.2.1, let us first establish some helpful lemmas.

2.2.2.1 Important lemmas

The following lemma show the boundedness of the sequence $\{\tilde{X}_t^m, m \geq 0\}$.

Lemma 2.2.1. [39] *Under the assumptions of Theorem* 2.2.1*, for all* $m \geq 1$*, there exists a positive constant* J_1 *such that*

$$\mathbb{E}|\tilde{X}_t^m|^2 \leq J_1, \quad \text{for all } u \leq t \leq T.$$

Proof. Using the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, one obtains

$$|\tilde{X}_t^m|^2 \leq 3\left(|x|^2 + \left|\int_u^t \tilde{X}_s^{m-1}\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s\right|^2 + \left|\int_u^t \tilde{X}_s^{m-1} f(\tilde{X}_s^{m-1}-(1-Z_s))dY_s\right|^2\right).$$

To simplify notation, we denote

$$R_s = \left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right), \tag{2.9}$$

$$\tilde{f}(\tilde{X}_s^{m-1}) = \tilde{X}_s^{m-1} f(\tilde{X}_s^{m-1}-(1-Z_s)). \tag{2.10}$$

By taking expectations on both sides and using the Burkholder-Davis-Gundy inequality (1.1), we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^m|^2\right] &\leq 3\left(|x|^2 + \mathbb{E}\left[\sup_{u\leq r\leq t}\left|\int_u^r \tilde{X}_s^{m-1}R_s dN_s\right|^2\right] + \mathbb{E}\left[\sup_{u\leq r\leq t}\left|\int_u^r \tilde{f}(\tilde{X}_s^{m-1})dY_s\right|^2\right]\right)\\
&\leq 3|x|^2 + 3C_1\mathbb{E}\left[\int_u^t |\tilde{X}_s^{m-1}|^2|R_s|^2 ds\right] + 3C_2\mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s^{m-1})|^2 ds\right]\\
&\leq 3|x|^2 + 3C_1\mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^{m-1}|^2\int_u^t |R_s|^2 ds\right] + 3C_2\mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s^{m-1})|^2 ds\right],
\end{aligned}$$

such that

$$\tilde{S}_1 = \mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^{m-1}|^2\int_u^t |R_s|^2 ds\right],$$

$$\tilde{S}_2 = \mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s^{m-1})|^2 ds\right].$$

We should start with the first term, $\tilde{S}_1$. By Young's inequality (1.3), we have

$$\tilde{S}_1 \leq \frac{1}{2}\left(\mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^{m-1}|^2\right]^2 + \mathbb{E}\left[\int_u^t |R_s|^2 ds\right]^2\right).$$

In addition, we have the quantity $\mathbb{E}\left[\int_u^t |R_s|^2 ds\right] < \infty$, and noting that $\mathbb{E}\left[\int_u^t |R_s|^2 ds\right] =$

M, we obtain

$$\begin{aligned}\tilde{S}_1 &\leq \frac{1}{2}\mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^{m-1}|^2\right]^2 + \frac{1}{2}M^2\\ &\leq \frac{1}{2}C_3 + \frac{1}{2}M^2.\end{aligned}$$

Hence, we derive

$$\tilde{S}_1 = \mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^{m-1}|^2\int_u^t |R_s|^2 ds\right] \leq \frac{1}{2}C_3 + \frac{1}{2}M^2. \tag{2.11}$$

Let us move on to the second term, $\tilde{S}_2$. According to (2.10), since $\tilde{X}_s^{m-1}$ is continuous and f satisfies the linear growth condition (2.7), there exists a constant $\tilde{K}$ such that

$$|\tilde{f}(\tilde{X}_s^{m-1})|^2 \leq \tilde{K}(1+|\tilde{X}_s^{m-1}|^2).$$

Consequently, we deduce

$$\tilde{S}_2 \leq \tilde{K}\left[\int_u^t \left(1+\mathbb{E}|\tilde{X}_s^{m-1}|^2\right)ds\right]. \tag{2.12}$$

From (2.11) and (2.12), it follows that

$$\begin{aligned}\mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^{m}|^2\right] &\leq 3|x|^2 + 3C_1\left(\frac{1}{2}C_3 + \frac{1}{2}M^2\right) + 3C_2\tilde{K}\left[\int_u^t \left(1+\mathbb{E}|\tilde{X}_s^{m-1}|^2\right)ds\right]\\ &\leq 3|x|^2 + 3C_1\left(\frac{1}{2}C_3 + \frac{1}{2}M^2\right) + 3C_2\tilde{K}(T-u) + 3C_2\tilde{K}\int_u^t \mathbb{E}|\tilde{X}_s^{m-1}|^2 ds.\end{aligned}$$

Note that for any $l \geq 1$

$$\max_{1\leq m\leq l}\mathbb{E}|\tilde{X}_s^{m-1}|^2 = \max\left\{\mathbb{E}|\tilde{X}_s^{0}|^2, \max_{1\leq m\leq l}\mathbb{E}\left|\tilde{X}_s^{m}\right|^2\right\} \leq \mathbb{E}|\tilde{X}_s^{0}|^2 + \max_{1\leq m\leq l}\mathbb{E}|\tilde{X}_s^{m}|^2. \tag{2.13}$$

Hence, we obtain

$$
\begin{aligned}
\max_{1\le m\le l}\mathbb{E}\left[\sup_{u\le r\le t}|\tilde{X}^m_r|^2\right] &\le 3|x|^2+3C_1\left(\frac{1}{2}C_3+\frac{1}{2}M^2\right)+3C_2\tilde{K}(T-u)+3C_2\tilde{K}\int_u^t\max_{1\le m\le l}\mathbb{E}|\tilde{X}^{m-1}_s|^2ds\\
&\le 3|x|^2+3C_1\left(\frac{1}{2}C_3+\frac{1}{2}M^2\right)+3C_2\tilde{K}(T-u)+3C_2\tilde{K}(T-u)\mathbb{E}|\tilde{X}^0_s|^2\\
&+3C_2\tilde{K}\int_u^t\max_{1\le m\le l}\mathbb{E}|\tilde{X}^m_s|^2ds\\
&\le 3|x|^2+3C_1\left(\frac{1}{2}C_3+\frac{1}{2}M^2\right)+3C_2\tilde{K}(T-u)(1+\mathbb{E}|\tilde{X}^0_s|^2)\\
&+3C_2\tilde{K}\int_u^t\max_{1\le m\le l}\mathbb{E}\left[\sup_{u\le r\le s}|\tilde{X}^m_r|^2\right]ds.
\end{aligned}
$$

Then,

$$
\max_{1\le m\le l}\mathbb{E}\left[\sup_{u\le r\le t}|\tilde{X}^m_r|^2\right]\le \alpha+\beta\int_u^t\max_{1\le m\le l}\mathbb{E}\left[\sup_{u\le r\le s}|\tilde{X}^m_r|^2\right]ds,
$$

where $\alpha=3|x|^2+3C_1\left(\frac{1}{2}C_3+\frac{1}{2}M^2\right)+3C_2\tilde{K}(T-u)(1+\mathbb{E}|\tilde{X}^0_s|^2)$ and $\beta=3C_2\tilde{K}$.

Since $\mathbb{E}\left[\sup_{u\le r\le t}|\tilde{X}^0_r|^2\right]=|x|^2$, the Gronwall inequality (Lemma 1.1.1) yields

$$
\begin{aligned}
\max_{1\le m\le l}\mathbb{E}\left[\sup_{u\le r\le t}|\tilde{X}^m_r|^2\right] &\le \alpha\exp(\beta(T-u))\\
&\le J_1,
\end{aligned}
$$

where $J_1=\alpha\exp(\beta(T-u))>0$.

Since l is arbitrary, we obtain

$$
\mathbb{E}|\tilde{X}^m_t|^2\le J_1,\quad u\le t\le T,\ m\ge 1.
$$

This completes the proof of Lemma 2.2.1. □

Lemma 2.2.2. [39] *Under the assumptions of Theorem 2.2.1, for all $m,k\ge 1$, there exists a positive constant J_2 such that*

$$
\mathbb{E}\left[\sup_{u\le r\le t}|\tilde{X}^{m+k}_r-\tilde{X}^m_r|^2\right]\le J_2\int_u^t\tilde{\lambda}\left(\mathbb{E}\left[\sup_{u\le r\le s}|\tilde{X}^{m+k-1}_r-\tilde{X}^{m-1}_r|^2\right]\right)ds,\quad \textit{for all } u\le t\le T,
$$

where $\tilde{\lambda}(v) = \lambda(v) + v$ *is a concave non-decreasing continuous function from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$ *such that* $\tilde{\lambda}(0) = 0$, $\tilde{\lambda}(v) > 0$ *for* $v > 0$ *and* $\int_{0^+} \frac{dv}{\tilde{\lambda}(v)} = \infty$.

Proof. For all $m, k \geq 1$, $u \leq t \leq T$, we can obtain from (2.8)

$$\begin{aligned}
\tilde{X}_t^{m+k} - \tilde{X}_t^m &= \int_u^t \tilde{X}_s^{m+k-1}\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s + \int_u^t \tilde{X}_s^{m+k-1} f(\tilde{X}_s^{m+k-1} - (1-Z_s))dY_s \\
&\quad - \int_u^t \tilde{X}_s^{m-1}\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s - \int_u^t \tilde{X}_s^{m-1} f(\tilde{X}_s^{m-1} - (1-Z_s))dY_s \\
&= \int_u^t \left[(\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1})\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\right]dN_s \\
&\quad + \int_u^t \left[\tilde{X}_s^{m+k-1} f(\tilde{X}_s^{m+k-1} - (1-Z_s)) - \tilde{X}_s^{m-1} f(\tilde{X}_s^{m-1} - (1-Z_s))\right]dY_s.
\end{aligned}$$

Let us set

$$R_s = \left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right), \tag{2.14}$$

$$\tilde{f}(\tilde{X}_s^{m+k-1}) = \tilde{X}_s^{m+k-1} f(\tilde{X}_s^{m+k-1} - (1-Z_s)), \tag{2.15}$$

$$\tilde{f}(\tilde{X}_s^{m-1}) = \tilde{X}_s^{m-1} f(\tilde{X}_s^{m-1} - (1-Z_s)). \tag{2.16}$$

Taking expectation on both sides and using the basic inequality $(a+b)^2 \leq 2(a^2+b^2)$ and Burkholder-Davis-Gundy inequality (1.1), we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r^{m+k} - \tilde{X}_r^m|^2\right] &\leq 2\mathbb{E}\left[\sup_{u\leq r\leq t}\left|\int_u^r \left(\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}\right)(R_s)\, dN_s\right|^2\right] \\
&\quad + 2\mathbb{E}\left[\sup_{u\leq r\leq t}\left|\int_u^r \left(\tilde{f}(\tilde{X}_s^{m+k-1}) - \tilde{f}(\tilde{X}_s^{m-1})\right) dY_s\right|^2\right] \\
&\leq 2C_4\mathbb{E}\left[\int_u^t |\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2 |R_s|^2 ds\right] \\
&\quad + 2C_5\mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s^{m+k-1}) - \tilde{f}(\tilde{X}_s^{m-1})|^2 ds\right],
\end{aligned}$$

such that

$$\tilde{P}_1 = \mathbb{E}\left[\int_u^t |\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2 |R_s|^2 ds\right],$$

$$\tilde{P}_2 = \mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s^{m+k-1}) - \tilde{f}(\tilde{X}_s^{m-1})|^2 ds\right].$$

For the first term $\tilde{P}_1$, according to (2.14), we observe that $|R_s| = \frac{|-e^{-\Lambda_s}|}{|1-Z_{s\wedge\tau_n}|} \leq n$ because Λ_s is a given continuous increasing process null at the origin and Z_s is a continuous process. Therefore, we have

$$\tilde{P}_1 = \mathbb{E}\left[\int_u^t |\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2 |R_s|^2 ds\right] \leq n^2 \int_u^t \mathbb{E}|\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2 ds. \tag{2.17}$$

For the second term $\tilde{P}_2$, referring to equations (2.15) and (2.16), and since f satisfies the non-Lipschitz condition (2.6), we obtain

$$|\tilde{f}(\tilde{X}_s^{m+k-1}) - \tilde{f}(\tilde{X}_s^{m-1})|^2 \leq \lambda(|\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2).$$

Consequently, we derive

$$\tilde{P}_2 \leq \int_u^t \mathbb{E}(\lambda(|\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2))ds. \tag{2.18}$$

Combining (2.17) and (2.18), we deduce

$$\begin{aligned}\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r^{m+k} - \tilde{X}_r^{m}|^2\right] \leq 2C_4 n^2 \int_u^t \mathbb{E}|\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2 ds \\ + 2C_5 \int_u^t \mathbb{E}(\lambda(|\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2))ds.\end{aligned}$$

Therefore,

$$\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r^{m+k} - \tilde{X}_r^{m}|^2\right] \leq J_2 \int_u^t \mathbb{E}(\tilde{\lambda}(|\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2))ds,$$

where $J_2 = \max(2C_4 n^2, 2C_5)$ and $\tilde{\lambda}(v) = \lambda(v) + v$ is a concave non-decreasing continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\tilde{\lambda}(0) = 0$, $\tilde{\lambda}(v) > 0$ for $v > 0$ and $\int_{0+} \frac{dv}{\tilde{\lambda}(v)} = \infty$.

Applying Jensen's inequality (1.7), we get

$$\mathbb{E}\left[\sup_{u \le r \le t} |\tilde{X}_r^{m+k} - \tilde{X}_r^m|^2\right] \le J_2 \int_u^t \tilde{\lambda}(\mathbb{E}|\tilde{X}_s^{m+k-1} - \tilde{X}_s^{m-1}|^2) ds$$

$$\le J_2 \int_u^t \tilde{\lambda}\left(\mathbb{E}\left[\sup_{u \le r \le s} |\tilde{X}_r^{m+k-1} - \tilde{X}_r^{m-1}|^2\right]\right) ds.$$

This completes the proof of Lemma 2.2.2. □

Lemma 2.2.3. [39] *Under the assumptions of Theorem* 2.2.1*, for all* $m, k \ge 1$*, there exists a positive constant* J_3 *such that*

$$\mathbb{E}\left[\sup_{u \le r \le t} |\tilde{X}_r^{m+k} - \tilde{X}_r^m|^2\right] \le J_3(t-u), \quad \text{for all } u \le t \le T.$$

Proof. By Lemma 2.2.1 and Lemma 2.2.2, we obtain

$$\mathbb{E}\left[\sup_{u \le r \le t} |\tilde{X}_r^{m+k} - \tilde{X}_r^m|^2\right] \le J_2 \int_u^t \tilde{\lambda}\left(\mathbb{E}\left[\sup_{u \le r \le s} |\tilde{X}_r^{m+k-1} - \tilde{X}_r^{m-1}|^2\right]\right) ds$$

$$\le J_2 \int_u^t \tilde{\lambda}(4J_1) ds$$

$$\le J_2 \tilde{\lambda}(4J_1)(t-u)$$

$$\le J_3(t-u),$$

where $J_3 = J_2 \tilde{\lambda}(4J_1)$.

This completes the proof of Lemma 2.2.3. □

For any $m, k \ge 1$, the recursive function is defined as follows:

$$\phi_t^1 = J_3(t-u),$$

$$\phi_t^{m+1} = J_2 \int_u^t \tilde{\lambda}\left(\phi_s^m\right) ds,$$

$$\phi_t^{m,k} = \mathbb{E}\left[\sup_{u \le r \le t} \left|\tilde{X}_r^{m+k} - \tilde{X}_r^m\right|^2\right].$$

Let us choose $T_1 \in [u,T]$ such that

$$J_2\tilde{\lambda}\left(J_3(t-u)\right) \leq J_3 \quad \text{for all} \quad t \in [u,T_1]. \tag{2.19}$$

Lemma 2.2.4. [39] *Under the assumptions of Theorem 2.2.1, for all $m,k \geq 1$, there exists a positive $T_1 \in [u,T]$ such that*

$$0 \leq \phi_t^{m,k} \leq \phi_t^m \leq \phi_t^{m-1} \leq \cdots \leq \phi_t^1, \quad \textit{for all } t \in [u,T_1]\,. \tag{2.20}$$

Proof. By Lemma 2.2.3 and induction in the exponent m, we have

$$\begin{aligned}
\phi_t^{1,k} &= \mathbb{E}\left[\sup_{u\leq r\leq t}\left|\tilde{X}_r^{1+k}-\tilde{X}_r^1\right|^2\right] \leq J_3(t-u) = \phi_t^1.\\
\phi_t^{2,k} &= \mathbb{E}\left[\sup_{u\leq r\leq t}\left|\tilde{X}_r^{2+k}-\tilde{X}_r^2\right|^2\right]\\
&\leq J_2\int_u^t \tilde{\lambda}\left(\mathbb{E}\left[\sup_{u\leq r\leq s}\left|\tilde{X}_r^{1+k}-\tilde{X}_r^1\right|^2\right]\right)ds\\
&\leq J_2\int_u^t \tilde{\lambda}\left(\phi_s^{1,k}\right)ds\\
&\leq J_2\int_u^t \tilde{\lambda}\left(\phi_s^{1}\right)ds = \phi_t^2.
\end{aligned}$$

From (2.19), we obtain

$$\phi_t^2 = J_2\int_u^t \tilde{\lambda}\left(\phi_s^1\right)ds \leq \int_u^t J_2\tilde{\lambda}\left(J_3(t-u)\right)ds \leq J_3(t-u) = \phi_t^1.$$

Consequently, we deduce that

$$\phi_t^{2,k} \leq \phi_t^2 \leq \phi_t^1, \quad \text{for all } \ t \in [u,T_1]\,.$$

We suppose that (2.20) holds for some $m \geq 1$. For $m+1$, we can easily check Lemma

2.2.4. In fact,

$$\begin{aligned}
\phi_t^{m+1,k} &= \mathbb{E}\left(\sup_{u\le r\le t}\left|\tilde{X}_r^{m+k+1}-\tilde{X}_r^{m+1}\right|^2\right)\\
&\le J_2\int_u^t \tilde{\lambda}\left(\mathbb{E}\left[\sup_{u\le r\le t}\left|\tilde{X}_r^{m+k}-\tilde{X}_r^{m}\right|^2\right]\right)ds\\
&\le J_2\int_u^t \tilde{\lambda}\left(\phi_s^{m,k}\right)ds\\
&\le J_2\int_u^t \tilde{\lambda}\left(\phi_s^{m}\right)ds=\phi_t^{m+1}.
\end{aligned}$$

In addition, we have

$$\phi_t^{m+1}=J_2\int_u^t \tilde{\lambda}\left(\phi_s^{m}\right)ds\le J_2\int_u^t \tilde{\lambda}\left(\phi_s^{m-1}\right)ds=\phi_t^{m}.$$

Therefore, we conclude that

$$\phi_t^{m+1,k}\le\phi_t^{m+1}\le\phi_t^{m},\quad \text{for all }\ t\in[u,T_1]\,.$$

This completes the proof of Lemma 2.2.4. □

2.2.2.2 Proof of the main Theorem

By using the previous lemmas, we can now demonstrate Theorem 2.2.1. We start by establishing the uniqueness result.

Proof of Theorem 2.2.1. **(Uniqueness)**. Let both $\tilde{X}_t$ and $\tilde{X}'_t$ be two solutions to the natural equation (2.4) with the same initial condition $\tilde{X}_u=\tilde{X}'_u$. Thus, we have

$$\begin{aligned}
\tilde{X}_t-\tilde{X}'_t &= \int_u^t \tilde{X}_s\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s+\int_u^t \tilde{X}_s f(\tilde{X}_s-(1-Z_s))dY_s\\
&-\int_u^t \tilde{X}'_s\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s-\int_u^t \tilde{X}'_s f(\tilde{X}'_s-(1-Z_s))dY_s\\
&=\int_u^t\left[(\tilde{X}_s-\tilde{X}'_s)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\right]dN_s\\
&+\int_u^t\left[\tilde{X}_s f(\tilde{X}_s-(1-Z_s))-\tilde{X}'_s f(\tilde{X}'_s-(1-Z_s))\right]dY_s.
\end{aligned}$$

We put

$$R_s = \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right), \tag{2.21}$$

$$\tilde{f}(\tilde{X}_s) = \tilde{X}_s f(\tilde{X}_s - (1 - Z_s)), \tag{2.22}$$

$$\tilde{f}(\tilde{X}'_s) = \tilde{X}'_s f(\tilde{X}'_s - (1 - Z_s)). \tag{2.23}$$

Taking expectation on both sides, and by the Burkholder-Davis-Gundy inequality (1.1) and the fundamental inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r - \tilde{X}'_r|^2\right] &\leq 2\mathbb{E}\left[\sup_{u\leq r\leq t} \left| \int_u^r (\tilde{X}_s - \tilde{X}'_s)\,(R_s)\,dN_s \right|^2\right] \\
&+ 2\mathbb{E}\left[\sup_{u\leq r\leq t} \left| \int_u^r \left(\tilde{f}(\tilde{X}_s) - \tilde{f}(\tilde{X}'_s)\right) dY_s \right|^2\right] \\
&\leq 2C_6\mathbb{E}\left[\int_u^t |\tilde{X}_s - \tilde{X}'_s|^2 |R_s|^2 ds\right] + 2C_7\mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s) - \tilde{f}(\tilde{X}'_s)|^2 ds\right],
\end{aligned}$$

such that

$$\begin{aligned}
\tilde{I}_1 &= \mathbb{E}\left[\int_u^t |\tilde{X}_s - \tilde{X}'_s|^2 |R_s|^2 ds\right], \\
\tilde{I}_2 &= \mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s) - \tilde{f}(\tilde{X}'_s)|^2 ds\right].
\end{aligned}$$

For the first term $\tilde{I}_1$, according to (2.21), we observe that $|R_s| = \frac{|-e^{-\Lambda_s}|}{|1-Z_{s\wedge\tau_n}|} \leq n$ because Λ_s is a given continuous increasing process null at the origin and Z_s is a continuous process. Moreover, we have

$$\tilde{I}_1 = \mathbb{E}\left[\int_u^t |\tilde{X}_s - \tilde{X}'_s|^2 |R_s|^2 ds\right] \leq n^2 \int_u^t \mathbb{E}|\tilde{X}_s - \tilde{X}'_s|^2 ds. \tag{2.24}$$

For $\tilde{I}_2$, referring to equations (2.22) and (2.23), and since f satisfies the non-Lipschitz condition (2.6), we obtain

$$|\tilde{f}(\tilde{X}_s) - \tilde{f}(\tilde{X}'_s)|^2 \leq \lambda(|\tilde{X}_s - \tilde{X}'_s|^2).$$

Thus, we derive

$$\tilde{I}_2 \leq \int_u^t \mathbb{E}(\lambda(|\tilde{X}_s - \tilde{X}'_s|^2))ds. \tag{2.25}$$

Combining (2.24) and (2.25), we deduce

$$\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r - \tilde{X}'_r|^2\right] \leq 2C_6 n^2 \int_u^t \mathbb{E}|\tilde{X}_s - \tilde{X}'_s|^2 ds + 2C_7 \int_u^t \mathbb{E}(\lambda(|\tilde{X}_s - \tilde{X}'_s|^2))ds.$$

Therefore,

$$\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r - \tilde{X}'_r|^2\right] \leq J_4 \int_u^t \mathbb{E}(\tilde{\lambda}(|\tilde{X}_s - \tilde{X}'_s|^2))ds,$$

where $J_4 = \max\left(2C_6 n^2, 2C_7\right)$ and $\tilde{\lambda}(v) = \lambda(v) + v$ is a concave non-decreasing continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\tilde{\lambda}(0) = 0$, $\tilde{\lambda}(v) > 0$ for $v > 0$ and $\int_{0+} \frac{dv}{\tilde{\lambda}(v)} = \infty$.

Using Jensen's inequality (1.7), we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r - \tilde{X}'_r|^2\right] &\leq J_4 \int_u^t \tilde{\lambda}(\mathbb{E}|\tilde{X}_s - \tilde{X}'_s|^2)ds \\ &\leq J_4 \int_u^t \tilde{\lambda}\left(\mathbb{E}\left[\sup_{u\leq r\leq s} |\tilde{X}_r - \tilde{X}'_r|^2\right]\right) ds.\end{aligned}$$

From Bihari's inequality (Lemma 1.1.2), it follows that

$$\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r - \tilde{X}'_r|^2\right] = 0, \quad u \leq t \leq T.$$

For all $t \in [u, T]$, the expression above signifies that $\tilde{X}_t = \tilde{X}'_t$ a.s. The proof for uniqueness is concluded.

(Proof of the existence). Initially, we note that it ϕ^m is continuous on $[u, T_1]$. For each $m \geq 1$, ϕ^m_t is non-increasing on the interval $[u, T_1]$. Hence, the function ϕ_t is defined from the dominated convergence theorem as follows:

$$\phi_t = \lim_{m\to\infty} \phi^m_t = \lim_{m\to\infty} J_2 \int_u^t \tilde{\lambda}\left(\phi^{m-1}_s\right) ds = J_2 \int_u^t \tilde{\lambda}(\phi_s)ds, \text{ for all } t \in [u, T_1]. \tag{2.26}$$

Therefore,

$$\phi_t \leq \phi_0 + J_2 \int_u^t \tilde{\lambda}(\phi_s)ds.$$

Since $\phi_0 = 0$, the Bihari inequality (Lemma 1.1.2) yields $\phi_t = 0$ for all $u \leq t \leq T_1$.

According to Lemma 2.2.4, we can derive that for all $u \leq t \leq T_1$,

$$\phi_t^{m,k} \leq \phi_t^m \to 0 \quad \text{as} \quad m \to \infty,$$

namely,

$$\mathbb{E}|\tilde{X}_t^{m+k} - \tilde{X}_t^m|^2 \to 0 \quad \text{as} \quad m \to \infty.$$

Due to the continuity of both processes $\tilde{X}_t$ and Z_t and by the property of the function to $\lambda(\cdot)$ along with the completeness of L^2, we can infer that for all $u \leq t \leq T_1$

$$f(\tilde{X}_t^m - (1 - Z_t)) \xrightarrow{L^2} f(\tilde{X}_t - (1 - Z_t)) \quad \text{as} \quad m \to \infty,$$

such that

$$\tilde{X}_t^m \xrightarrow{L^2} \tilde{X}_t \quad \text{as} \quad m \to \infty.$$

Upon taking limits on both sides of equation (2.8), we achieve for all $u \leq t \leq T_1$ that

$$\lim_{m\to\infty} \tilde{X}_t^m = x + \lim_{m\to\infty} \int_u^t \tilde{X}_s^{m-1} \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \lim_{m\to\infty} \int_u^t \tilde{X}_s^{m-1} f(\tilde{X}_s^{m-1} - (1 - Z_s))dY_s,$$

namely,

$$\tilde{X}_t = x + \int_u^t \tilde{X}_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right) dN_s + \int_u^t \tilde{X}_s f(\tilde{X}_s - (1 - Z_s))dY_s.$$

The presented expression suggests that $\tilde{X}_t$ is the only solution of the natural equation (2.4) on $[u, T_1]$. Therefore, one can deduce by iteration that the natural equation (2.4) has a unique solution on $[u, T]$.

This completes the proof of Theorem 2.2.1. □

Remark 2.2.4. *The existence and uniqueness of the solution to the natural equation* (2.4) *are defined on a finite interval* $[u, T]$ *by Theorem* 2.2.1*. If all additions of the existence and uniqueness theorem are met on every finite sub-interval of* $[u, +\infty)$*, then the natural equation* (2.4) *will have a unique solution* $\tilde{X}_t$ *on the whole interval* $(-\infty, +\infty)$*. So, the proof is similar to that in Theorem* 2.2.1*.*

2.3 Stability of solutions for the natural equation w.r.t. initial value

In this section, we establish the mean-square stability of solutions with respect to the initial value. To this end, we first provide a definition of mean-square stability.

Definition 2.3.1. [39] *A solution* $\tilde{X}_t^x$ *of the natural equation* (2.4) *with initial value* $\tilde{X}_u = x$ *is said to be stable in mean-square if for all* $\varepsilon > 0$ *there is an* $\sigma > 0$ *such that*

$$\mathbb{E}|\tilde{X}_t^x - \tilde{X}_t^y|^2 \leq \varepsilon \quad \text{when } \mathbb{E}|x-y|^2 < \sigma,$$

where $\tilde{X}_t^y$ *is another solution of the natural equation* (2.4) *with the initial value* $\tilde{X}_u = y$*.*

2.3.1 Main result

The stability theorem of solutions for the natural equation (2.4) with the initial value $\tilde{X}_u = x$ is now presented.

Theorem 2.3.1. [39] *Suppose that the assumptions of Theorem* 2.2.1 *are true. Let* $\tilde{X}_t^x$ *and* $\tilde{X}_t^y$ *be two solutions of the natural equation* (2.4) *with initial values* x *and* y *respectively. Then, the solution of the natural equation* (2.4) *is stable in mean-square.*

In order to obtain the stability theorem of solutions, we need the following lemmas.

2.3.1.1 Important lemmas

First, we introduce the extended Bihari's inequality which was introduced in [49]

Lemma 2.3.1. [52] *Let $T > 0$ and $z_0 \geq 0$, and let $z(t), q(t)$ be two continuous functions on $[0, T]$. Let $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave continuous and non-decreasing function such that $\lambda(v) > 0$ for $v > 0$. If*

$$z(t) \leq z_0 + \int_t^T q(s)\lambda(z(s))\mathrm{d}s, \quad \textit{for all} \quad 0 \leq t \leq T,$$

then for $t \in [0, T]$,

$$z(t) \leq F^{-1}\left(F(z_0) + \int_t^T q(s)ds\right),$$

so that

$$F(z_0) + \int_t^T q(s)ds \in \mathrm{Dom}\left(F^{-1}\right),$$

where $F(v) = \int_1^v \frac{ds}{\lambda(s)}, v \geq 0$ and F^{-1} is the inverse function of F.

Lemma 2.3.2. [52] *Assume the assumptions of Lemma 2.3.1 hold. Let for $t \in [0, T]$, $q(t) \geq 0$. If for all $\varepsilon > 0$, there exists $t_1 \geq 0$ such that $0 \leq z_0 < \varepsilon$ it satisfies*

$$\int_{t_1}^T q(s)ds \leq \int_{z_0}^{\varepsilon} \frac{ds}{\lambda(s)}.$$

Then for all $t \in [t_1, T]$,

$$z(t) \leq \varepsilon,$$

holds.

Proof. For every $\varepsilon > 0$, let $f = F(v) = \int_1^v \frac{ds}{\lambda(s)}$.
By Lemma 2.3.1, we have

$$z(t) \leq F^{-1}\left(F\left(z_0\right) + \int_t^T q(s)ds\right).$$

If $\lambda(s)$ is a continuous non-decreasing function, then the inverse function of $v = F^{-1}(f)$ is also a non-decreasing function.

For $0 \leqslant v \leqslant \varepsilon$, we have

$$F(v) = \int_1^\varepsilon \frac{ds}{\lambda(s)} + \int_\varepsilon^v \frac{ds}{\lambda(s)} = F(\varepsilon) + \int_\varepsilon^v \frac{ds}{\lambda(s)}.$$

For $0 \leq z_0 < \varepsilon$, we can choose $t_1 \geq 0$ such that

$$\int_{t_1}^T q(s)ds \leq \int_{z_0}^\varepsilon \frac{ds}{\lambda(s)}.$$

Hence, for all $t \in [t_1, T]$,

$$\int_t^T q(s)ds \leq \int_{z_0}^\varepsilon \frac{ds}{\lambda(s)}.$$

Therefore,

$$F(z_0) + \int_t^T q(s)ds \leq F(z_0) + \int_{z_0}^\varepsilon \frac{ds}{\lambda(s)} = F(\varepsilon).$$

Thus, we obtain

$$z(t) \leq F^{-1}\left(F(z_0) + \int_t^T q(s)ds\right) \leq F^{-1}\left(F(z_0) + \int_{z_0}^\varepsilon \frac{ds}{\lambda(s)}\right) \leq F^{-1}(F(\varepsilon)) \leq \varepsilon.$$

This completes the proof of Lemma 2.3.2. □

2.3.1.2 Proof of the main Theorem

With the help of the previous lemmas, we can now demonstrate Theorem 2.3.1.

Proof. According to the assumptions of Theorem 2.2.1, $\tilde{X}_t^x$ and $\tilde{X}_t^y$ are both solutions of the natural equation (2.4) with initial values x and y, respectively, which are defined as follows:

$$\tilde{X}_t^x = x + \int_u^t \tilde{X}_s^x \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \tilde{X}_s^x f(\tilde{X}_s^x - (1 - Z_s))dY_s, \quad u \leq t \leq T, \quad \text{a.s.} \tag{2.27}$$

$$\tilde{X}_t^y = y + \int_u^t \tilde{X}_s^y \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right) dN_s + \int_u^t \tilde{X}_s^y f(\tilde{X}_s^y - (1 - Z_s))dY_s, \quad u \leq t \leq T, \quad \text{a.s.} \tag{2.28}$$

Subtracting the Eq. (2.27) from the Eq. (2.28), we obtain

$$\begin{aligned} \tilde{X}_t^x - \tilde{X}_t^y = x - y &+ \int_u^t \left[(\tilde{X}_s^x - \tilde{X}_s^y)\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right)\right] dN_s \\ &+ \int_u^t \left[\tilde{X}_s^x f(\tilde{X}_s^x - (1 - Z_s)) - \tilde{X}_s^y f(\tilde{X}_s^y - (1 - Z_s))\right] dY_s, \quad u \leq t \leq T, \quad \text{a.s.} \end{aligned} \tag{2.29}$$

We put

$$R_s = \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right), \tag{2.30}$$

$$\tilde{f}(\tilde{X}_t^x) = \tilde{X}_s^x f(\tilde{X}_s^x - (1 - Z_s)), \tag{2.31}$$

$$\tilde{f}(\tilde{X}_t^y) = \tilde{X}_s^y f(\tilde{X}_s^y - (1 - Z_s)). \tag{2.32}$$

We take the expectations on both sides, and using Burkholder-Davis-Gundy (BDG) inequality (1.1) and the fundamental inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, we obtain

$$\begin{aligned} \mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r^x - \tilde{X}_r^y|^2\right] &\leq 3\mathbb{E}|x - y|^2 + 3\mathbb{E}\left[\sup_{u\leq r\leq t}\left|\int_u^r (\tilde{X}_s^x - \tilde{X}_s^y)(R_s)\, dN_s\right|^2\right] \\ &+ 3\mathbb{E}\left[\sup_{u\leq r\leq t}\left|\int_u^r \left(\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)\right) dY_s\right|^2\right] \\ &\leq 3\mathbb{E}|x - y|^2 + 3C_8\mathbb{E}\left[\int_u^t |\tilde{X}_s^x - \tilde{X}_s^y|^2 |R_s|^2 ds\right] \\ &+ 3C_9\mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)|^2 ds\right] \\ &\leq 3\mathbb{E}|x - y|^2 + 3C_8\mathbb{E}\left[\sup_{u\leq r\leq t} |\tilde{X}_r^x - \tilde{X}_r^y|^2 \int_u^t |R_s|^2 ds\right] \\ &+ 3C_9\mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)|^2 ds\right], \end{aligned}$$

such that

$$\tilde{A}_1 = \mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^x - \tilde{X}_r^y|^2\int_u^t |R_s|^2 ds\right],$$

$$\tilde{A}_2 = \mathbb{E}\left[\int_u^t |\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)|^2 ds\right].$$

For the first term $\tilde{A}_1$, by Young's inequality (1.3), we have

$$\tilde{A}_1 \leq \frac{1}{2}\left(\mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^x - \tilde{X}_r^y|^2\right]^2 + \mathbb{E}\left[\int_u^t |R_s|^2 ds\right]^2\right).$$

Moreover, we have the quantity $\mathbb{E}\left[\int_u^t |R_s|^2 ds\right] < \infty$, we note that $\mathbb{E}\left[\int_u^t |R_s|^2 ds\right] = M$. Then,

$$\begin{aligned}\tilde{A}_1 &\leq \frac{1}{2}\mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^x - \tilde{X}_r^y|^2\right]^2 + \frac{1}{2}M^2\\ &\leq \frac{1}{2}C_{10} + \frac{1}{2}M^2.\end{aligned}$$

Hence, we deduce

$$\tilde{A}_1 = \mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^x - \tilde{X}_r^y|^2\int_u^t |R_s|^2 ds\right] \leq \frac{1}{2}C_{10} + \frac{1}{2}M^2. \tag{2.33}$$

For the second term $\tilde{A}_2$, according to (2.31) and (2.32), and since f satisfies the non-Lipschitz condition (2.6), we have

$$|\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)|^2 \leq \lambda(|\tilde{X}_s^x - \tilde{X}_s^y|^2).$$

Consequently, we derive

$$\tilde{A}_2 \leq \int_u^t \mathbb{E}(\lambda(|\tilde{X}_s^x - \tilde{X}_s^y|^2))ds. \tag{2.34}$$

Combining (2.33) and (2.34), we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{u\leq r\leq t}|\tilde{X}_r^x - \tilde{X}_r^y|^2\right] &\leq 3\mathbb{E}|x-y|^2 + 3C_8\left(\frac{1}{2}C_{10} + \frac{1}{2}M^2\right)\\ &\quad + 3C_9\int_u^t \lambda\left(\mathbb{E}\left[\sup_{u\leq r\leq s}|\tilde{X}_r^x - \tilde{X}_r^y|^2\right]\right)ds.\end{aligned}$$

Let $\lambda_1(v) = 3C_9\lambda(v)$. So, $\lambda_1(v)$ is clearly a concave non-decreasing continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\lambda_1(0) = 0$ and $\int_{0+} \frac{dv}{\lambda_1(v)} = \infty$. Moreover, let $z(t) = \mathbb{E}(\sup_{u\le r\le t} |\tilde{X}^x_r - \tilde{X}^y_r|^2)$, $q(t) = 1$ and $z_0 = 3\mathbb{E}|x-y|^2 + 3C_8\left(\frac{1}{2}C_{10} + \frac{1}{2}M_1^2\right) \ge 0$ with $M_1 = \mathbb{E}\left[\int_u^T |R_s|^2 ds\right] \ge M$. Hence, for any $\varepsilon > 0$, $\tilde{\varepsilon} = \frac{\varepsilon}{4}$ with ε arbitrary. From Lemma 2.3.2, we have

$$\int_{z_0}^{\tilde{\varepsilon}} \frac{dv}{\lambda_1(v)} \ge \int_u^T q(s)ds = T - u, \quad \text{as} \quad z_0 \le \tilde{\varepsilon}.$$

Thus, for any $t \in [u, T]$, we derive

$$z(t) \le \tilde{\varepsilon},$$

holds.
So,

$$\mathbb{E}\left[\sup_{u\le r\le t} |\tilde{X}^x_r - \tilde{X}^y_r|^2\right] \le \tilde{\varepsilon}.$$

This completes the proof of Theorem 2.3.1. □

2.4 Concluding remarks

The main contribution of this chapter is the examination of a crucial financial model the so-called natural model focusing on its representation through the significant stochastic differential equation known as the natural equation. The main objective was to investigate the existence and uniqueness of the solution to the natural equation (2.4) under non-Lipschitz conditions and to establish the mean-square stability of solutions to the considered equation.

The obtained results in this chapter show that the natural equation admits a unique solution under the non-Lipschitz conditions, which makes the analysis more challenging, we were nevertheless able to establish the desired results. These results significantly extend the applicability of the natural model in financial mathematics, offering a more flexible and realistic modeling framework for markets exhibiting irregular dynamics.

The findings of this chapter provide a solid mathematical foundation for the natural model and significantly broaden its applicability in financial modeling. The proposed analysis allows the model to accommodate a wider class of stochastic dynamics that better reflect realistic market behaviours. As a result, the natural model becomes a strong and adaptable framework for the theoretical investigation of stochastic systems and defaultable financial markets.

Chapter 3

Stochastic flows for one-dimensional natural equations with non-Lipschitz coefficients

In stochastic analysis, the study of stochastic flows for stochastic differential equations (SDEs) has attracted significant attention. This area of research offers a framework for examining mathematical concepts like regularity and stability as well as practical applications in disciplines like biology, physics, and finance.

The study of stochastic flows for stochastic differential equations (SDEs) aims to investigate the stability of the solution with respect to the initial condition together with its regularity in both time and space variables. Under suitable assumptions on the coefficients, such stability properties combined with appropriate regularity in time allow the application of Kolmogorov's continuity criterion, which yields the existence of a modification of the solution that is jointly continuous in (t, x). In the case of Lipschitz coefficients, this can be obtained by the powerful Gronwall inequality. The situation changes completely, however, in the non-Lipschitz case since the use of Gronwall inequality meets an obvious difficulty. This joint continuity reflects the regular dependence of trajectories on both time and initial conditions and constitutes a key tool in the analysis of qualitative properties of stochastic systems.

Furthermore, there has been a lot of interest in stochastic analysis on the study of the homeomorphism property of stochastic flows generated by stochastic differential equations

(SDEs). For a large class of SDEs, the associated solution map generates a stochastic flow of homeomorphisms. More precisely, for almost every sample path ω, the mapping $x \longmapsto X_t(x,\omega)$ is continuous, one-to-one and onto, and its inverse mapping is also continuous. This property plays an important role in the qualitative analysis of stochastic dynamical systems and provides valuable information on the behavior and stability of the trajectories.

Our objectives in the present chapter are to examine the bicontinuity and homeomorphism property of stochastic flows of the solution for the natural equation (3.2) in the one-dimensional case with non-Lipschitz coefficients.

This chapter is organized as follows. In Section 3.1, we present the formulation of the problem and the hypotheses used throughout this chapter. In Section 3.2, we will demonstrate that the solution for the one-dimensional natural equation (3.2) has a bicontinuous modification in (t,x). In Section 3.3, we will prove the homeomorphism property of the solution of the natural equation (3.2) with non-Lipschitz coefficients. We end the chapter with some concluding remarks.

3.1 Presentation of the problem and hypothesis

In this section, we present our problem together with the hypotheses used throughout this chapter. We begin by recalling the natural equation in the one-dimensional case announced in the chapter above.

Let $(\Omega,\mathbb{F}=(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a filtered probability space. For any continuous increasing process Λ null at the origin, for any continuous non negative local martingale N, for any continuous local martingale Y, for any azéma supermartingale such that $Z_t = N_t e^{-\Lambda_t}$ satisfies $0 < Z_t < 1$, $t > 0$, for any Lipschitz function f on $\mathbb{R}$ null at the origin, there exist a random variable τ such that the family of conditional expectations $dX_t^u = \mathbb{Q}\left[\tau \leq u/\mathcal{F}_t\right], 0 < u,t < \infty$, satisfies the following one-dimensional stochastic differential equation:

$$(\natural_u) : \begin{cases} dX_t = X_t\left(-\frac{e^{-\Lambda_t}}{1-Z_t}dN_t + f(X_t-(1-Z_t))dY_t\right), & t\in[u,\infty), \\ X_u = x, \end{cases} \tag{3.1}$$

where x is the initial condition that is $\mathcal{F}_u$-measurable random variable.

Our goal is to add a technical change to Equation (3.1). We know whether the quantity $\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)$ is finite or not. So, we introduce the stopping time: $\tau_n = \inf\left\{t : 1 - Z_t < \frac{1}{n}\right\}$. Hence, we assume the process $\tilde{X}$ rather than X in the above Equation (3.1).

The changed equation can now be expressed as follows:

$$d\tilde{X}_t = \tilde{X}_t\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right)dN_t + \tilde{X}_t f(\tilde{X}_t - (1 - Z_t))dY_t, \quad u \leq t < \infty, \tag{3.2}$$

such that $\tilde{X}_t = X_t$, for all $t \leq \tau_n$ and $n \in \mathbb{N}$.

Equation (3.2) can be expressed in integral form as follows:

$$\tilde{X}_t = x + \int_u^t \tilde{X}_s\left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}}\right)dN_s + \int_u^t \tilde{X}_s f(\tilde{X}_s - (1 - Z_s))dY_s, \quad s \in [u, t], \tag{3.3}$$

with the initial value $\tilde{X}_u = x$.

In order to attain the bicontinuous modification and homeomorphism property of the solution for the natural equation (3.2) with the initial value x, let us suppose the following:

Hypotheses (H')

(H'1) We assume that the stochastic integral $\int_u^t \frac{e^{-\Lambda_s}}{1-Z_s}dN_s$, $u \leq t < \infty$, exists and defines a continuous local martingale. (see [27] for a similar assumption)

(H'2) Suppose that f is bounded and satisfying

$$|f(x) - f(y)|^2 \leq \kappa_{2,\theta}^2(|x - y|), \tag{3.4}$$

and

$$|f(x) - f(y)| \leq \kappa_{1,\theta}(|x - y|), \tag{3.5}$$

where $\kappa_{1,\theta}(\cdot)$ and $\kappa_{2,\theta}(\cdot)$ are two concave functions.

(H'3) There exists a constant $K > 0$ such that for all $x \in \mathbb{R}$, f satisfies the linear growth condition

$$|f(x)|^2 \leq K(1 + |x|^2). \tag{3.6}$$

We present the following fundamental example of a concave function.

Example 3.1.1. [50] *Notice that for some* $0 < \theta < 1/\mathrm{e}$*, we define a concave function as follows:*

$$\kappa_\theta(x) = \begin{cases} x \log x^{-1}, & x \le \theta, \\ \theta \log \theta^{-1} + (\log \theta^{-1} - 1)(x - \theta), & x > \theta. \end{cases}$$

Setting $x_0 = \theta$*, we obtain*

$$F(x) = \log\left(\frac{\log \theta}{\log x}\right), \quad 0 < x < \theta.$$

Then,

$$F^{-1}(x) = \exp\{\log \theta \cdot \exp\{-x\}\}, \quad x < 0.$$

If $z(0) < \theta$*, replacing this in* (1.6) *gives*

$$z(t) \le (z(0))^{\exp\left\{-\int_0^t q(s)ds\right\}}. \tag{3.7}$$

For $0 < \theta < 1/\mathrm{e}$*, let* $\kappa_{1,\theta}$*,* $\kappa_{2,\theta}$ *be two concave functions that are defined for* $j = 1, 2$ *by*

$$\kappa_{j,\theta}(x) = \begin{cases} x \left[\log x^{-1}\right]^{1/j}, & x \le \theta, \\ \left(\left[\log \theta^{-1}\right]^{1/j} - \frac{1}{j}\left[\log \theta^{-1}\right]^{(1/j)-1}\right) x + \frac{1}{j}\left[\log \theta^{-1}\right]^{(1/j)-1} \theta, & x > \theta. \end{cases}$$

3.2 Bicontinuous modification of the solution

In this section, we establish the bicontinuity property of the solution for the one-dimensional natural equation (3.2) with non-Lipschitz coefficients based on Kolmogorov's continuity criterion and Bihari's inequality. Before presenting our theorem, we shall mention a few remarks.

Remark 3.2.1. *Based on the Kolmogorov theorem and Gronwall's lemma, F. Benziadi and A. Kandouci have demonstrated the continuity property of the solution of the natural equation with Lipschitz coefficients in the one-dimensional case* (see [6])*.*

Remark 3.2.2. *Recall that, under assumptions of Theorem* 2.2.1*, the existence and uniqueness results of the natural equation* (3.2) *have been established* (see [39])*.*

3.2.1 Main result

The bicontinuity theorem of the solution for the one-dimensional natural equation (3.2) is now presented under the above hypotheses.

Theorem 3.2.1. [40] *Assume that* **(H'1)**, **(H'2)** *and* **(H'3)** *hold, and* $\tilde{X}_t^x$ *is the solution of the natural equation* (3.2). *Then exists a bicontinuous modification of the random field* $(t,x) \longmapsto \tilde{X}_t^x$.

To prove Theorem 3.2.1, let us first establish some helpful lemmas.

3.2.1.1 Important lemmas

We need the following elementary lemma.

Lemma 3.2.1. [50]

1. *For* $j = 1, 2$, $\kappa_{j,\theta}$ *is a decreasing function in* θ. *Specifically, if* $1 > \theta_1 > \theta_2$, *then* $\kappa_{j,\theta_1} \le \kappa_{j,\theta_2}$.

2. *For any* $p > 1$ *and* θ *sufficiently small, we have*

$$x^p \kappa_{j,\theta}^j(x) \le \frac{1}{j+p} \kappa_{1,\theta^{j+p}} \left(x^{j+p}\right), \quad j = 1, 2. \tag{3.8}$$

Proof. The first assertion is straightforward. Let us consider the second one. It is clear that both sides coincide when $x \le \theta$. For $x > \theta$, we need to prove that

$$\begin{aligned} h_j(x) = &\left(\log \theta^{-1} - \frac{1}{j+p}\right) x^{j+p} + \frac{\theta^{j+p}}{j+p} \\ &- x^p \left(\left((\log \theta^{-1})^{\frac{1}{j}} - \frac{1}{j}(\log \theta^{-1})^{\frac{1}{j}-1}\right) x + \frac{1}{j}(\log \theta^{-1})^{\frac{1}{j}-1}\theta\right)^j \ge 0. \end{aligned}$$

Set

$$B = \left([\log \theta^{-1}]^{\frac{1}{j}} - \frac{1}{j}[\log \theta^{-1}]^{\frac{1}{j}-1}\right).$$

Then we have

$$h_j'(x) = \big((j+p)\log\theta^{-1}-1\big)\,x^{j+p-1} - px^{p-1}\left(Bx+\frac{1}{j}[\log\theta^{-1}]^{\frac{1}{j}-1}\theta\right)^j - jBx^p\left(Bx+\frac{1}{j}[\log\theta^{-1}]^{\frac{1}{j}-1}\theta\right)^{j-1}.$$

When $j=1$, we obtain

$$\begin{aligned} h_1'(x) &= x^{p-1}\left(x\big((1+p)\log\theta^{-1}-1\big) - p\big((\log\theta^{-1}-1)x+\theta\big) - (\log\theta^{-1}-1)x\right) \\ &= x^{p-1}\left(p(x-\theta)\right) \geq 0. \end{aligned}$$

When $j=2$, a careful computation shows that the derivative can be written

$$h_2'(x) = \left[(p+1) - \frac{1}{2\log\theta^{-1}}\right]\theta(x-\theta) + \left[(p+1) - \frac{p+2}{4\log\theta^{-1}}\right](x-\theta)^2 \geq 0,$$

if θ is sufficiently small. Then the lemma follows from $h_j(\theta)=0$.

This completes the proof of Lemma 3.2.1 □

We state now the following version of the Kolmogorov continuity criterion.

Lemma 3.2.2. [50] *Let $J_1, J_2 \subset \mathbb{R}$ two closed intervals and $\{X(s,t),(s,t)\in J_1\times J_2\}$ a stochastically continuous process. For $n\in\mathbb{N}$, let*

$$X_n(s,t) = (X(s,t)\wedge n)\vee(-n).$$

If for every n there exists $p_n, C_n, \alpha_n > 0$ such that

$$\mathbb{E}\left[\sup_{s\in J_1}|X_n(s,t)-X_n(s,t')|^{p_n}\right] \leq C_n\,|t-t'|^{1+\alpha_n}, \quad \forall t,t'\in J_2,$$

then X has a bicontinuous modification $\widehat{X}$. In particular, if $p=p_n>1$ and $\alpha=\alpha_n>0$ are independent of n, then the paths $J_2\ni t\to\widehat{X}(\cdot,t)\in C(J_1)$ are β-Hölder continuous for every $\beta<\alpha p^{-1}$.

Proof. Let $\widehat{X}_n$ be a bicontinuous modification of X_n, and let I denote the set of all rational points within the rectangle $J_1\times J_2$.

Define the sets as follows:

$$C_1 = \left\{\omega : \forall n\in\mathbb{N},\ (s,t)\mapsto\widehat{X}_n(s,t,\omega) \text{ is continuous}\right\},$$

$$C_2 = \left\{ \omega : \forall n,\ \forall (s,t) \in I,\ \widehat{X}_n(s,t,\omega) = X_n(s,t,\omega) \right\},$$

$$C_3 = \bigcup_{n=1}^{\infty} B_n,$$

where

$$B_n = \left\{ \omega : \forall (s,t) \in I,\ |\widehat{X}_n(s,t,\omega)| < n \right\}.$$

Note that the sequence $C_2 \cap B_n$ is increasing with respect to n, since for each n,

$$C_2 \cap B_n \subset C_2 \cap B_{n+1}.$$

Moreover

$$C_2 \cap C_3 = C_2 \bigcap_{(s,t)\in I} \{\omega : X(s,t,\omega) < \infty\}.$$

Therefore, the set $C = C_1 \cap C_2 \cap C_3$ has probability one. For each $\omega \in C$, define

$$\widehat{X}(s,t,\omega) = \widehat{X}_n(s,t,\omega) \quad \text{if } \omega \in B_n.$$

We now check that $\widehat{X}$ is well-defined. Indeed, if $\omega \in C_1 \cap C_2 \cap B_n$ and $(s,t) \in I$, then

$$\widehat{X}_{n+1}(s,t,\omega) = X_{n+1}(s,t,\omega) = X(s,t,\omega) = X_n(s,t,\omega) = \widehat{X}_n(s,t,\omega).$$

By continuity, it follows that

$$\widehat{X}_{n+1}(s,t,\omega) = \widehat{X}_n(s,t,\omega), \quad \text{for all} \quad (s,t).$$

Thus, $\widehat{X}$ is well-defined. Additionally, for every $\omega \in C$ and $(s,t) \in I$, we have

$$\widehat{X}(s,t,\omega) = X(s,t,\omega).$$

Since $J_1 \times J_2 \ni (s,t) \mapsto \widehat{X}(s,t,\omega)$ is continuous for $\omega \in C$, and by the almost sure continuity of $\widehat{X}(s,t)$ together with the stochastic continuity of $X(s,t)$, we conclude that

$\widehat{X}$ is a modification of X.

For the second one, we set

$$D = \left\{ \omega : \forall n \in \mathbb{N}, \sup_{\substack{s \in J_1,\ t,t' \in J_2 \\ t \neq t'}} \frac{|\widehat{X}_n(s,t,\omega) - \widehat{X}_n(s,t',\omega)|}{|t-t'|^{\beta}} < \infty \right\}.$$

Then, for each $\omega \in C \cap D$, the mapping $J_1 \times J_2 \ni (s,t) \mapsto \widehat{X}(s,t,\omega)$ is bicontinuous. Therefore, $\sup_{(s,t)\in J_1 \times J_2} \widehat{X}(s,t,\omega)$ is bounded by some L. Hence, $\widehat{X}(s,t,\omega) = \widehat{X}_L(s,t,\omega)$ for all $(s,t) \in J_1 \times J_2$, and

$$\sup_{\substack{s \in J_1,\ t,t' \in J_2 \\ t \neq t'}} \frac{|\widehat{X}(s,t,\omega) - \widehat{X}(s,t',\omega)|}{|t-t'|^{\beta}} < \infty.$$

Since $C \cap D$ is of full measure, the desired result holds.

This completes the proof of Lemma 3.2.2. □

We present the following lemma on the non-confluence property.

Lemma 3.2.3. [40] *Assume that* **(H'1)**, **(H'2)** *and* **(H'3)** *hold, then the natural equation* (3.2) *has a unique solution $\tilde{X}_t^x$ with the initial value x. In addition, the following non-confluence property holds:*

$$x > y \Rightarrow \mathbb{P}\left(\tilde{X}_t^x > \tilde{X}_t^y, u \leq t < \infty\right) = 1. \tag{3.9}$$

Proof. Let $x > y$, the solution of the natural equation

$$\begin{cases} d\tilde{X}^{(m)}(t,x) = \tilde{X}^{(m)}(t,x)\left(-\dfrac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}}dN_t + f(\tilde{X}^{(m)}(t,x) - (1-Z_t))dY_t\right), \\ \tilde{X}^{(m)}(u,x) = x, \end{cases}$$

has the property that

$$\mathbb{P}\left(\tilde{X}^{(m)}(t,x) > \tilde{X}^{(m)}(t,y), u \leq t < \infty\right) = 1, \quad \forall m.$$

For $m \to \infty$, $\tilde{X}^{(m)}(t,x) \to \tilde{X}_t^x$, equation (3.9) holds.

This completes the proof of Lemma 3.2.3. □

We also need the following crucial Lemma.

Lemma 3.2.4. [40] *Assume that* **(H'1)**, **(H'2)** *and* **(H'3)** *hold, and $\tilde{X}_t^x$ is the unique strong solution of the natural equation* (3.2). *For any $p \geq 2$ and $t \geq u$, we have*

$$\mathbb{E}\left[\sup_{u\leq s\leq t}\left|\tilde{X}_s^x - \tilde{X}_s^y\right|^p\right] \leq \tilde{C}_p^1|x-y|^{p\cdot\exp\left\{-\tilde{C}_p^2(t-u)\right\}}, \quad \forall x,y\in\mathbb{R}, \tag{3.10}$$

where $\tilde{C}_p^1$ and $\tilde{C}_p^2$ are two positive constants.

Proof. Put $A_t = \tilde{X}_t^x - \tilde{X}_t^y, x > y$. By Lemma 3.2.3 we know that $A_t > 0$. Then

$$\tilde{X}_t^x = x + \int_u^t \tilde{X}_s^x\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s + \int_u^t \tilde{X}_s^x f(\tilde{X}_s^x - (1-Z_s))dY_s,$$

$$\tilde{X}_t^y = y + \int_u^t \tilde{X}_s^y\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s + \int_u^t \tilde{X}_s^y f(\tilde{X}_s^y - (1-Z_s))dY_s.$$

Therefore,

$$\begin{aligned}A_t = x - y &+ \int_u^t \left[(\tilde{X}_s^x - \tilde{X}_s^y)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\right]dN_s + \int_u^t \Big[\tilde{X}_s^x f(\tilde{X}_s^x - (1-Z_s)) \\ &- \tilde{X}_s^y f(\tilde{X}_s^y - (1-Z_s))\Big]\,dY_s.\end{aligned}$$

By applying Itô's formula to A_t^p, we obtain

$$\begin{aligned}A_t^p = A_0^p &+ \int_u^t pA_s^{p-1}\Bigg[(\tilde{X}_s^x - \tilde{X}_s^y)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s \\ &+ \Big(\tilde{X}_s^x f(\tilde{X}_s^x - (1-Z_s)) - \tilde{X}_s^y f(\tilde{X}_s^y - (1-Z_s))\Big)\,dY_s\Bigg] \\ &+ \frac{1}{2}\int_u^t p(p-1)A_s^{p-2}\Bigg[(\tilde{X}_s^x - \tilde{X}_s^y)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s \\ &+ \Big(\tilde{X}_s^x f(\tilde{X}_s^x - (1-Z_s)) - \tilde{X}_s^y f(\tilde{X}_s^y - (1-Z_s))\Big)\,dY_s\Bigg]^2.\end{aligned}$$

Moreover,

$$
\begin{aligned}
A_t^p &= A_0^p + \int_u^t pA_s^{p-1}(\tilde{X}_s^x - \tilde{X}_s^y)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right) dN_s \\
&+ \int_u^t pA_s^{p-1}\left[\tilde{X}_s^x f(\tilde{X}_s^x - (1-Z_s)) - \tilde{X}_s^y f(\tilde{X}_s^y - (1-Z_s))\right] dY_s \\
&+ \frac{1}{2}\int_u^t p(p-1)A_s^{p-2}(\tilde{X}_s^x - \tilde{X}_s^y)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2 ds \\
&+ \frac{1}{2}\int_u^t p(p-1)A_s^{p-2}\left[\tilde{X}_s^x f(\tilde{X}_s^x - (1-Z_s)) - \tilde{X}_s^y f(\tilde{X}_s^y - (1-Z_s))\right]^2 ds \\
&+ \frac{1}{2}\int_u^t 2p(p-1)A_s^{p-2}(\tilde{X}_s^x - \tilde{X}_s^y)\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right) \\
&\quad \times \left[\tilde{X}_s^x f(\tilde{X}_s^x - (1-Z_s)) - \tilde{X}_s^y f(\tilde{X}_s^y - (1-Z_s))\right] d<N,Y>_s .
\end{aligned}
$$

To simplify notation, we denote

$$
R_s = \left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right), \tag{3.11}
$$

$$
\tilde{f}(\tilde{X}_s^x) = \tilde{X}_s^x f(\tilde{X}_s^x - (1-Z_s)), \tag{3.12}
$$

$$
\tilde{f}(\tilde{X}_s^y) = \tilde{X}_s^y f(\tilde{X}_s^y - (1-Z_s)). \tag{3.13}
$$

Thus,

$$
\begin{aligned}
A_t^p &= A_0^p + \int_u^t pA_s^{p-1}(\tilde{X}_s^x - \tilde{X}_s^y)R_s dN_s + \int_u^t pA_s^{p-1}\left[\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)\right] dY_s \\
&+ \frac{1}{2}\int_u^t p(p-1)A_s^{p-2}(\tilde{X}_s^x - \tilde{X}_s^y)^2 R_s^2 ds + \frac{1}{2}\int_u^t p(p-1)A_s^{p-2}\left[\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)\right]^2 ds \\
&+ \int_u^t p(p-1)A_s^{p-2}(\tilde{X}_s^x - \tilde{X}_s^y)R_s\left[\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)\right] d\langle N,Y\rangle_s.
\end{aligned}
$$

Then,

$$
|A_t|^p = |A_0|^p + B_t^1 + B_t^c + B_t^2 + B_t^d + B_t^j,
$$

where

$$
\begin{aligned}
B_t^1 &= \int_u^t p|A_s|^{p-1}(\tilde{X}_s^x - \tilde{X}_s^y)R_s dN_s, \\
B_t^2 &= \int_u^t p|A_s|^{p-1}\left[\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)\right] dY_s, \\
B_t^c &= \frac{1}{2}\int_u^t p(p-1)|A_s|^{p-2}(\tilde{X}_s^x - \tilde{X}_s^y)^2 R_s^2 ds, \\
B_t^d &= \frac{1}{2}\int_u^t p(p-1)|A_s|^{p-2}\left[\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)\right]^2 ds, \\
B_t^j &= \int_u^t p(p-1)|A_s|^{p-2}(\tilde{X}_s^x - \tilde{X}_s^y)R_s\left[\tilde{f}(\tilde{X}_s^x) - \tilde{f}(\tilde{X}_s^y)\right] d\langle N, Y\rangle_s.
\end{aligned}
$$

For B_t^1, by the Burkholder-Davis-Gundy inequality (1.1), we obtain

$$
\mathbb{E}\left(\sup_{u\le s\le t}|B_s^1|\right) \le pC\mathbb{E}\left(\int_u^t |A_s|^{2p}|R_s|^2 ds\right)^{\frac{1}{2}}.
$$

From equation (3.11), we observe that $|R_s| = \frac{|-e^{-\Lambda_s}|}{|1-Z_{s\wedge\tau_n}|} \le n$ because Λ_s is a given continuous increasing process null at the origin and Z_s is a continuous process. By Young's inequality (1.3), we have

$$
\begin{aligned}
\mathbb{E}\left(\sup_{u\le s\le t}|B_s^1|\right) &\le pCn\mathbb{E}\left(\int_u^t |A_s|^{2p} ds\right)^{\frac{1}{2}} \\
&\le C_p^1\mathbb{E}\left(\sup_{u\le s\le t}|A_s|^p \int_u^t |A_s|^p ds\right)^{\frac{1}{2}} \\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{u\le s\le t}|A_s|^p\right] + C_p^1\mathbb{E}\left[\int_u^t |A_s|^p ds\right] \\
&\le \frac{1}{4}h(t) + C_p^1\mathbb{E}\left[\int_u^t |A_s|^p ds\right],
\end{aligned} \tag{3.14}
$$

where $C_p^1 = \frac{pCn}{4}$.

For B_t^2, according to (3.12) and (3.13), and since f satisfies the equation (3.4) and by

Young's inequality (1.3) and Burkholder-Davis-Gundy inequality (1.1), we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\le s\le t}|B_t^2|\right) &\le pC\mathbb{E}\left(\int_u^t |A_s|^{2p-2}\left|\tilde{f}(\tilde{X}_s^x)-\tilde{f}(\tilde{X}_s^y)\right|^2 ds\right)^{\frac{1}{2}}\\
&\le C_p^2\mathbb{E}\left(\sup_{u\le s\le t}|A_s|^p\int_u^t |A_s|^{p-2}\kappa_{2,\theta}^2(|A_s|)ds\right)^{\frac{1}{2}}\\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{u\le s\le t}|A_s|^p\right]+C_p^2\int_u^t\mathbb{E}\left(|A_s|^{p-2}\kappa_{2,\theta}^2(|A_s|)\right)ds\\
&\le \frac{1}{4}h(t)+C_p^2\int_u^t\mathbb{E}\left(|A_s|^{p-2}\kappa_{2,\theta}^2(|A_s|)\right)ds, \qquad (3.15)
\end{aligned}$$

where $C_p^2=\frac{pC}{4}$.

For B_t^c, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\le s\le t}|B_s^c|\right) &\le \frac{1}{2}p(p-1)\mathbb{E}\left(\int_u^t |A_s|^p\,|R_s|^2 ds\right)\\
&\le \frac{1}{2}p(p-1)n^2\mathbb{E}\left(\int_u^t |A_s|^p\,ds\right)\\
&\le C_p^3\mathbb{E}\left[\int_u^t |A_s|^p\,ds\right], \qquad (3.16)
\end{aligned}$$

where $C_p^3=\frac{1}{2}p(p-1)n^2$.

For B_t^d, according to (3.12) and (3.13), and since f satisfies the equation (3.4), we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\le s\le t}|B_s^d|\right) &\le \frac{1}{2}p(p-1)\int_u^t\mathbb{E}\left(|A_s|^{p-2}\kappa_{2,\theta}^2(|A_s|)\right)ds\\
&\le C_p^4\int_u^t\mathbb{E}\left(|A_s|^{p-2}\kappa_{2,\theta}^2(|A_s|)\right)ds, \qquad (3.17)
\end{aligned}$$

where $C_p^4=\frac{1}{2}p(p-1)$.

For B_t^j, according to (3.12) and (3.13), and since f satisfies the equation (3.5) and by

Lemma 1.1.4, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\le s\le t}|B_s^j|\right) &\le p(p-1)nC\mathbb{E}\left(\int_u^t |A_s|^{p-1}\left|\tilde f(\tilde X_s^x)-\tilde f(\tilde X_s^y)\right| d\langle N,Y\rangle_s\right)\\
&\le C_p^5\mathbb{E}\left(\int_u^t |A_s|^{p-1}\left|\tilde f(\tilde X_s^x)-\tilde f(\tilde X_s^y)\right| \sqrt{d\langle N\rangle_s}\sqrt{d\langle Y\rangle_s}\right)\\
&\le C_p^5\mathbb{E}\left(\int_u^t |A_s|^{p-1}\left|\tilde f(\tilde X_s^x)-\tilde f(\tilde X_s^y)\right| ds\right)\\
&\le C_p^5\mathbb{E}\left(\int_u^t |A_s|^{p-1}\kappa_{1,\theta}(|A_s|)ds\right)\\
&\le C_p^5\int_u^t \mathbb{E}\left(|A_s|^{p-1}\kappa_{1,\theta}(|A_s|)ds\right).
\end{aligned} \tag{3.18}$$

Combining (3.14)-(3.18) gives that

$$\begin{aligned}
h(t):=\mathbb{E}\left[\sup_{u\le s\le t}|A_s|^p\right] &\le A_0^p+\frac14 h(t)+C_p^1\mathbb{E}\left[\int_u^t|A_s|^p\,ds\right]+\frac14 h(t)\\
&+C_p^2\int_u^t\mathbb{E}\left(|A_s|^{p-2}\kappa_{2,\theta}^2(|A_s|)\right)ds+C_p^3\mathbb{E}\left[\int_u^t|A_s|^p\,ds\right]\\
&+C_p^4\int_u^t\mathbb{E}\left(|A_s|^{p-2}\kappa_{2,\theta}^2(|A_s|)\right)ds+C_p^5\int_u^t\mathbb{E}\left(|A_s|^{p-1}\kappa_{1,\theta}(|A_s|)ds\right)\\
&\le A_0^p+C_p^6\int_u^t\mathbb{E}\,|A_s|^p\,ds+C_p^7\int_u^t\mathbb{E}\left(|A_s|^{p-2}\kappa_{2,\theta}^2(|A_s|)\right)ds+\frac12 h(t)\\
&+C_p^5\int_u^t\mathbb{E}\left(|A_s|^{p-1}\kappa_{1,\theta}(|A_s|)ds\right),
\end{aligned}$$

where $C_p^6=C_p^1+C_p^3$ and $C_p^7=C_p^2+C_p^4$.

From Lemma 3.2.1, we obtain

$$\begin{aligned}
h(t) &\le 2A_0^p+2C_p^6\int_u^t\mathbb{E}\,|A_s|^p\,ds+\frac{2C_p^7}{p}\int_u^t\mathbb{E}\left(\kappa_{1,\theta^p}(|A_s|^p)\right)ds+\frac{2C_p^5}{p}\int_u^t\mathbb{E}\left(\kappa_{1,\theta^p}(|A_s|^p)\right)ds\\
&\le 2A_0^p+2C_p^6\int_u^t\mathbb{E}\,|A_s|^p\,ds+C_p^8\int_u^t\mathbb{E}\left(\kappa_{1,\theta^p}(|A_s|^p)\right)ds\\
&\le 2A_0^p+C_p^9\int_u^t\mathbb{E}\left(\tilde\kappa_{1,\theta^p}(|A_s|^p)\right)ds,
\end{aligned}$$

where $C_p^8 = \frac{2C_p^7}{p} + \frac{2C_p^5}{p}$, $C_p^9 = \max(2C_p^6, C_p^8)$ and $\tilde{\kappa}_{1,\theta^p}(x) = \kappa_{1,\theta^p}(x) + x$ is a concave function decreasing in θ.

By Jensen's inequality (1.7), we obtain

$$h(t) \leq 2A_0^p + C_p^9 \int_u^t \tilde{\kappa}_{1,\theta^p}\left(\mathbb{E}(|A_s|^p)\right) ds$$

$$\leq \tilde{C}_p^1 A_0^p + \tilde{C}_p^2 \int_u^t \tilde{\kappa}_{1,\theta^p}\left(h(s)\right) ds,$$

where $\tilde{C}_p^1 = 2$ and $\tilde{C}_p^2 = C_p^9$.

Finally, by the equation (3.7), we have

$$\mathbb{E}\left[\sup_{u \leq s \leq t} |A_t|^p\right] \leq \tilde{C}_p^1 A_0^{p \cdot \exp\left\{-\tilde{C}_p^2 (t-u)\right\}}.$$

Thus,

$$\mathbb{E}\left[\sup_{u \leq s \leq t} \left|\tilde{X}_s^x - \tilde{X}_s^y\right|^p\right] \leq \tilde{C}_p^1 |x-y|^{p \cdot \exp\left\{-\tilde{C}_p^2 (t-u)\right\}}, \quad \forall x, y \in \mathbb{R}.$$

This completes the proof of Lemma 3.2.4. $\square$

3.2.1.2 Proof of the main Theorem

Since $\tilde{X}_t^x$ is continuous with respect to t, by the above lemmas, one may prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Let $p \geq 2$ and let $\left(\tilde{X}_t^x\right)_{t \geq u}$ be the solution of the natural equation (3.2) with $\tilde{X}_u^x = x$. For every $t \in \left[u, u + \frac{\log p}{\tilde{C}_p^2}\right)$, the mapping $x \longmapsto \tilde{X}_t^x$ has a β-Hölder continuous modification for $\beta < e^{-\tilde{C}_p^2 (t-u)} - \frac{1}{p}$. Moreover, by Lemma 3.2.4 and Kolmogorov's criterion Lemma 3.2.2, $\tilde{X}_t^x$ has a bicontinuous modification in $(t, x) \in [u, T] \times \mathbb{R}$.

This completes the proof of Theorem 3.2.1. $\square$

3.3 Homeomorphism property of the solution

In this section, we prove that for almost all ω, the mapping $x \longmapsto \tilde{X}_t^x(\omega)$ defines a homeomorphism on $\mathbb{R}$. To achieve this, we follow the approach introduced by Yamada and Ogura [65], which requires proving that the mapping $x \longmapsto \tilde{X}_t^x(\omega)$ is strictly increasing and continuous w.r.t. x, that its inverse mapping is also strictly increasing and continuous w.r.t. x, and that $\lim\limits_{|x|\to\infty} \left|\tilde{X}_t^x(\omega)\right| = \infty$, for almost all ω. Before presenting our theorem, we shall mention a few remarks.

Remark 3.3.1. *In the one-dimensional case, F. Benziadi demonstrated the homeomorphism property of stochastic flows of the natural equation with Lipschitz coefficients* (see [5]).

Remark 3.3.2. *Under the assumptions of Theorem* 3.2.1*, we recall that the solution of the natural equation* (3.2) *has a bicontinuous modification in* $(t, x) \in [u, T] \times \mathbb{R}$.

3.3.1 Main result

Under the above hypotheses, the homeomorphism theorem of the solution for the one-dimensional natural equation (3.2) is now given.

Theorem 3.3.1. [40] *Assume that* **(H'1)**, **(H'2)** *and* **(H'3)** *hold and* $\tilde{X}_t^x$ *is the solution of the natural equation* (3.2). *Then, for almost all* ω*, for each* $t \geq u$*, the mapping* $\mathbb{R} \ni x \longmapsto \tilde{X}_t^x(\omega) \in \mathbb{R}$ *is a homeomorphism on* $\mathbb{R}$.

To prove Theorem 3.3.1, we first establish an important lemma, which plays a fundamental role in proving the surjectivity property.

Lemma 3.3.1. [40] *Assume that* **(H'1)**, **(H'2)** *and* **(H'3)** *hold, and* $\tilde{X}_t^x$ *is the solution of the natural equation* (3.2). *Then, for every* $\alpha \in \mathbb{R}$*, there exists a constant* $C_{T,u,\alpha}$ *such that,*

$$\mathbb{E}\left[\sup_{u\leq s\leq t}\left(1+|\tilde{X}_t^x|^2\right)^{\alpha}\right] \leq C_{T,u,\alpha}\left(1+|x|^2\right)^{\alpha}, \quad t\in[u,T], \quad x\in\mathbb{R}.$$

Proof. Let $u < T < \infty$. By Itô's formula, we have

$$
\begin{aligned}
\left(1+|\tilde{X}_t^x|^2\right)^{\alpha} &= (1+|x|^2)^{\alpha} + \int_u^t 2\alpha\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1}\tilde{X}_s^x \\
&\quad \times \left[\tilde{X}_s^x\Big(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\Big)dN_s + \tilde{X}_s^x f\big(\tilde{X}_s^x-(1-Z_s)\big)dY_s\right] \\
&\quad + \int_u^t \left[\alpha\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1} + 2\alpha(\alpha-1)(\tilde{X}_s^x)^2\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-2}\right] \\
&\quad \times \left[\tilde{X}_s^x\Big(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\Big)dN_s + \tilde{X}_s^x f\big(\tilde{X}_s^x-(1-Z_s)\big)dY_s\right]^2.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\left(1+|\tilde{X}_t^x|^2\right)^{\alpha} &= (1+|x|^2)^{\alpha} + \int_u^t 2\alpha\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1}(\tilde{X}_s^x)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)dN_s \\
&\quad + \int_u^t 2\alpha\tilde{X}_s^x\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1}\tilde{X}_s^x f(\tilde{X}_s^x-(1-Z_s))dY_s \\
&\quad + \int_u^t \left[\alpha\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1} + 2\alpha(\alpha-1)(\tilde{X}_s^x)^2\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-2}\right] \\
&\quad \times \left[(\tilde{X}_s^x)^2\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)^2\right]ds \\
&\quad + \int_u^t \left[\alpha\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1} + 2\alpha(\alpha-1)(\tilde{X}_s^x)^2\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-2}\right] \\
&\quad \times \left[\left[\tilde{X}_s^x f(\tilde{X}_s^x-(1-Z_s))\right]^2\right]ds \\
&\quad + \int_u^t \left[\alpha\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1} + 2\alpha(\alpha-1)(\tilde{X}_s^x)^2\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-2}\right] \\
&\quad \times \left[2\times\tilde{X}_s^x\left(-\frac{e^{-\Lambda_s}}{1-Z_{s\wedge\tau_n}}\right)\left[\tilde{X}_s^x f(\tilde{X}_s^x-(1-Z_s))\right]\right]d\langle N,Y\rangle_s.
\end{aligned}
$$

Noting

$$R_s = \left(-\frac{e^{-\Lambda_s}}{1 - Z_{s\wedge\tau_n}} \right), \tag{3.19}$$

$$\tilde{f}(\tilde{X}_s^x) = \tilde{X}_s^x f(\tilde{X}_s^x - (1 - Z_s)). \tag{3.20}$$

Hence,

$$\begin{aligned}
\left(1 + |\tilde{X}_t^x|^2\right)^\alpha = (1 + |x|^2)^\alpha &+ \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} (\tilde{X}_s^x)^2 R_s dN_s \\
&+ \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} \tilde{X}_s^x \tilde{f}(\tilde{X}_s^x) dY_s \\
&+ \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} (\tilde{X}_s^x)^2 (R_s)^2 ds \\
&+ \int_u^t 2\alpha(\alpha - 1)(\tilde{X}_s^x)^4 \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-2} (R_s)^2 ds \\
&+ \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} \left[\tilde{f}(\tilde{X}_s^x)\right]^2 ds \\
&+ \int_u^t 2\alpha(\alpha - 1)(\tilde{X}_s^x)^2 \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-2} \left[\tilde{f}(\tilde{X}_s^x)\right]^2 ds \\
&+ \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} \tilde{X}_s^x R_s \left[\tilde{f}(\tilde{X}_s^x)\right] d\langle N, Y\rangle_s \\
&+ \int_u^t 4\alpha(\alpha - 1)(\tilde{X}_s^x)^3 \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-2} R_s \left[\tilde{f}(\tilde{X}_s^x)\right] d\langle N, Y\rangle_s.
\end{aligned}$$

Then,

$$\left(1 + |\tilde{X}_t^x|^2\right)^\alpha = (1 + |x|^2)^\alpha + A_t^1 + A_t^2 + A_t^3 + A_t^4 + A_t^5 + A_t^6 + A_t^7 + A_t^8,$$

where

$$A_t^1 = \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} (\tilde{X}_s^x)^2 R_s \, dN_s,$$

$$A_t^2 = \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} \tilde{X}_s^x \tilde{f}(\tilde{X}_s^x) \, dY_s,$$

$$A_t^3 = \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} (\tilde{X}_s^x)^2 (R_s)^2 \, ds,$$

$$A_t^4 = \int_u^t 2\alpha(\alpha-1)(\tilde{X}_s^x)^4 \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-2} (R_s)^2 \, ds,$$

$$A_t^5 = \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} \left[\tilde{f}(\tilde{X}_s^x)\right]^2 \, ds,$$

$$A_t^6 = \int_u^t 2\alpha(\alpha-1)(\tilde{X}_s^x)^2 \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-2} \left[\tilde{f}(\tilde{X}_s^x)\right]^2 \, ds,$$

$$A_t^7 = \int_u^t 2\alpha \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-1} \tilde{X}_s^x R_s \left[\tilde{f}(\tilde{X}_s^x)\right] \, d\langle N, Y\rangle_s,$$

$$A_t^8 = \int_u^t 4\alpha(\alpha-1)(\tilde{X}_s^x)^3 \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha-2} R_s \left[\tilde{f}(\tilde{X}_s^x)\right] \, d\langle N, Y\rangle_s.$$

For A_t^1, by the Burkholder-Davis-Gundy inequality (1.1), Young's inequality (1.3), and the elementary inequality $|x|^2 \leq 1 + |x|^2$, we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{u \leq s \leq t} |A_s^1|\right) &\leq 2\alpha C \mathbb{E}\left(\int_u^t \left(1 + |\tilde{X}_s^x|^2\right)^{2\alpha-2} |\tilde{X}_s^x|^4 |R_s|^2 ds\right)^{\frac{1}{2}} \\
&\leq 2\alpha C n \mathbb{E}\left(\int_u^t \left(1 + |\tilde{X}_s^x|^2\right)^{2\alpha} ds\right)^{\frac{1}{2}} \\
&\leq 2\alpha C n \mathbb{E}\left(\sup_{u \leq s \leq t} \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha} \int_u^t \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha} ds\right)^{\frac{1}{2}} \\
&\leq \frac{1}{4}\mathbb{E}\left[\sup_{u \leq s \leq t} \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha}\right] + C_\alpha^1 \mathbb{E}\left[\int_u^t \left(1 + |\tilde{X}_s^x|^2\right)^{\alpha} ds\right], \qquad (3.21)
\end{aligned}$$

where $C_\alpha^1 = \frac{\alpha C n}{2}$.

For A_t^2, using the Burkholder-Davis-Gundy inequality (1.1), we have

$$\mathbb{E}\left(\sup_{u\le s\le t}|A_s^2|\right) \le 2\alpha C\mathbb{E}\left(\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{2\alpha-2}|\tilde{X}_s^x|^2|\tilde{f}(\tilde{X}_s^x)|^2 ds\right)^{\frac{1}{2}},$$

according to (3.20), and since f satisfies the equation (3.6), by Young's inequality (1.3) and the elementary inequality $|x|^2 \le 1+|x|^2$, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\le s\le t}|A_s^2|\right) &\le 2\alpha CK\mathbb{E}\left(\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{2\alpha} ds\right)^{\frac{1}{2}} \\
&\le 2\alpha CK\mathbb{E}\left(\sup_{u\le s\le t}\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{\alpha} ds\right)^{\frac{1}{2}} \\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{u\le s\le t}\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}\right] + C_\alpha^2\mathbb{E}\left[\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{\alpha} ds\right], \qquad (3.22)
\end{aligned}$$

where $C_\alpha^2 = \frac{\alpha CK}{2}$.

For A_t^3, by the elementary inequality $|x|^2 \le 1+|x|^2$, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\le s\le t}|A_s^3|\right) &\le 2\alpha\mathbb{E}\left(\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1}|\tilde{X}_s^x|^2|R_s|^2 ds\right) \\
&\le 2\alpha n^2\mathbb{E}\left(\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1}|\tilde{X}_s^x|^2 ds\right) \\
&\le C_\alpha^3\mathbb{E}\left(\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{\alpha} ds\right), \qquad (3.23)
\end{aligned}$$

where $C_\alpha^3 = 2\alpha n^2$.

For A_t^4, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{u\le s\le t}|A_s^4|\right) &\le 2\alpha(\alpha-1)\mathbb{E}\left(\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{\alpha-2}|\tilde{X}_s^x|^4|R_s|^2 ds\right) \\
&\le 2\alpha(\alpha-1)n^2\mathbb{E}\left(\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{\alpha} ds\right) \\
&\le C_\alpha^4\mathbb{E}\left(\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^{\alpha} ds\right), \qquad (3.24)
\end{aligned}$$

where $C_\alpha^4 = 2\alpha(\alpha-1)n^2$.

For A_t^5, according to (3.20), and since f satisfies the equation (3.6), we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{u\le s\le t}|A_s^5|\right) &\le 2\alpha\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1}|\tilde{f}(\tilde{X}_s^x)|^2ds\right)\\ &\le 2\alpha K\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right)\\ &\le C_\alpha^5\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right),\end{aligned} \tag{3.25}$$

where $C_\alpha^5 = 2\alpha K$.

For A_t^6, according to (3.20), and since f satisfies the equation (3.6) and by the elementary inequality $|x|^2 \le 1+|x|^2$, we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{u\le s\le t}|A_s^6|\right) &\le 2\alpha(\alpha-1)\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-2}|\tilde{X}_s^x|^2|\tilde{f}(\tilde{X}_s^x)|^2ds\right)\\ &\le 2\alpha(\alpha-1)K\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right)\\ &\le C_\alpha^6\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right),\end{aligned} \tag{3.26}$$

where $C_\alpha^6 = 2\alpha(\alpha-1)K$.

For A_t^7, according to (3.20), and since f satisfies the linear growth condition and by Lemma 1.1.4, we obtain

$$\begin{aligned}\mathbb{E}\left(\sup_{u\le s\le t}|A_s^7|\right) &\le 2\alpha\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-1}|\tilde{X}_s^x||R_s||\tilde{f}(\tilde{X}_s^x)|d\langle N,Y\rangle_s\right)\\ &\le 2\alpha nC'K'\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}\sqrt{d\langle N\rangle_s}\sqrt{d\langle Y\rangle_s}\right)\\ &\le C_\alpha^7\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right),\end{aligned} \tag{3.27}$$

where $C_\alpha^7 = 2\alpha nC'K'$.

For A_t^8, Lemma 1.1.4 gives us

$$\begin{aligned}\mathbb{E}\left(\sup_{u\le s\le t}|A_s^8|\right) &\le 4\alpha(\alpha-1)\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha-2}|\tilde{X}_s^x|^3|R_s||\tilde{f}(\tilde{X}_s^x)|d\langle N,Y\rangle_s\right)\\ &\le 4\alpha(\alpha-1)nC'K'\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}\sqrt{d\langle N\rangle_s}\sqrt{d\langle Y\rangle_s}\right)\\ &\le C_\alpha^8\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right), \qquad (3.28)\end{aligned}$$

where $C_\alpha^8 = 4\alpha(\alpha-1)nC'K'$.

Combining (3.21)-(3.28) gives that

$$\begin{aligned}g(t,x) := \mathbb{E}\left[\sup_{u\le s\le t}\left(1+|\tilde{X}_t^x|^2\right)^{\alpha}\right] &\le (1+|x|^2)^\alpha + \mathbb{E}\left[\sup_{u\le s\le t}|A_s^1|\right] + \mathbb{E}\left[\sup_{u\le s\le t}|A_s^2|\right]\\ &+ \mathbb{E}\left[\sup_{u\le s\le t}|A_s^3|\right] + \mathbb{E}\left[\sup_{u\le s\le t}|A_s^4|\right] + \mathbb{E}\left[\sup_{u\le s\le t}|A_s^5|\right]\\ &+ \mathbb{E}\left[\sup_{u\le s\le t}|A_s^6|\right] + \mathbb{E}\left[\sup_{u\le s\le t}|A_s^7|\right] + \mathbb{E}\left[\sup_{u\le s\le t}|A_s^8|\right].\end{aligned}$$

Thus,

$$\begin{aligned}g(t,x) := \mathbb{E}\left[\sup_{u\le s\le t}\left(1+|\tilde{X}_t^x|^2\right)^{\alpha}\right] &\le (1+|x|^2)^\alpha + \frac{1}{4}\mathbb{E}\left[\sup_{u\le s\le t}\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}\right]\\ &+ C_\alpha^1\mathbb{E}\left[\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right] + \frac{1}{4}\mathbb{E}\left[\sup_{u\le s\le t}\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}\right]\\ &+ C_\alpha^2\mathbb{E}\left[\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right] + C_\alpha^3\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right)\\ &+ C_\alpha^4\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right) + C_\alpha^5\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right)\\ &+ C_\alpha^6\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right) + C_\alpha^7\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right)\\ &+ C_\alpha^8\mathbb{E}\left(\int_u^t\left(1+|\tilde{X}_s^x|^2\right)^{\alpha}ds\right).\end{aligned}$$

Therefore, we have

$$g(t,x) := \mathbb{E}\left[\sup_{u\le s\le t}\left(1+|\tilde{X}_t^x|^2\right)^\alpha\right] \le (1+|x|^2)^\alpha + C_\alpha^9 \mathbb{E}\left[\int_u^t \left(1+|\tilde{X}_s^x|^2\right)^\alpha ds\right] + \frac{1}{2}g(x,t),$$

where $C_\alpha^9 = C_\alpha^1 + C_\alpha^2 + C_\alpha^3 + C_\alpha^4 + C_\alpha^5 + C_\alpha^6 + C_\alpha^7 + C_\alpha^8$.

Hence, we obtain

$$\begin{aligned} g(t,x) := \mathbb{E}\left[\sup_{u\le s\le t}\left(1+|\tilde{X}_t^x|^2\right)^\alpha\right] &\le 2(1+|x|^2)^\alpha + 2C_\alpha^9 \int_u^t \mathbb{E}\left(1+|\tilde{X}_s^x|^2\right)^\alpha ds \\ &\le 2(1+|x|^2)^\alpha + 2C_\alpha^9 \int_u^t g(s,x)ds. \end{aligned}$$

By Gronwall's inequality (Lemma 1.1.1), we obtain

$$\begin{aligned} g(t,x) &\le 2(1+|x|^2)^\alpha \exp\left(2C_\alpha^9(T-u)\right) \\ &\le C_{T,u,\alpha}(1+|x|^2)^\alpha. \end{aligned}$$

It follows that,

$$\mathbb{E}\left[\sup_{u\le s\le t}\left(1+|\tilde{X}_t^x|^2\right)^\alpha\right] \le C_{T,u,\alpha}(1+|x|^2)^\alpha, \quad t\in[u,T], \quad x\in\mathbb{R},$$

where $C_{T,u,\alpha} = 2\exp\left(2C_\alpha^9(T-u)\right)$.

This completes the proof of Lemma 3.3.1. □

3.3.1.1 Proof of the main Theorem

By using the above surjectivity property lemma, we can now prove Theorem 3.3.1.

Proof of Theorem 3.3.1. First, we show that the mapping $x \longmapsto \tilde{X}_t^x$ is strictly increasing and continuous w.r.t. x for each $t \ge u$ and for almost all ω.

Let $\mathbb{Q}$ denote the set of all rational numbers. By Lemma 3.2.3,

$$\mathbb{P}(\tilde{X}_t^r < \tilde{X}_t^q, \quad u\le t<\infty, \quad r<q, \quad r,q\in\mathbb{Q}) = 1.$$

From this, we obtain that $x \longmapsto \tilde{X}_t^x$ is strictly increasing on $\mathbb{Q}$. By continuity w.r.t. x, it follows that $x \longmapsto \tilde{X}_t^x$ is strictly increasing and continuous w.r.t. $x \in \mathbb{R}$ for each $t \geq u$ and for almost all ω. Moreover, its inverse mapping is strictly increasing and continuous w.r.t. x for each $t \geq u$ and for almost all ω.

Hence, from Theorem 3.2.1, the mapping

$$\mathbb{R} \ni x \longmapsto \tilde{X}_t^x(\omega) \in \mathbb{R}, \quad \text{for all } t \geq u, \quad \text{for almost all } \omega,$$

is a continuous injection for each $t \geq u$. It follows that, for almost all ω, the mapping $x \longmapsto \tilde{X}_t^x(\omega)$ is one-to-one for all $t \geq u$.

Second, we show that

$$\lim_{|x|\to\infty} |\tilde{X}_t^x(\omega)| = \infty \quad \text{for all } t \geq u, \text{ for almost all } \omega, \text{ holds.} \tag{3.29}$$

By the above Lemma 3.3.1, in the case where $\alpha = -2$, we have

$$\lim_{|x|\to\infty} \mathbb{E}\left[\sup_{u\leq t\leq T} \frac{1}{\left(1+|\tilde{X}_t^x|^2\right)^2}\right] = 0.$$

Fatou's lemma implies that

$$\liminf_{|x|\to\infty} \sup_{u\leq t\leq T} \frac{1}{\left(1+|\tilde{X}_t^x|^2\right)^2} = 0, \quad \text{a.s.} \tag{3.30}$$

We now show that

$$\lim_{x\to\infty} \inf_{u\leq t\leq T} \tilde{X}_t^x = +\infty.$$

From (3.30), we deduce that

$$\limsup_{x\to\infty} \inf_{u\leq t\leq T} |\tilde{X}_t^x| = \infty, \quad \text{a.s.} \tag{3.31}$$

By the increasing property of the mapping $x \longmapsto \tilde{X}_t^x$, we have

$$\inf_{u\leq t\leq T} |\tilde{X}_t^x| \leq \inf_{u\leq t\leq T} \left(\tilde{X}_t^x \vee |\tilde{X}_t^0|\right) \leq \inf_{u\leq t\leq T} \tilde{X}_t^x \vee \sup_{u\leq t\leq T} |\tilde{X}_t^0|, \quad \text{for all } x \geq 0, \text{ a.s.} \tag{3.32}$$

Combining (3.31) and (3.32) gives that

$$\limsup_{x\to\infty} \inf_{u\le t\le T} \tilde{X}_t^x = +\infty, \quad \text{a.s.} \tag{3.33}$$

Where $\sup\limits_{u\le t\le T} |\tilde{X}_t^0| < \infty$, a.s. holds.

Noting that the mapping $x \longmapsto \inf\limits_{u\le t\le T} \tilde{X}_t^x$ is increasing, we deduce from (3.33)

$$\lim_{x\to\infty} \inf_{u\le t\le T} \tilde{X}_t^x(\omega) = +\infty, \quad \text{for almost all } \omega.$$

By analogous arguments, one can also show that

$$\lim_{x\to-\infty} \sup_{u\le t\le T} \tilde{X}_t^x(\omega) = -\infty, \quad \text{for almost all } \omega.$$

Consequently, we obtain that

$$\lim_{|x|\to\infty} |\tilde{X}_t^x(\omega)| = \infty \quad \text{for all } t \ge u, \text{ for almost all } \omega.$$

Hence, the mapping $x \longmapsto \tilde{X}_t^x(\omega)$ is an onto mapping from $\mathbb{R}$ to $\mathbb{R}$ for each $t \ge u$ and for almost all ω.

Since the mapping $x \longmapsto \tilde{X}_t^x(\omega)$ is a continuous and bijective mapping (one-to-one and onto mapping) from $\mathbb{R}$ to $\mathbb{R}$ for each $t \ge u$ and for almost all ω and its inverse mapping is also continuous for each $t \ge u$ and for almost all ω. As a result, for all $t \ge u$ and for almost all ω, the mapping $x \longmapsto \tilde{X}_t^x(\omega)$ is a homeomorphism on $\mathbb{R}$.

This completes the proof of Theorem 3.3.1. □

3.4 Concluding remarks

The main contribution of this chapter is the study of a financial stochastic differential equation known as the natural equation. The objective was to examine the qualitative properties of the stochastic flows such as bicontinuity and homeomorphism of the solution for the natural equation in the one-dimensional case under non-Lipschitz coefficients.

Under Lipschitz coefficients, these properties are classically obtained using standard tools such as the Gronwall inequality, which ensures stability and regular dependence on initial data. However, in the non-Lipschitz coefficients, this approach is no longer applicable, and the analysis becomes much more complex. The weaker regularity of the coefficients makes the problem more challenging and requires more refined techniques to establish the continuity of the solution.

By Kolmogorov's criterion and Bihari's inequality, we are able to overcome these difficulties and establish the bicontinuous modification and the homeomorphism property of stochastic flows of the solution for the natural equation. These results show that even under weaker non-Lipschitz conditions the solution retains strong regularity properties.

The findings of this chapter provide that the bicontinuity and homeomorphism properties ensure that the natural model behaves in a stable and consistent way with respect to the initial data. This is particularly relevant in credit risk modeling, where the accuracy and stability of the model are essential for describing default events and assessing financial risk. Hence, proving these results under weaker assumptions increases the reliability and applicability of the model in practice.

Conclusion and perspectives

Conclusion

Throughout this thesis, we focused on the study of an important financial model, the so-called natural model, with particular emphasis on its representation through the significant stochastic differential equation known as the natural equation. The principal contributions and findings of this thesis can be summarized as follows:

In the first contribution of this thesis, we applied the successive approximation method to prove the existence and uniqueness of the solution to the natural equation under weaker conditions, termed the non-Lipschitz conditions. Moreover, we studied the mean-square stability of solutions for the natural equation with the same conditions. The results obtained show that the solution of the natural equation can still exist and be unique when the classical Lipschitz condition is relaxed.

In the second contribution of this thesis, based on Kolmogorov's continuity criterion and Bihari's inequality, we studied the bicontinuity property of the solution for the natural equation in the one-dimensional case under non-Lipschitz coefficients. In addition, we established the homeomorphism property of the solution using the approach of Yamada and Ourga. These results contribute to a better understanding of stochastic flows under non-Lipschitz coefficients. Furthermore, proving this property of stochastic flows under non-Lipschitz coefficients was particularly challenging due to the inability to apply Gronwall's inequality.

Perspectives

The study carried out in this thesis can be used to develop new research on stochastic flows under non-Lipschitz conditions. For future work, several interesting issues remain to be addressed, such as:

• Demonstrates additional properties of the stochastic flows, such as the differentiability of the solution of the natural equation in the one-dimensional case under non-Lipschitz conditions.

• Extend the study of stochastic flows of the solution for the natural equation in the multi-dimensional case.

• Introduce numerical simulations which will be carried out to illustrate the theoretical results obtained.

• Demonstrates the stochastic flows of solutions to non-Lipschitz natural equations driven by Lévy noise.

• Demonstrates the stochastic flows of solutions of natural equations on manifolds.

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« حول تدفق معادلة تفاضلية عشوائية »

الملخص:
في هذه الاطروحة، نركز على نموذج مالي بارز يُعرف بالنموذج الطبيعي. وعلى وجه التحديد، يتم تمثيل هذا النموذج بمعادلة تفاضلية عشوائية تسمى المعادلة الطبيعية، والتي تلعب دورًا مهما في بحثنا. أولاً، نقوم بدراسة وجود وتفرد الحلول للمعادلة الطبيعية في حالة أحادية البعد في ظل شرط أضعف من شرط ليبشيتز، والتي تسمى بشرط غير ليبشيتز. بالإضافة إلى ذلك، ندرس استقرار متوسط مربع حلول المعادلة الطبيعية. ثانياً، ندرس خصائص الاستمرارية الثنائية والهومومورفية للتدفقات العشوائية لحل المعادلة الطبيعية أحادية البعد ذات المعاملات غير ليبشيتز فيما يتعلق بالقيمة الأولية بناءً على معيار كولموغوروف.
كلمات مفتاحية: التدفقات العشوائية، المعاملات غير الليبشيتزية، المعادلات التفاضلية العشوائية، الوجود و التفرد، السوق المالية، الاستمرارية الثنائية، الهومومورفية ، متراجحة بيهاري، مبرهنة كولموغوروف.

« Sur le flot d'une équation différentielle stochastique »

Résumé :
Dans cette thèse, nous nous concentrons sur un modèle financier important, connu sous le nom de modèle naturel. Plus précisément, ce modèle est décrit par une équation différentielle stochastique appelée équation naturelle, qui joue un rôle important dans notre étude. Dans un premier temps, nous étudions l'existence et l'unicité des solutions de l'équation naturelle dans le cas unidimensionnel sous une condition plus faible que la condition de Lipschitz, appelée condition de non-Lipschitz. De plus, nous étudions la stabilité en moyenne quadratique des solutions de l'équation naturelle. Dans un second temps, nous examinons les propriétés de bicontinuité et d'homéomorphisme des flots stochastiques de la solution de l'équation naturelle unidimensionnelle à coefficients non lipschitziens, par rapport à la condition initiale, en nous basant sur le critère de Kolmogorov.

Mots clés : Flots stochastiques, coefficients non lipschitziens, équations différentielles stochastiques, existence et unicité, marché financier, bicontinuité, homéomorphisme, inégalité de Bihari, théorème de Kolmogorov.

« On the flow of a stochastic differential equation »

Abstract :
In this thesis, we focus on a prominent financial model, known as the natural model. Specifically, this model is represented by a stochastic differential equation called the natural equation, which plays an important role in our research. Firstly, we investigate the existence and uniqueness of solutions to the natural equation in the one-dimensional case under a condition weaker than the Lipschitz condition, termed the non-Lipschitz condition. Additionally, we study the mean-square stability of the solutions to the natural equation. Secondly, we study the bicontinuity and homeomorphism properties of stochastic flows of the solution to the one-dimensional natural equation with non-Lipschitz coefficients with respect to the initial value based on Kolmogorov's criterion.

Key words : Stochastic flows, Non-Lipschitz coefficients, Stochastic differential equations, Existence and uniqueness, Financial market, Bicontinuity, Homeomorphism, Bihari inequality, Kolmogorov's theorem.