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**Faculté des Mathématiques, Informatique et Télécommunications**

# Thèse

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Par :

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Thème :

# Contribution to the non-parametric estimation of semi-Markov processes

Thèse soutenue le 02 juin 2026 devant le jury composé de :

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# ** Dedication **

***In the name of Allah, the Most Gracious, the Most Merciful.***

*All praise is due to* ***Allah****, who granted me the strength, patience, and faith to*
*reach this milestone. With His help, I have gathered the sleepless nights,*
*the long hours, and the challenges of this journey between the covers*
*of this humble thesis.*

*To the light of my life, my beloved parents your prayers were my shield,*
*your sacrifices my foundation, and your endless support my motivation.*
*You have walked beside me in every step, lifted me when I was weary,*
*and believed in me even in silence. May Allah protect you,*
*reward you, and bless you always.*

*This work is a reflection of your love and a small gift in return*
*for everything you've done for me.*

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***In the name of Allah, the Most Gracious, the Most Merciful.***

*All praise be to **Allah**, Lord of the worlds, and peace and blessings be upon His noble Prophet Muhammad, the master of creation and the seal of messengers.*

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*Finally, I express my gratitude to all those who contributed, directly or indirectly, to the success of this work. May **Allah** bless you all.*

## الملخص:

يتناول هذا العمل في جزئه الأول تقدير التوزيع المستقر لعملية شبه ماركوف مستمرة الزمن ذات فضاء حالات منتهٍ، وذلك بالاعتماد على منهجية لا معلمية. وقد تم بناء مقدّرات نواة لكل من التوزيع المستقر، ومتوسط زمن الإقامة، ومتوسط زمن العود. كما تم إثبات التقارب شبه المؤكَّد لهذه المقدّرات، إلى جانب دراسة خاصية الطبيعية التقاربية لها.

كما نقدّم مقدّرات نواة لمجموعة من الخصائص الأساسية لعمليات شبه ماركوف المستمرة، بما في ذلك أزمنة الإقامة الشرطية وغير الشرطية، وكذلك نواة شبه ماركوف. ويهدف ذلك إلى توظيف هذه المقدّرات في تحليل وإثبات الخصائص التقاربية للمقدّرات المطوَّرة.

أما في الجزء الثاني، فنبحث في تقييم أداء الأنظمة المعتمدة على نماذج شبه ماركوف، حيث تم اقتراح مقدّر نواة لمؤشر الأداء، ثم دراسة خصائصه التقاربية بشكل مفصّل.

ولإظهار فعالية النتائج النظرية، تم دعم كل قسم بأمثلة عددية توضيحية.

## الكلمات الرئيسية:

عمليات شبه ماركوف ؛ التوزيع المستقر؛ أداء النظام ؛ مقدار النواة ؛ التقدير اللا معلمي ؛ الاتساق ؛ الوضع الطبيعي المقارب.

**Résumé**:

Le présent travail porte dans un premier temps sur l'estimation de la distribution stationnaire d'un processus semi markovien à temps continu et à espace d'états fini (SMP) par une méthode non paramétrique. Nous présentons la construction des estimateurs à noyau pour la distribution stationnaire, ainsi que pour la moyenne du temps de séjour et la récurrence moyenne. Ensuite, nous établissons la convergence forte et la normalité asymptotique des estimateurs proposés.

Nous donnons des estimateurs à noyau des principales caractéristiques d'un processus semi-markovien en temps continu, telles que les temps de séjour conditionnels et inconditionnels, ainsi que le noyau semi-markovien. L'objectif principal est d'utiliser ces estimateurs afin d'établir les propriétés asymptotiques des estimateurs construits.

Dans un second temps, nous étudions la performance des systèmes semi-markoviens. Nous introduisons un estimateur à noyau de la performance. Nous analysons ensuite les propriétés asymptotiques des estimateurs proposés.

Afin de prouver l'efficacité de nos résultats théoriques, chaque partie est illustrée à travers un exemple numérique.

**Mots clés**: Processus semi-markoviens; Estimateur à noyau; Distribution stationnaire; Performance du système; Consistance; Normalité asymptotique.

**Abstract**:

The present work firstly concerns the estimation of the stationary distribution of a continuous-time semi-Markov process with a finite state space (SMP) using a non-parametric method. We present the construction of kernel estimators for the stationary distribution, as well as for the mean sojourn time and the mean recurrence time. Next, we establish the almost sure convergence and the asymptotic normality of the proposed estimators.

We provide kernel estimators of the main characteristics of a continuous-time semi-Markov process, such as the conditional and unconditional sojourn times, as well as the semi-Markov kernel. The primary objective is to use these estimators to establish the asymptotic properties of the constructed estimators.

In a second stage, we study the performance of semi-Markov systems. We introduce a kernel estimator of performance. We then analyze the asymptotic properties of the proposed estimators.

To demonstrate the effectiveness of our theoretical results, each part is illustrated with a numerical example.

**Keywords**: Semi-Markov processes; Kernel estimator; Stationary distribution; Performance; Consistency; Asymptotic normality.

# List of Works

### Publications

- Kernel Smoothing of The Mean Performance For Homogeneous Continuous Time Semi-Markov Process, (2025), T. Hamlat, F. Mokhtari, S. Rahmani, Reliability: Theory & Applications, 20 (1 (82)), 397-412.

- Hamlat Tayeb, Fatiha Mokhtari and Saâdia Rahmani. Kernel smoothing of the stationary distribution for continuous time semi-Markov process, submitted, 2025.

### Presentations in International Seminars

- Participation in the International Symposium MOAD 2022 (Methods and Tools for Decision Support), held in Bejaia, Algeria, from 15 to 17 November 2022. Presentation title: ‘Nonparametric Estimation of Performability for Homogeneous Continuous-Time Semi-Markov Models’.

- Participation in the First International Workshop on Statistics and Their Applications (IWSA 2023), held in Saida, Algeria, from 1 to 2 March 2023. Presentation title: ‘Nonparametric Estimation of the Stationary Distribution for Discrete- and Continuous-Time Semi-Markov Processes’.

- Participation in the 25th International Pure Mathematics Conference (IPMC 2025), held in Islamabad, Pakistan, from 29 to 31 August 2025. Presentation title: ‘Nonparametric Estimation of Performability in Homogeneous Continuous-Time Semi-Markov Processes’.

- Participation in the International Conference on Mathematics and Computers with Applications (ICMCA 2025), held in Istanbul, Türkiye, from 16 to 18 July 2025. Presentation title: ‘Estimation of the Stationary Distribution for Continuous-Time Semi-Markov Models: A Nonparametric Approach with Applications’.

### Presentations in National Seminars

- Participation in the First National Conference on Applied Mathematics, Statistics and Their Applications (NCAMSA 2024), held in Batna, Algeria, from 18 to 19

November 2024. Presentation title: ‘Nonparametric Estimation of the Stationary Distribution in Continuous-Time Semi-Markov Processes’.

- Participation in the 4th National Conference on Mathematics and Their Applications (CNMA 2024), held in Mila, Algeria, from 7 to 8 December 2024. Presentation title: ‘Nonparametric Estimation of Performability in Homogeneous Continuous-Time Semi-Markov Processes’.
- Participation in the First National Conference on Mathematics and Their Applications (NCMA 2025), held at the University of Adrar, Algeria, on 4–5 November 2025. Presentation title: ‘Empirical Estimation of the Expected Cumulative Operational Time in Finite Semi-Markov Systems’.

### Member of the Research Project

- I have served as a member of the PRFU research project (Project Code: C00L03UN200120220004), entitled *Functional Statistics and Queue Model Study*, approved on 01 January 2022.

### Scientific and Academic Activities

- Oral Presentation: “Concept of Semi-Markov Process”, presented during the Student Week, 16 May 2023, LMSSA, Saida, Algeria.
- Oral Presentation: “Empirical Estimation of the Stationary Distribution for Discrete and Continuous-Time Semi-Markov Processes”, presented during the Applied Mathematics Week, 4 March 2024.
- Poster Presentation: “Semi-Markov Processes: Theory and Basic Concepts”, presented at the Student Day, 19 May (Saida, Algeria).
- Research Internship (France): Internship carried out at the Laboratory of Applied Mathematics of Compiègne (LMAC), University of Technology of Compiègne (UTC), under the supervision of Professors Salim Bouzebda and Nikolas Limnios, from 20 November to 19 December 2024.
- Research Internship (France): Internship carried out at the Laboratory of Applied Mathematics of Compiègne (LMAC), University of Technology of Compiègne (UTC), under the supervision of Professor Nikolas Limnios, from 15 October to 13 November 2025.

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# Notations

### Abbreviations

| | |
|---|---|
| MP | Markov process |
| MC | Markov chain |
| SMP | Semi-Markov process |
| SMC | Semi-Markov chain |
| MRP | Markov renewal process |
| EMC | Embedded Markov chain |
| SMM | Semi-Markov models |
| HSMM | hidden semi-Markov models |
| KDE | Kernel density estimation |
| SLLN | Strong law of large numbers |
| CLT | Central limit theorem |
| CI | Confidence interval |
| LCL | Lower confidence bound |
| UCL | Upper confidence bound |
| MSE | Mean squared error |
| $r.v.$ | Random variable |

### Sets

| | |
|---|---|
| $\mathbb{N}$ | Set of natural numbers |
| $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ | Set of strictly positive natural numbers |
| $\mathbb{R}_+ = [0, \infty)$ | Set of nonnegative real numbers |
| $\mathbb{R}_+^* = (0, \infty)$ | Set of strictly positive real numbers |
| $\Omega$ | Set of all possible outcomes |
| $E = \{1, \dots, s\}$ | Finite state space |
| $U$ | Operating states of the semi-Markov system |
| $D$ | Failure states of the semi-Markov system |

### Convergence

| | |
|---|---|
| $\xrightarrow{a.s.}$ | Almost sure convergence (strong consistency) |
| $\xrightarrow{\mathcal{D}}$ | Convergence in distribution |
| $\xrightarrow{\mathbb{P}}$ | Convergence in probability |

## Probabilistic

| | |
|---|---|
| $(\Omega, \mathcal{F}, \mathbb{P})$ | Probability space |
| $(\Omega, \mathcal{F})$ | Measurable space |
| $(E, \mathscr{F})$ | Measurable space |
| $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ | Separable metric space |
| $\mathcal{B}(\mathcal{S})$ | Borel $\sigma$-algebra on $\mathcal{S}$ |
| $\mathcal{F}$ | $\sigma$-algebra on $\Omega$ |
| $\mathscr{F}$ | $\sigma$-algebra on E |
| $\mathbb{P}$ | Probability |
| $\mathbb{P}_i(\cdot)$ | Conditional probability $\mathbb{P}(\cdot \mid J_0 = i)$ |
| $\mathbb{E}$ | Expectation |
| $\mathbb{E}_i$ | Conditional expectation corresponding to $\mathbb{P}_i$ |
| $Z := (Z_t)_{t \in \mathbb{R}_+}$ | Semi-Markov process (SMP) |
| $(J, S) := (J_n, S_n)_{n \in \mathbb{N}}$ | Markov renewal process (MRP) |
| $J := (J_n)_{n \in \mathbb{N}}$ | Visited states, embedded Markov chain (EMC) |
| $S := (S_n)_{n \in \mathbb{N}}$ | Jump times of the semi-Markov process |
| $X := (X_n)_{n \in \mathbb{N}}$ | Sojourn times between successive jumps |
| $(S_n^i)_{i \in E, n \in \mathbb{N}}$ | Renewal process of successive times of visits to state $i$ |
| $\alpha := (\alpha_i)_{i \in E}$ | Initial law of the semi-Markov process and of the embedded Markov chain |
| $\nu := (\nu_i)_{i \in E}$ | Stationary law of the embedded Markov chain |
| $\pi := (\pi_i)_{i \in E}$ | Stationary law of the semi-Markov process |
| $p := (p_{ij})_{i,j \in E}$ | Transition matrix of the embedded Markov chain |
| $q(\cdot) := (q_{ij}(\cdot))_{i,j \in E}$ | Density of the semi-Markov kernel |
| $Q(\cdot) := (Q_{ij}(\cdot))_{i,j \in E}$ | Semi-Markov kernel |
| $F(\cdot) := (F_{ij}(\cdot))_{i,j \in E}$ | Sojourn time distribution, in state $i$ before visiting state $j$ |
| $H(\cdot) := (H_i(\cdot))_{i \in E}$ | Sojourn time distribution in state $i$ |
| $\overline{H}(\cdot) := (\overline{H}_i(\cdot))_{i \in E}$ | Survival function in state $i$ |
| $\mathrm{diag}(H_i(\cdot))_{i \in E}$ | Matrix of sojourn time distribution functions |
| $G(\cdot)$ | Kernel function |
| $K(\cdot)$ | Derivative of kernel function $G$ |
| $h(\cdot)$ | Smoothing parameter |
| $Q_1 * Q_2$ | Stieltjes convolution of $Q_1, Q_2$ |
| $Q^{(n)}$ | n-fold Stieltjes convolution of Q |
| $P(\cdot) := (P_{ij}(\cdot))_{i,j \in E}$ | Transition function of the semi-Markov process |
| $\Psi(\cdot) := (\Psi_{ij}(\cdot))_{i,j \in E}$ | Markov renewal function |
| $m := (m_i)_{i \in E}$ | Mean sojourn time in state i |
| $\overline{m}$ | Mean sojourn times of the semi-Markov process |
| $\mu_{ij}$ | Mean recurrence time from state $i$ to state $j$, for semi-Markov process |

| | |
|---|---|
| $\mu^*_{ij}$ | Mean recurrence time from state $i$ to state $j$, for the embedded Markov chain |
| $M$ | Fixed censoring time |
| $\mathcal{M}_E$ | Set of real matrices on $E \times E$ |
| $\mathcal{M}_E(\mathbb{N})$ | Matrix-valued functions defined on $\mathbb{N}$, with values in $\mathcal{M}_E$ |
| $\mathcal{Y}(M)$ | Sample path of the semi-Markov process |
| $N(M)$ | Number of jumps of $Z$ in the time interval $[1, M]$ |
| $N_i(M)$ | Number of visits to state $i$ of the semi-Markov process, up to time $M$ |
| $N_{ij}(M)$ | Number of transitions from state $i$ to state $j$, of the semi-Markov process, up to time $M$ |
| $N_{ij}(t, M)$ | Number of transitions from state $i$ to state $j$ of the semi-Markov process, up to time $M$, with sojourn time in state $i$ less than or equal to $t$ |

### Performance and mean performance of a semi-Markov system

| | |
|---|---|
| $\Phi(\cdot)$ | Performance |
| $\overline{\Phi}(\cdot)$ | Mean performance |
| $\mathcal{W}(\cdot)$ | Cumulative functional |
| $\overline{\mathcal{W}}(\cdot)$ | Expected cumulative functional |
| $\mathcal{G}(t, \cdot)$ | Marginal distribution of performance function at time $t$ |

### Various symbols

| | |
|---|---|
| $\delta_{ij}$ | Kronecker symbol, i.e., $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise |
| $\mathbb{1}_A$ | Indicator function of $A$ |
| $*$ | Convolution product |
| $\sim$ | Follows |
| $\approx$ | Approximately equal |
| $\mathcal{N}(0, \sigma^2)$ | Normal distribution with mean 0 and variance $\sigma^2$ |
| $\mathcal{N}(0, 1)$ | Standard normal random variable with mean 0 and variance 1 |

# General Introduction

Stochastic processes constitute a fundamental framework for modeling systems that evolve over time under uncertainty. A key characteristic of certain stochastic processes is their limited memory property, meaning that the future evolution of the system depends only on its present state, not on its past trajectory. Such processes offer powerful tools for capturing the dynamics of various real world phenomena, including those modeled by Markov and semi-Markov processes.

A Markov process (MP), named by the Russian mathematician Andrey Markov, is characterized by its memoryless property. In such processes, the future state of the system depends solely on its current state and not on the sequence of states that preceded it. This property, known as the Markov property, simplifies the analysis and computation of the process's behavior. Markov processes have been extensively studied over the past several decades, leading to a rich body of theoretical results and practical applications. They are widely used to model systems where the time spent in each state follows an exponential distribution in continuous time or a geometric distribution in discrete time.

Although these models are intuitive and computationally efficient, their restrictive assumptions on the distribution of sojourn times limit their applicability in many real-world scenarios.

To overcome the limitations of Markov models, semi-Markov processes (SMP) were introduced in the mid-20th century, with key contributions from Levy [1954], Smith [1955] and Takács [1954]. These models generalize the Markov framework by allowing the sojourn time in each state to follow an arbitrary probability distribution. This additional flexibility makes semi-Markov processes suitable for modeling systems where state transitions occur at irregular intervals or are influenced by external factors. Unlike Markov processes, the memoryless property in semi-Markov models (SMM) applies to the sojourn time within a state rather than the calendar time. This distinction enables semi-Markov processes to capture a broader range of temporal dynamics while retaining the markovian dependency structure between states. A distinctive feature of SMPs is that the sojourn time in a given state is not limited to geometric or exponential distributions; rather, it can follow any distribution on the positive real axis. Pyke [1961a], Pyke [1961b] was the first who established the foundations of the theory of SMP by describing it as a generalization of Markov jump processes as well as renewal processes. Since then, significant developments have been achieved starting from Pyke and Schaufele [1964] and Çinlar

[1969] to Wu et al. [2021] and Janssen and Manca [2007] and references therein, resulting in a comprehensive framework for both discrete-time and continuous-time settings.

A versatility of Markov and semi-Markov processes has led to their adoption in numerous domains. In queuing systems, Korolyuk et al. [1975] in the study of queuing networks, SMPs help in understanding systems where service times and inter-arrival times are not exponentially distributed, providing a more general framework than traditional Markov models. In reliability engineering Limnios and Oprisan [2001], SMPs are utilized to model and analyze the reliability of complex systems, allowing for the assessment of system performance and failure probabilities over time. In finance and credit risk D'Amico et al. [2006], SMPs are applied to model credit rating migrations and the timing of defaults, offering a nuanced approach to credit risk assessment. In speech recognition Yu [2010], hidden semi-Markov models (HSMMs), an extension of SMPs, are used in speech recognition to model temporal dependencies and durations of phonetic units more effectively than traditional hidden Markov models. In biology, Gorissen and Vanderzande [2011] used SMPs to model biological processes, such as animal movement patterns and population dynamics, where the time spent in various states is variable. In insurance mathematics Buchardt et al. [2015], SMPs are employed to model various insurance risks, including claim occurrences and policyholder behavior, enhancing the accuracy of risk assessments. In manufacturing systems Silvestrov and Manca [2017], SMPs model systems where the processing times are variable, aiding in the optimization of production schedules and maintenance planning. In physics Ertel et al. [2022], in physical systems, SMPs assist in modeling phenomena where particles or entities transition between states with non-exponential waiting times. In artificial intelligence and machine learning Ascione and Cuomo [2022], SMPs contribute to reinforcement learning algorithms, especially in scenarios where decision-making processes involve actions with variable durations.

Advancements in statistical inference methods for semi-Markov processes have emerged in response to growing interest in this area. Parametric approaches have been widely developed to estimate key parameters, such as transition probabilities and sojourn-time distributions. Early works by Moore and Pyke [1968] and Lagakos et al. [1978] laid the groundwork for maximum likelihood estimation (MLE) and empirical estimation techniques. These parametric methods are known for their desirable asymptotic properties, but they may encounter difficulties when dealing with small sample sizes or when the underlying distributions are difficult to specify. In contrast, nonparametric methods have been introduced to address these challenges. Studies such as those by Ouhbi and Limnios [1999] expanded nonparametric estimation techniques to handle more complex scenarios, including nonlinear functional and continuous-time processes. While nonparametric methods provide greater robustness and flexibility, they often come with increased computational complexity. Recent research has also explored hybrid approaches that combine the strengths of both parametric and nonparametric methods, enabling more accurate and reliable modeling of semi-Markov processes.

In the last three decades, the estimation of semi-Markov processes has been a fundamental area of research in applied probability and stochastic modeling. Various methodologies have been developed to enhance the accuracy and applicability of estimation techniques for SMPs. Ouhbi and Limnios [1996] developed nonparametric estimation for the semi-Markov kernel, advancing reliability analysis and stochastic modeling. A year later, Dewanji [1997] introduced methods for estimating sojourn time distributions in cyclic semi-Markov processes at equilibrium, offering insights into time-dependent behaviors in stochastic systems. Around the same year, Ouhbi and Limnios [1997a] studied the asymptotic behavior of the Markov renewal matrix and its application to semi-Markov processes, providing deeper theoretical insights into renewal probabilities. Additionally, in the same year, Ouhbi and Limnios [1997b] applied semi-Markov processes to estimate the reliability of complex systems, demonstrating their practical utility in engineering and risk analysis. In 1999, Ouhbi and Limnios [1999] extended nonparametric estimation techniques based on hazard rate functions, increasing their flexibility in real-world applications. In 2006, Barbu and Limnios [2006] introduced maximum likelihood estimation methods for hidden semi-Markov models, contributing to parameter estimation in systems with unobservable states. In the same year, Girardin and Limnios [2006] explored entropy in semi-Markov processes with Borel state spaces, linking entropy concepts to long-term behavior and invariance principles. In 2012, Votsi et al. [2012] applied semi-Markov models to estimate earthquake occurrences, enhancing seismic risk assessment. Dumitrescu et al. [2016] introduced minimum divergence estimators for the Radon-Nikodym derivatives of the semi-Markov kernel, providing new tools for estimating transition dynamics in complex stochastic processes. In 2021, Asanjarani et al. [2022] compared major estimation approaches for sojourn times and transition intensities in semi-Markov multi-state models, offering insights into model selection for real-world applications. Finally, Ayhar et al. [2022] proposed kernel estimators for key characteristics of continuous-time semi-Markov processes and established their asymptotic properties, including consistency and asymptotic normality. They illustrate their approach with a three-state system and provide numerical evaluations. These advancements significantly strengthen the theoretical and practical understanding of SMP estimation, leading to more reliable and adaptable models for real-world stochastic processes.

The study of stationary distributions for semi-Markov processes plays a crucial role in understanding their long-term behavior and applications in various domains. A stationary distribution of an SMP describes the limiting probabilities of occupying different states as time approaches infinity, generalizing the steady-state concept of Markov chains (MC) by incorporating sojourn time distributions Limnios et al. [2005]. In 2012, Barbu et al. [2012] further expands on the theoretical framework of stationary distributions, providing analytical approaches for determining these distributions in practical settings.

Beyond stationary properties, performance and performability analysis of SMP has gained significant attention, particularly in reliability and operational modeling. Limnios et al. [2006] explored the role of SMP in evaluating system performance, addressing both transient and steady-state reliability metrics. Additionally, Ouhbi et al. [2007] introduced the concept of expected cumulative operational time, which quantifies the total time a system remains operational within a given period, serving as a key measure in reliability assessment. These contributions collectively advance both the theoretical and applied understanding of semi-Markov processes in the fields of performance evaluation and long-term system behavior analysis. Main results presented in this thesis build upon these foundational works, extending their methodologies and applying them to new contexts within the framework of semi-Markov processes.

Several R packages have been developed to facilitate the modeling, estimation, and analysis of semi-Markov processes in both discrete and continuous settings. A `hsmm` package Bulla et al. [2010] specializes in hidden semi-Markov models, providing methods for simulation and maximum likelihood estimation while offering more flexible sojourn time distributions compared to standard hidden Markov models. A `SemiMarkov` package Król and Saint-Pierre [2015] provides parametric maximum likelihood estimation for continuous time semi-Markov models, supporting different sojourn time distributions such as exponential, Weibull, and exponentiation Weibull. For discrete time semi-Markov processes, the `SMM` package Barbu et al. [2018] offers parametric and nonparametric estimation, along with simulation capabilities using various discrete distributions such as Geometric, Poisson, and Negative Binomial. Extending the functionalities of `SMM`, the `smmR` package Barbu et al. [2023] includes additional tools for computing reliability, maintainability, availability, and failure rates in multi-state semi-Markov models. These packages collectively contribute to the practical implementation and study of semi-Markov processes across various domains.

Most existing estimation techniques for continuous-time semi-Markov processes rely on empirical methods. While straightforward and widely used, empirical distribution functions are step functions and may fail to provide accurate approximations when the true underlying distribution is continuous. This limitation motivates the use of smoothing techniques, particularly kernel-based methods, to construct continuous estimators that better reflect the underlying probabilistic structure.

Kernel density estimation (KDE), introduced by Rosenblatt [1956] and Parzen [1962], is a classical nonparametric approach for estimating the probability density function of a continuous random variable. The kernel estimator smooths the empirical distribution by spreading the probability mass over a neighborhood defined by a kernel function and a bandwidth parameter. This technique has well-established theoretical properties, including pointwise consistency and asymptotic normality, as shown by Silverman [2018], and others. Bandwidth selection is a crucial component in practice and is addressed through various strategies, such as plug-in and cross-validation methods Sarda [1993], Altman and Leger [1995], Bowman et al. [1998], Polansky and Baker [2000].

Despite the broad applicability of KDE, its integration into the semi-Markov framework has been relatively limited. Shamsuddinov [2001] studied kernel estimation for sojourn time densities under the assumption that the distribution depends only on the current state, with a fixed bandwidth and without investigating asymptotic normality. Dumitrescu et al. [2016] proposed kernel estimators for more general semi-Markov models and studied their convergence in the space of integrable functions. Related developments for Markov processes include the works of Laksaci and Yousfate [2002], Roussas [1989, 1990], and Athreya and Atuncar [1998], while Atuncar et al. [2008] investigated consistency of estimators in the context of reliability.

The goal of this thesis is to develop and study kernel estimators tailored for continuous-time semi-Markov processes. We construct estimators for the stationary distribution and the associated functions, the mean performance and the cumulative operational time. We also establish the asymptotic properties of these estimators, namely uniform strong consistency and asymptotic normality. Furthermore, we obtain that the main theoretical results are illustrated through a numerical example of a three state continuous-time semi-Markov system.

In this thesis, we develop nonparametric kernel-based estimators for the stationary distribution and the performance function of continuous-time semi-Markov processes. We investigate their theoretical behavior, establishing results concerning their consistency and asymptotic normality. These estimators are further illustrated through numerical simulations and applied case studies.

# Contributions of the Thesis

## Objectives

The primary objectives of this thesis are summarized as follows:

(i) The construction of kernel estimators for the stationary distribution and related key quantities. In this context, we establish fundamental asymptotic properties of these estimators, including strong consistency and asymptotic normality. This contribution addresses a notable gap in the literature by applying the well-established Parzen–Rosenblatt nonparametric framework to semi-Markov models, thereby advancing the statistical theory of these stochastic systems. The theoretical findings are supported by a numerical example that illustrates the asymptotic behavior of the proposed estimators.

(ii) The investigation of the modeling of system performance within the semi-Markov framework from a nonparametric estimation perspective. We present the nonparametric kernel estimators of the mean performance and the cumulative operational time. These estimators are derived using kernel-based methods applied to the underlying transition characteristics, and their convergence properties are rigorously analyzed. The main theoretical results are validated through a numerical example involving a three-state continuous-time semi-Markov system.

## Outline of the Thesis

In chapter 1, we introduces the essential mathematical framework required for the analysis of semi-Markov processes. It includes notational conventions and a review of foundational concepts such as continuous-time Markov processes and Markov renewal theory. These preliminaries are indispensable for the formal development of the semi-Markov framework presented in subsequent chapters.

In chapter 2, we present the theoretical foundations and construction of continuous-time semi-Markov processes. These models generalize both Markov and renewal processes by allowing non-exponential sojourn time distributions, thereby relaxing the memoryless property inherent to classical Markov models. The formal definition of semi-Markov processes is presented, followed by a discussion of their probabilistic structure and a survey of relevant application domains. The chapter concludes with an overview of numerical techniques and Monte Carlo simulation methods for analyzing such processes when closed-form expressions are not available.

In chapter 3, the focus shifts to the statistical estimation of semi-Markov characteristics. We present a selection of results from nonparametric estimation theory that provide the necessary theoretical tools for the construction and analysis of estimators. Empirical and kernel estimators are proposed for key quantities such as the semi-Markov kernel and sojourn time distributions, under the assumption of a finite state space. The asymptotic properties of these estimators, including strong consistency and asymptotic normality, are rigorously established.

In chapter 4, we introduce the kernel estimation of the stationary distribution of semi-Markov processes. Building upon existing work in the literature, we investigate nonparametric estimators for the stationary law in continuous time and study their asymptotic behavior. Particular attention is given to empirical and kernel estimators with their convergence properties. Theoretical contributions related to strong consistency, asymptotic normality, and large deviations are discussed in the context of finite state space models.

In chapter 5, we present the modeling and estimation of performance and performability measures in systems represented by homogeneous semi-Markov processes. A reward-based framework is adopted in which each state is associated with a performance level or reward rate. Kernel estimators for performance-related quantities are introduced, and their asymptotic properties are studied.

Finally, the thesis concludes with a general summary of the main findings and suggests possible directions for future research.

# Chapter 1

# Fundamental Concepts

This chapter presents the notations and essential prerequisites for describing the continuous-time semi-Markov model. We provide the fundamental concepts and probabilistic properties of continuous-time Markov processes and continuous-time Markov renewal theory.

Consider a finite state space $E = \{1, \ldots, s\}$, representing the set of possible states between which the system may transition over time. Let $(\Omega, \mathcal{F}, \mathbb{P})$ represent a probability space, where $\Omega$ is the set of all possible outcomes, $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$, representing the collection of events, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. Furthermore, $(E, \mathscr{F})$ denote a measurable space, where $\mathscr{F}$ is the associated $\sigma$-algebra of measurable sets. We denote by $\mathcal{M}_E$ the set of real matrices defined on $E \times E$, and by $\mathcal{M}_E(\mathbb{N})$ the set of matrix-valued functions on $\mathbb{N}$ with values in $\mathcal{M}_E$. Finally, let $I$ be a parameter set, typically a subset of $\mathbb{R}_+$, representing time or some other continuous index.

## 1.1 Background

**Definition 1.1.1.** *Doob [1953] **(Stochastic process)***

*A stochastic process is a family of random variables $\{X(t), t \in I\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $E$. For every $t \in I, X(t)$ is a random variable $X(t) : \Omega \mapsto E$, whose value for the outcome $\omega \in \Omega$ is noted $X(t, \omega)$. If instead of $t$ we fix an $\omega \in \Omega$, we obtain the function $X(\cdot, \omega) : I \mapsto E$ which is called a trajectory or a path function or a sample function of the process.*

The set $E$ is called the state space of the stochastic process $X = (X(t), t \in \mathrm{I})$. It may be denoted by $X_t$ instead of $X(t)$ (respectively, $X_n$ if $I = \mathbb{N}$ ).

An exponential distribution is commonly used to model the time between independent events occurring at a constant rate. It has applications in various fields, including reliability, biology, and queuing theory. It is defined as follows:

**Definition 1.1.2.** *Ross [2014] **(Exponential Distribution)**A random variable $X$ :*

$\Omega \mapsto [0,\infty]$ *has exponential distribution of parameter* $\lambda$ $(0 \leq \lambda < \infty)$ *if*

$$\mathbb{P}(X > t) = e^{-\lambda t} \quad \text{for all } t \geq 0.$$

*We write* $X \sim Exp(\lambda)$ *for short. If* $\lambda > 0$*, then* $X$ *has density function*

$$f_X(t) = \lambda e^{-\lambda t} \mathbb{1}_{\{t \geq 0\}}.$$

*The mean of* $X$ *is given by*

$$\mathbb{E}(X) = \int_0^{\infty} \mathbb{P}(X > t)dt = \lambda^{-1}.$$

The exponential distribution is essential in the theory of continuous-time Markov processes, as highlighted by the following theorem.

**Theorem 1.1.1.** *Norris [1997]* ***(Memoryless Property)*** *A random variable* $X : \Omega \mapsto (0,\infty]$ *has an exponential distribution if and only if it has the following memoryless property:*

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t) \quad \text{for all } s, t \geq 0.$$

**Proof of Theorem 1.1.1.** See appendix 5.4.3. □

## 1.2 Discrete-time Markov chain

Let $(J_n)_{n\in\mathbb{N}}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in a measurable space $(E, \mathscr{F})$.

**Definition 1.2.1.** ***(Markov Chain)*** *Let* $J = (J_n)_{n\geq 0}$ *be a sequence of random variables defined on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in a finite space state* $E$*. We say that* $J$ *is a Markov Chain (MC) if, for all* $i_1, i_2, \ldots, i_{n+1} \in E$*, we have:*

$$\mathbb{P}(\underbrace{J_{n+1} = i_{n+1}}_{\textit{The future}} \mid \underbrace{J_1 = i_1, \ldots, J_n = i_n}_{\textit{The past and the present}}) = \mathbb{P}(\underbrace{J_{n+1} = i_{n+1}}_{\textit{The future}} \mid \underbrace{J_n = i_n}_{\textit{The present}}).$$

**Definition 1.2.2.** ***(Homogeneous Markov Chain)*** *A Markov chain is homogeneous if, for all* $n \geq 0$*,* $i, j \in E$*:*

$$\mathbb{P}(J_{n+1} = i \mid J_n = j) = \mathbb{P}(J_1 = i \mid J_0 = j).$$

*In this case, we define*

$$p_{ij} = \mathbb{P}(J_1 = i \mid J_0 = j),$$

*for all* $i, j \in E$*, and* $p_{ij}$ *is called the transition probability.*

**Definition 1.2.3.** ***(Transition Matrix)*** *The matrix* $\boldsymbol{p} = (p_{ij})_{i,j\in E}$ *is a stochastic matrix, i.e., for all* $i, j \in E$*,* $\boldsymbol{p} \geq 0$*, and for all* $i \in E$*,* $\sum_{j\in E} p_{ij} = 1$*.*

**Remark 1.2.1.** *The function* $p_{ij}^{(n)} := \mathbb{P}(J_n = j \mid J_0 = i)$ *is called the n-step transition function. It describes the probability of transitioning from state* $i$ *to state* $j$ *over* $n$ *steps, providing insight into how the Markov chain evolves over time.*

**Definition 1.2.4.** *Let* $\alpha = (\alpha_1, \dots, \alpha_s)$ *be the initial distribution of the chain, that is the distribution of* $J_0$*, is defined by*

$$\alpha_i = \mathbb{P}\left(J_0 = i\right).$$

**Definition 1.2.5.** ***(Sojourn Time)*** *Let* $(J_n)_{n\geq 0}$ *be a homogeneous Markov chain with transition matrix* $\mathbf{p} = (p_{i,j})_{i,j\in E}$*. We denote by* $T_i$ *the random variable equal to the sojourn time in state* $i$*. For every* $k \in \mathbb{N}^*$*,*

$$\mathbb{P}\left(T_i = k\right) = \mathbb{P}\left(J_{n+1} = i, \dots, J_{n+k} = i, J_{n+k+1} \neq i \mid J_n = i\right).$$

**Proposition 1.2.1.** *Pyke and Schaufele [1964] Let* $(J_n)_{n\geq 0}$ *be a Markov chain with transition function* $p$*. Then:*

$$\mathbb{P}\left(T_i = k\right) = (1 - p_{ii})p_{ii}^{(k)} \text{ (geometric distribution)}$$

*and on the other hand, if* $p_{ij} \neq 1$*, we have, for* $j \neq i$*:*

$$\mathbb{P}\left(J_{n+1} = j \mid J_n = i, J_{n+1} \neq i\right) = \frac{p_{ij}}{1 - p_{ii}}.$$

*Suppose* $p_{ii} \neq 1$*. If the chain is in state* $i$ *at a fixed (non-random) time* $n$*, it remains in that state for a geometrically distributed number of steps with parameter* $1 - p_{ii}$*. However, it's worth noting that we can only deduce the sojourn time in state* $i$ *as a geometric distribution with parameter* $1 - p_{ii}$*.*

## 1.3 Classification of States

In this section, we introduce the classification of states in Markov chain. By analyzing these classifications, we gain insight into the long-term behavior and stability of the system. Understanding these concepts is essential for predicting the future dynamics of Markov chain in various applications.

**Definition 1.3.1.** ***(Accessible State)*** *We say that state* $j$ *is accessible from state* $i$*, written as* $i \to j$ *if* $p_{ij}^{(n)} > 0$*. We assume every state is accessible from itself since* $p_{ii}^{(0)} = 1$*.*

**Definition 1.3.2.** ***(Communicate State)*** *Two states* $i$ *and* $j$ *are said to communicate, written as* $i \leftrightarrow j$ *if they are accessible from each other. In other words,*

$$i \leftrightarrow j \text{ means } i \to j \text{ and } j \to i.$$

**Definition 1.3.3.** ***(Irreducible Markov Chain)*** *A Markov chain is said to be irreducible if all states communicate with each other.*

**Definition 1.3.4.** ***(Recurrent State and Positive Recurrent)*** *A state is said to be recurrent if, any time that we leave that state, we will return to that state in the future with probability one. On the other hand, if the probability of returning is less than one, the state is called transient. Here, we provide a formal definition, for any state $i$, we define*

$$G_{ii} = \mathbb{P}\left(J_n = i,\ \text{for some } n \geq 1 \mid J_0 = i\right).$$

*Furthermore, a state $i$ is recurrent if $G_{ii} = 1$, and it is transient if $G_{ii} < 1$.*

*A state $i$ is called positive recurrent if it is recurrent $(G_{ii} = 1)$ and the expected return time to state $i$ is finite. More formally, let $S_i$ be the return time to state $i$, defined as the smallest $n > 0$ such that $J_n = i$. The state $i$ is positive recurrent if:*

$$\mathbb{E}[S_i \mid J_0 = i] < \infty.$$

**Definition 1.3.5.** ***(Periodic, Aperiodic State)*** *A state $i \in E$ is said to be periodic of period $d > 1$, or $d$-periodic, if $d$ is equal to the greatest common divisor of all $n$ such that $\mathbb{P}\left(J_{n+1} = i \mid J_1 = i\right) > 0$. If $d = 1$, then the state $i$ is said to be aperiodic.*

**Definition 1.3.6.** ***(Ergodic State)*** *An aperiodic recurrent state is called ergodic. An irreducible Markov chain with one state ergodic (and then all states ergodic) is called ergodic.*

**Definition 1.3.7.** ***(Stationary Distribution)*** *A probability distribution $\nu$ on $E$ is said to be stationary or invariant for the Markov chain $(J_n)_{n\geq 0}$ if, for any $j \in E$*

$$\sum_{i\in E} \nu(i)p_{ij} = \nu(j),$$

*or, in matrix form,*

$$\nu \mathbf{p} = \nu,$$

*where $\nu = (\nu(1), \ldots, \nu(s))$ is a row vector of the stationary distribution.*

**Theorem 1.3.1.** *Norris [1997]* ***(Existence and Uniqueness)*** *Let $(J_n)_{n\geq 0}$ be a discrete-time Markov chain with state space $E$, and suppose that the chain is irreducible and recurrent. Fix an arbitrary state $i \in E$. For each $j \in E$, we have*

$$\begin{aligned} \mathcal{D}_j &:= \mathbb{E}_i(\text{ number of visits to } j \text{ during a cycle around } i) \\ &= \mathbb{E}_i\left[\sum_{n\in\mathbb{N}} \mathbb{1}\left\{X_n = j, \mathcal{V}_i \geq n\right\}\right] \\ &= \sum_{n\in\mathbb{N}} \mathbb{P}_i\left(X_n = j, \mathcal{V}_i \geq n\right), \end{aligned}$$

*where $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot|J_0 = i)$, $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot \mid J_0 = i]$, and $\mathcal{V}_i = \inf\{n > 0 : X_n = i\}$ is the first return time to state $i$. Note that $\mathcal{D}_i = 1$ because the cycle ends with the first return, at* *time $\mathcal{V}_i$, to state $i$. Furthermore, $\mathbb{1}_{\{X_n=j,\mathcal{V}_i\geq n\}}$ is an indicator function, which is defined by*

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

*Then the vector $\mathcal{D} = (\mathcal{D}_j)_{j\in E}$ satisfies the following properties:*

$$\mathcal{D}_j = \sum_{i\in E} \mathcal{D}_i p_{ij}, \quad \forall j \in E,$$

*and*

$$\sum_{j\in E} \mathcal{D}_j = \mathbb{E}_i[\mathcal{V}_i].$$

*If the state $i$ is positive recurrent. Then*

$$\nu_j := \frac{\mathcal{D}_j}{\mathbb{E}_i[\mathcal{V}_i]}, \quad j \in E,$$

*defines a stationary probability distribution for the chain.*

## 1.4 Continuous-time Markov process

Let $(J_t)_{t\geqslant 0}$ be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in the measurable space $(E, \mathscr{F})$.

**Definition 1.4.1.** ***(Markov Process)*** *A stochastic process $\{J_t : t \in \mathbb{R}_+\}$ with a finite state space $E$ is said to be a Markov process, if for all $i, j, i_0, i_1, \ldots, i_{n-1} \in E$ and $t_0, t_1, \ldots, t_n, t_{n+1} \in \mathbb{R}_+$ such that $0 \leqslant t_0 < t_1 < \cdots < t_n < t_{n+1}$,*

$$\mathbb{P}\left(J_{t_{n+1}} = j \mid J_{t_n} = i, J_{t_{n-1}} = i_{n-1}, \ldots, J_{t_0} = i_0\right) = \mathbb{P}(J_{t_{n+1}} = j \mid J_{t_n} = i).$$

If $t_0, t_1, \ldots, t_{n-1}$ are taken as moments from the past, $t_n$ as the current time, and $t_{n+1}$ as a future moment, the equation indicates that the probability of a future state depends exclusively on the present state, without influence from prior states. This means that the future behavior of the Markov process is determined only by its current state. For this reason, Markov processes are classified as stochastic processes with the memoryless property.

**Definition 1.4.2.** ***(Homogeneous Markov process)*** *The Markov process $(J_t)_{t\in\mathbb{R}_+}$ is called homogeneous, if for all $i, j \in E$ and $t_{n+1}, t_n \in \mathbb{R}_+$, such that $0 \leqslant t_0 < t_1 < \cdots < t_n < t_{n+1}$,*

$$\mathbb{P}(J_{t_{n+1}} = j \mid J_{t_n} = i) = \mathbb{P}(J_{t_1} = j \mid J_{t_0} = i).$$

**Definition 1.4.3.** ***(Transition function)*** *Let $(J_t)_{t\in\mathbb{R}_+}$ be a homogeneous continuous-time Markov process with state space $E$. The functions defined on $\mathbb{R}_+$ by*

$$t \mapsto p_{ij}(t) := \mathbb{P}(J_{t_{n+1}} = j \mid J_{t_n} = i), \quad i, j \in E,$$

*are called transition functions of the process. The matrix $\mathbf{p}(t) = (p_{ij}(t))_{i,j\in E}$ is called the transition matrix (possibly infinite).*

**Lemma 1.4.1.** ***(Chapman-Kolmogorov equations)*** *For all $s \geq 0$ and $t \geq 0$,*

$$p_{ij}(s+t) = \sum_{u \in E} p_{iu}(s) p_{uj}(t). \tag{1.1}$$

***Proof of Lemma 1.4.1.*** *See appendix 5.4.3.* □

Using matrix notation, we write $\mathbf{p}(t)$ for the square matrix of transition probabilities $(p_{ij}(t))$, and call it the transition function. In matrix notation, the Chapman-Kolmogorov equations reduce to a simple relation among the transition functions involving matrix multiplication:

$$\mathbf{p}(s+t) = \mathbf{p}(s)\mathbf{p}(t), \tag{1.2}$$

for all $s \geq 0$ and $t \geq 0$.

It is important to recognize that (1.2) means (1.1). From the perspective of abstract algebra, equation (1.2) says that the transition function has a semi-group property, where the single operation is matrix multiplication.

**Lemma 1.4.2.** *Kijima [1997]* ***(Positive Transition Probabilities)*** *For an irreducible Markov process, $p_{ij}(t) > 0$ for all $i, j$ and $t > 0$.*

***Proof of Lemma 1.4.2.*** *See appendix 5.4.3.* □

**Proposition 1.4.1.** *Howard [1971] Let the random variable $T_i$ be the waiting time in state $i$. Using the Chapman-Kolmogorov equation (see Iosifescu [2014] and Doob [1953]), $T_i$ follows an exponential distribution with a parameter $\lambda_i > 0$,*

$$G_i(t) = \mathbb{P}(T_i \leq t) = 1 - e^{-\lambda_i t}, \quad t \geq 0, \quad i \in E.$$

**Definition 1.4.4.** ***(Generator Matrix)*** *Let us define a generator matrix as a matrix $\boldsymbol{A} = (a_{ij})_{i,j \in E}$ on $E$ with entries*

$$a_{ij} := \begin{cases} -\lambda_i \cdot (1 - p_{ii}) & i = j, \\ \lambda_i \cdot p_{ij} & i \neq j, \end{cases}$$

*where $\lambda_i$ is called the exponential rate of the system. In particular, the relation*

$$a_{ii} = -\sum_{j \neq i} a_{ij}$$

*holds for all $i \in E$.*

*Furthermore, the $(i, j)$-th entry of the generator $\boldsymbol{A}$ is called the infinitesimal transition rate from state $i$ to state $j$.*

The specification of the initial distribution $\alpha$ completes the characterization of the process and enables an explicit expression of the finite-dimensional marginal distributions, as given in the following theorem.

**Theorem 1.4.1.** Anderson [2012] *For a Markov process* $(J_t)_{t \geqslant 0}$ *with initial distribution* $\alpha$ *and time instances* $0 < t_1 < \ldots < t_n, n \in \mathbb{N}$, *the equation*

$$\mathbb{P}\left(J_{t_1} = j_1, \ldots, J_{t_n} = j_n\right) \equiv \sum_{i \in E} \alpha_i p_{ij_1}\left(t_1\right) p_{j_1 j_2}\left(t_2 - t_1\right) \ldots p_{j_{n-1} j_n}\left(t_n - t_{n-1}\right),$$

*holds for all* $j_1, \ldots, j_n \in E$.

An initial distribution $\alpha$ is said to be stationary if the process $J$ is stationary, which means that

$$\mathbb{P}\left(J_{t_1} = j_1, \ldots, J_{t_n} = j_n\right) = \mathbb{P}\left(J_{t_1+s} = j_1, \ldots, J_{t_n+s} = j_n\right),$$

holds for all $n \in \mathbb{N}$, where $0 \leq t_1 < \ldots < t_n$, states $j_1, \ldots, j_n \in E$, and $s \geq 0$.

**Theorem 1.4.2.** *A distribution* $\pi$ *on* $E$ *is stationary if and only if* $\pi \boldsymbol{A} = 0$ *holds.*

***Proof of Theorem 1.4.2.*** *See appendix 5.4.3.* □

Before proceeding, we establish the existence and uniqueness of a stationary distribution for a Markov process in the following theorem.

**Theorem 1.4.3.** Kijima [1997] *Let* $\{J_t : t \geqslant 0\}$ *be an irreducible and positive recurrent Markov process. Further assume that*

$$\check{\lambda} := \inf\left\{\lambda_i : i \in E\right\} > 0,$$

*where* $\lambda_i = -a_{ii}$ *denotes the total rate of leaving state* $i$, *and* $\mathbf{A} = (a_{ij})$ *is the generator matrix of the process. The condition* $\check{\lambda} > 0$ *ensures that no state has arbitrarily small exit rate. Then, there exists a unique stationary distribution for* $J_t$.

**Definition 1.4.5.** *A Markov process is said to be regular if it is irreducible and positive recurrent.*

The following proposition focus on two key results regarding the asymptotic behavior of a Markov process.

**Proposition 1.4.2.** Norris [1997] *If* $J_t$ *is a regular Markov process, then the limit*

$$\lim_{t \to \infty} \mathbb{P}\left(J_t = j\right) = \pi_j,$$

*of the marginal distribution at time* $t$ *tends to the stationary distribution as* $t$ *tends to infinity. Further the limit*

$$\lim_{t \to \infty} p_{ij}(t) = \pi_j,$$

*holds for all* $i, j \in E$ *and is independent of* $i$.

They therefore fulfill a system of linear equations

$$\sum_{i \in E} \pi_i \lambda_{ij} = 0, j \in E, \quad \sum_{j \in E} \pi_j = 1.$$

To obtain the limit distribution of the process, it is necessary to resolve the specified system of linear equations.

## 1.5 Markov Renewal Process

Markov renewal processes are a class of stochastic models that combine aspects of Markov processes and renewal theory. These processes describe systems that reset at random intervals, with the transitions between states following the memoryless property of Markov processes. Let us consider:

- $E = \{1, \ldots, s\}$ the state space.
- The stochastic process $J = (J_n)_{n \geq 0}$ with state space $E$, where $J_n$ denotes the state of the system at the $n$th jump time.
- The stochastic process $S = (S_n)_{n \geq 0}$ with state space $\mathbb{N}$, where $S_n$ denotes the time of the $n$th jump, with $S_0 = 0$ and $0 < S_1 < S_2 < \ldots < S_n < S_{n+1} < \ldots$.

**Definition 1.5.1.** *Under the above assumptions, the sequence $S = (S_n; n \in \mathbb{N})$ is called a renewal process. The times $S_n$ are called renewal times.*

A Markov renewal process is a bivariate stochastic process $(J_n, S_n)$. The process has to satisfy the following formula:

$$\mathbb{P}\left(J_{\mathrm{n}+1} = j, S_{\mathrm{n}+1} - S_{\mathrm{n}} \leq t \mid J_0, J_1, \ldots, J_{\mathrm{n}}, S_0, S_1, \ldots, S_{\mathrm{n}}\right) = \mathbb{P}\left(J_{\mathrm{n}+1} = j, S_{\mathrm{n}+1} - S_{\mathrm{n}} \leq t \mid J_{\mathrm{n}}\right), \tag{1.3}$$

for all $j \in E$, all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$.

Moreover, if Equation (1.3) is independent of $n$, $(J_n, S_n)$ is considered to be time homogeneous Markov renewal process.

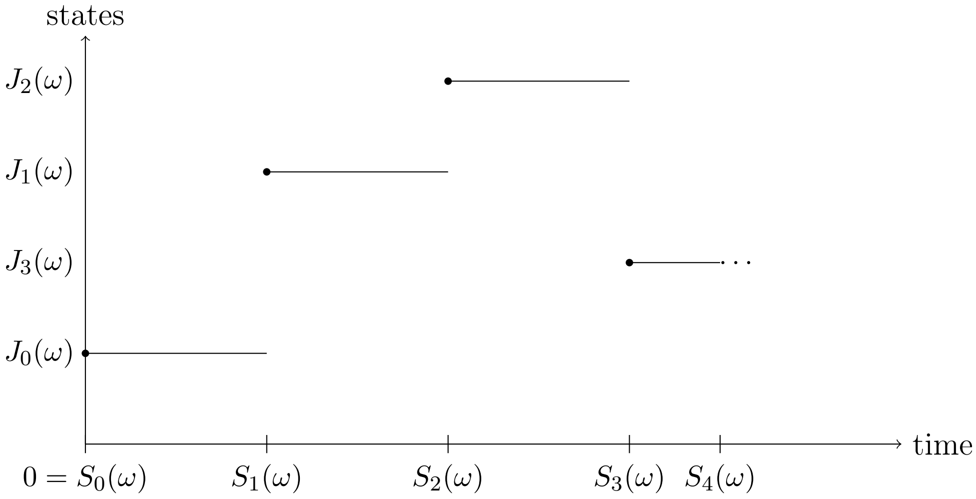


Figure 1.1: A sample path of a Markov renewal process.

Figure 1.1 shows a sample path of the process, where each jump occurs at time $S_n(\omega)$ and the process enters state $J_n(\omega)$. The horizontal segments represent the sojourn times in each state, and the black dots mark the transition instants. This path illustrates the piecewise-constant nature of the Markov renewal process over time.

From an operational point of view, an important characteristic of the considered system at time $t$ is the total number of replacements occurring on the interval $[0, t]$. Note that, the initial component is excluded from consideration.

If $N(t)$ represents the random variable defined above, for $n \geq 1$:

$$N(t) > n - 1 \Leftrightarrow S_n \geq t.$$

$N(t)$ is the counting process of the renewal process.

**Definition 1.5.2.** *We call a renewal equation the equation of the form*

$$h = g + F * h, \tag{1.4}$$

*where $F$ is a distribution function on $\mathbb{R}_+$, and $h, g$ are real-valued functions defined on $\mathbb{R}$. Here, $F$ is an unknown function, and $g$ is a given function.*

# Chapter 2

# Semi-Markov Process

This chapter is dedicated to the presentation of the semi-Markov model, a powerful generalization of Markov and renewal processes that allows for arbitrary sojourn time distributions. Unlike classical continuous-time Markov chains, semi-Markov processes (SMPs) are not restricted by the memoryless property, making them more suitable for modeling complex real-world systems where the timing of transitions plays a critical role. We begin by introducing the formal definitions and basic structure of continuous-time semi-Markov processes, which will serve as a foundation for the remainder of the thesis. Furthermore, we highlight the wide range of applications of SMPs across various domains. Finally, we introduce some numerical methods and simulation techniques that are useful for analyzing semi-Markov processes when analytical solutions are intractable.

## 2.1 Continuous-Time Semi-Markov Framework

Let $\mathbb{R}_+$ be the set of nonnegative real numbers $[0, \infty)$, $E$ be the finite state space. Let $J := (J_n)_{n\in\mathbb{N}}$ be the sequence of consecutive states visited by $Z = (Z_t)_{t\in\mathbb{R}_+}$, and $S := (S_n)_{n\in\mathbb{N}}$ the corresponding jump times of $Z$. We further denote by $X_0 = S_0 = 0$, and by $X_1, X_2, \dots$ the sequence defined by $X_n := S_n - S_{n-1}$, for $n \in \mathbb{N}^*$, the sojourn times in these states.

**Definition 2.1.1.** ***(Continuous-Time Semi-Markov Process)*** *Consider a Markov-renewal process $\{(J_n, S_n) : n \in \mathbb{N}\}$ defined on a complete probability space and with state space $E$. The stochastic process $(Z_t)_{t\in\mathbb{R}_+}$ defined by*

$$Z_t = J_{N(t)}, \tag{2.1}$$

*is called a semi-Markov process (SMP) where $N(t) := \sup\{n \geq 0 \mid S_n \leq t\}$ is the counting process of the SMP up to time $t$.*

**Definition 2.1.2.** *The semi-Markov process $Z$ is said to be regular if*

$$\mathbb{P}_i(N(t) < \infty) = 1,$$

*for any $t \geq 0$ and any $i \in E$.*

**Lemma 2.1.1.** Limnios and Oprisan [2001] *For regular semi-Markov processes, we have $S_n \leq S_{n+1}$, for any $n \in \mathbb{N}$, and $S_n \to \infty$.*

## 2.2 Related Quantities of Continuous-Time Semi-Markov Process

In order to analyze the behavior and performance of a continuous-time semi-Markov process (SMP), it is essential to introduce and study several random variables and functions associated with its evolution. These quantities provide key insights into the structure and dynamics of the process.

**Definition 2.2.1.** *Let us denote by* $\mathbf{Q}(t) = [Q_{ij}(t), i, j \in E]$, $t \geq 0$, *the semi-Markov kernel of* $Z$, *defined by*

$$Q_{ij}(t) = \mathbb{P}\left(J_{n+1} = j, X_{n+1} \leq t \mid J_n = i\right), \tag{2.2}$$

*which are absolutely continuous with respect to the Lebesgue measure, and let* $q_{ij}(t)$ *be the corresponding Radon-Nikodym derivative.*

Let us define the transition probabilities from state $i$ to state $j$ of the embedded Markov chain $(J_n)_{n\geq 0}$, denoted by $\mathbf{p} = (p_{ij})_{i,j\in E}$, as

$$p_{ij} = \lim_{t\to\infty} Q_{ij}(t) = Q_{ij}(\infty) = \mathbb{P}(J_{n+1} = j \mid J_n = i),$$

where we assume that the transition probabilities $\mathbf{p} = (p_{ij})_{i,j\in E}$ do not depend on the index $n$. Furthermore, the initial distribution, for all $i \in E$, is given by

$$\alpha_i := \mathbb{P}(J_0 = i) = \mathbb{P}(Z_0 = i). \tag{2.3}$$

Let us consider the renewal process $\left(S_n^i\right)_{n\geq 0}$ of successive times of visits to state $i$. Let $\mu_{ii}$ and $\mu_{ii}^*$ denote the mean recurrence times of the state $i$ in the MRP and in the corresponding Markov chain $(J_n)_{n\geq 0}$, respectively. Furthermore, $\mu_{ii}$ is the mean interarrival times of the eventual delayed renewal process $\left(S_n^i\right)$, i.e., $\mu_{ii} = \mathbb{E}\left[S_2^i - S_1^i\right]$ and $\mu_{ii}^* = \mathbb{E}\left[S_i^* | J_0 = i\right]$ with $S_i^* = \min\{n \geq 1, J_n = i\}$ is the first visit time to the state $i$.

**Definition 2.2.2.** *For all* $i, j \in E$ *and* $t \in \mathbb{R}_+$, *the conditional sojourn time distribution in state* $i$, *given that the next state to be visited is* $j$, *denoted* $F_{ij}$, *is defined by*

$$F_{ij}(t) := \mathbb{P}(X_{n+1} \leq t \mid J_n = i, J_{n+1} = j). \tag{2.4}$$

Meanwhile, the sojourn time distribution in state $i$, denoted $H_i$, is defined by

$$H_i(t) = \mathbb{P}\left(X_{n+1} \leq t \mid J_n = i\right) = \sum_{j\in E} Q_{ij}(t), \quad t \in \mathbb{R}_+.$$

Moreover, we define the corresponding survival function to $H_i$, denoted $\overline{H}_i$, by

$$\overline{H}_i(t) = 1 - H_i(t).$$

Assuming that the integral of the corresponding survival function is convergent. Then we have the following result:

$$\int_0^{+\infty} \overline{H}_i(t)dt = \int_0^{+\infty} td\overline{H}_i(t) = -\int_0^{+\infty} tdH_i(t).$$

**Remark 2.2.1.** *For any $i, j \in E$, the following observations hold:*

1. *In general, $Q_{ij}$ is a sub-distribution, i.e., $Q_{ij}(\infty) \leq 1$, hence, $H_i$ is a distribution function, $H_i(\infty) = 1$, and $Q_{ij}(0-) = H_i(0-) = 0$.*
2. *Another type of semi-Markov process can be obtained if $F_{ij}(\cdot)$ does not depend on $j$, i.e., $F_{ij}(t) \equiv H_i(t)$ and*
$$Q_{ij}(t) = p_{ij} H_i(t).$$

For the general case we can derive the following result.

**Proposition 2.2.1.** *For all $i, j \in E$ and $t \in \mathbb{R}_+$. It holds true that*

$$F_{ij}(t) = \frac{Q_{ij}(t)}{p_{ij}}. \tag{2.5}$$

***Proof of Proposition 2.2.1.*** *See appendix 5.4.3.* □

**Definition 2.2.3.** *Let us define the transition matrix $\mathbf{P}(t) = [P_{ij}(t) : i, j \in E]$ of the process $(Z_t)_{t\in\mathbb{R}_+}$, by*

$$P_{ij}(t) = \mathbb{P}\left(Z_t = j \mid Z_0 = i\right) = \mathbb{P}\left(J_{N(t)} = j \mid J_0 = i\right).$$

*Then the unconditional semi-Markov state probability is equal to*

$$\begin{aligned} P_j(t) &= \mathbb{P}\left(Z_t = j\right) = \mathbb{P}\left(J_{N(t)} = j\right) \\ &= \sum_{i=1}^{s} \mathbb{P}\left(J_{N(t)} = j \mid J_0 = i\right) \mathbb{P}\left(J_0 = i\right) \\ &= \sum_{i=1}^{s} \alpha_i P_{ij}(t). \end{aligned}$$

The transition matrix satisfies a Markov renewal equations, as a part of these equations the convolutions of functions which are introduced by Stieltjes and serve as the basis for matrix-Stieltjes convolutions.

**Definition 2.2.4.** *Let consider $g$ to be a locally bounded function and $G$ to be a real right continuous nondecreasing function both defined on $\mathbb{R}_+$, the Stieltjes convolution of the function $g$ with the function $G$ is defined by*

$$g * G(t) = \int_{\mathbb{R}} g(t-x)dG(x) = \int_0^t g(t-x)dG(x), \quad t \in \mathbb{R}_+.$$

*Furthermore, when $G$ and $F$ are cumulative distribution functions, we have*

$$G * F(t) = \int_0^t G(t-x)dF(x) = \int_0^t F(t-x)dG(x) = F * G(t).$$

**Definition 2.2.5.** ***((Stieltjes convolution))*** *Let $\mathcal{R}_i(t)$, $i \in E$, $t \geq 0$, be a real valued measurable function and $Q$ be a semi-Markov kernel. Then the Stieltjes convolution of $\mathcal{R}$ by $Q$ is defined as*

$$Q * \mathcal{R}_i(t) := \sum_{k\in E} \int_0^t Q_{ik}(ds)\mathcal{R}_k(t-s).$$

Now, consider the n-fold Stieltjes convolution of $Q$ by itself. For any $i, j \in E$,

$$Q_{ij}^{(n)}(t) = \begin{cases} \mathbb{1}_{\{i=j, t\in\mathbb{R}_+\}} & \text{if } n = 0, \\ Q_{ij}(t) & \text{if } n = 1, \\ \sum_{k\in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s) & \text{if } n \geq 2. \end{cases}$$

## 2.3 Markov Renewal Matrix

Let define $N(t)$, associated with the Markov renewal process $(J_n, S_n)_{n\in\mathbb{N}}$ (cf. Limnios and Oprisan [2001]), as the total number of transitions up to time $t$. It is given by:

$$N(t) = \sum_{j\in E} N_j(t),$$

where $N_j(t) := \sum_{n=0}^{N(M)} \mathbb{1}_{\{J_n=j\}} = \sum_{n=1}^{\infty} \mathbb{1}_{\{J_n=j, S_n\leq t\}}$ is the total number of visits to state $j$ up to time $t$.

The Markov renewal function denoted $\Psi_{ij}(\cdot), i, j \in E, t \geq 0$, is defined by

$$\begin{aligned} \Psi_{ij}(t) &:= \mathbb{E}_i\left[N_j(t)\right] = \mathbb{E}\left(N_j(t) \mid J_0 = i\right) \\ &= \sum_{n=0}^{\infty} \mathbb{P}\left(J_n = j, S_n \leq t \mid J_0 = i\right) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \end{aligned}$$

Hence, $\Psi_{ij}(t)$ is the expected number of visits from state $i$ to state $j$ up to time $t$.

The Markov renewal equation in matrix form is given by

$$\mathbf{\Psi}(t) = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}(t),$$

where $\mathbf{\Psi}(t) = [\Psi_{ij}(t) : i, j \in E]$.

The matrix renewal function $\mathbf{\Psi}(t)$ is the solution of the Markov renewal equation

$$\mathbf{\Psi}(t) = \mathbf{I}(t) + \mathbf{Q} \star \mathbf{\Psi}(t),$$

where $\mathbf{I}(t) = \mathbf{I}$ when $t \geq 0$ and $\mathbf{I}(t) = 0$ when $t < 0$.

The transition function $\mathbf{P}(t) = [P_{ij}(t) : i, j \in E]$ satisfies the following Markov renewal equation

$$\mathbf{P}(t) = \mathbf{I}(t) - \mathbf{H} + \mathbf{Q} * \mathbf{P}(t). \tag{2.6}$$

By solving the above Markov renewal equation (2.6), cf. Limnios and Oprisan [2001], it is seen that, in matrix notation, we have

$$\mathbf{P}(t) = (\mathbf{\Psi} \star (\mathbf{I} - \mathbf{H}))(t),$$

where $\mathbf{H}(t) = [H_i(t)]$ is the diagonal matrix of $ith$ entry $\sum_{j=1}^{s} Q_{ij}(t)$ and $\mathbf{1} = (1, 1, \ldots, 1)^t$.

It is also known, cf. Pyke [1961b], that the transition matrix function $\mathbf{P}(t) = [P_{ij}(t) : i, j \in E]$ of the semi-Markov process can be written as

$$P_{ij}(t) = \mathbb{1}_{\{i=j\}} \left(1 - \sum_{k\in E} Q_{ik}(t)\right) + \sum_{k\in E} \int_0^t P_{kj}(t-s) Q_{ik}(ds).$$

**Definition 2.3.1.** *For all $i, j \in E$, we define the function $G_{ij}$ by*

$$G_{ij}(t) = \mathbb{P}(N_j(t) > 0 \mid J_0 = i),$$

*which is the probability that state $j$ has been visited at least once by time $t$, given that the process starts in state $i$ at time $0$.*

## 2.4 Classification of States

Let $(\nu_i; i \in E)$ be an invariant measure for $\mathbf{p} = (p_{ij})_{i,j\in E}$, i.e., $\nu\mathbf{p} = \nu$.

**Definition 2.4.1.** *For all $i, j \in E$, the following statements hold:*

1. *Two states $i$ and $j$ are said to communicate if $i = j$ or $G_{ij}(\infty)G_{ji}(\infty) > 0$.*
2. *A state $i$ is said to be recurrent if $G_{ii}(\infty) = 1$, otherwise it is called transient.*
3. *A recurrent state $i$ is said to be a positive-recurrent state if $\mu_{ii} < \infty$ and null-recurrent if $\mu_{ii} = \infty$.*
4. *A state $i$ is said to be periodic with period $d > 0$ if $G_{ii}(\cdot)$ is arithmetic, i.e., concentrated on $\{nd : n \in \mathbb{N}\}$. In the opposite case it is called aperiodic.*

**Definition 2.4.2.** *A Markov Renewal Process (MRP) in which all states satisfy the following properties is described as:*

1. *Irreducible if all states communicate with each other;*
2. *Positive recurrent if all states are positive recurrent.*

**Proposition 2.4.1.** Limnios and Oprisan [2001] *For any $i, j \in E$, we have the following results:*

1. *A Markov Renewal Process (MRP) is irreducible if and only if its EMC is irreducible.*
2. *A state $i$ is recurrent (transient) in the MRP, if and only if it is recurrent (transient) in the EMC.*
3. *For an irreducible finite MRP, a state $i$ is positive recurrent in the MRP, if and only if it is recurrent in the EMC and if $m_j < \infty$.*
4. *If the EMC of an MRP is irreducible and recurrent, then all states are:*
    - *positive-recurrent if and only if $\sum_{i\in E} \nu_i m_i < \infty$;*
    - *null-recurrent if and only if $\sum_{i\in E} \nu_i m_i = \infty$.*

## 2.5 Applications of Semi-Markov Processes

Semi-Markov theory is one of the most productive subjects of stochastic processes to generate applications in real-life problems. While we cannot provide a comprehensive overview of all such applications, this area is particularly significant in fields such as economics, manpower models, insurance, finance (more recently), reliability, simulation, queuing, branching processes, medicine (including survival data), social sciences, language modeling, seismic risk analysis, biology, computer science, chromatography, and fluid mechanics. Significant results in these fields can be found in Janssen and Limnios [1999] and Janssen and Manca [2007].

To illustrate the potential of semi-Markov processes, we present four examples that highlight their applications in diverse domains.

### 2.5.1 Occupational Illness Insurance

This example involves occupational illness insurance, where partial or permanent disability may result. The amount of incapacitation allowance depends on the degree of disability recognized for the policyholder by the occupational health doctor, typically on a yearly basis, as the degree evolves over time.

Consider the invalidity degree as a stochastic process $(J_n, n \geq 0)$, where $J_n$ represents the value of this degree at time $n$. Introduce the random variable $S_n$, representing the time between two successive transitions from $J_{n-1}$ to $J_n$. These transitions are often observed through periodic medical inspections.

Assuming the process $(J_n, S_n)$ follows an SMC extends the Markov model and allows for a more detailed analysis of the illness progression.

### 2.5.2 Queuing Theory

Consider a queuing system where customers wait to be served by a single server. Customers are served according to the FIFO (first-in, first-out) discipline rule. Assume the capacity of the waiting room is infinite, meaning every customer waits until they are served. Time 0 is designated as the arrival of the first customer, who is immediately served.

As in the preceding example, suppose each customer has a type $J_n \in E = \{1, \ldots, s\}$ for all $n \geq 0$, which influences service times and/or interarrival times. Define the random variables $J_n$ as the type of the $(n+1)$th customer, $(B_n)_{n \geq 1}$ as the service time of the $n$th customer, and $(A_n)_{n \geq 1}$ as the interarrival time between the $n$th and $(n+1)$th customers.

A key problem in queuing theory is the study of successive waiting times $(W_n, n \geq 0)$, where $W_n$ represents the time the $(n+1)$th customer waits before being served, with $W_0 = 0$.

Queuing models start with assumptions about the sequences $(J_n, n \geq 0)$, $(B_n, n \geq 1)$, and $(A_n, n \geq 1)$, as well as their stochastic dependencies. For instance, in the M/SM/1 model, the assumptions are:

1. Successive interarrival times $(A_n, n \geq 1)$ form a Poisson process of parameter $\lambda$.
2. Successive service times $(B_n, n \geq 1)$ form an SMC with kernel $Q$.
3. The processes $(A_n, n \geq 1)$ and $(B_n, n \geq 1)$ are independent.

### 2.5.3 Claim Process in Insurance

Consider an insurance company covering $s$ types of risks or having $s$ different types of customers for the same risk, forming the set $E = \{1, \dots, s\}$. For example, in automobile insurance, we can distinguish three types of drivers: good, average, and bad. Thus, $E$ is a space consisting of three states: 1 for good, 2 for average, and 3 for bad.

Now, let $(S_n, n \geq 1)$ represent the sequence of successive observed claim amounts, $(X_n, n \geq 1)$ the sequence of interarrival times between two successive claims, and $(J_n, n \geq 1)$ the successive types of observed risks.

In the classical model of risk theory called the Cramer-Lundberg model (see Embrechts et al. [1997]), it is assumed there is only one type of risk, and the claim arrival process is a Poisson process of parameter $\lambda$. Thorin [1974] later extended this model to an arbitrary renewal process. In these classical models, the process of claim amounts is independent of the claim arrival process.

By considering a semi-Markov chain (SMC) for the two-dimensional processes $((J_n, S_n), n \geq 0)$ or $((J_n, X_n), n \geq 0)$, it becomes possible to introduce dependencies between successive claim amounts. This model was initially developed by Janssen and Manca [2007] and Janssen [1982] based on work of Miller [1962] and has since led to many extensions, such as those described in Asmussen [2000].

### 2.5.4 Reliability

Reliability analysis plays a crucial role in engineering and system design, ensuring that systems function as expected over time. It encompasses key concepts such as availability (the probability that a system is operational at a given time), maintainability (the probability of restoring a failed system within a given time), and dependability (a broader measure that includes reliability, availability, maintainability, and safety). However, in many real world applications, such as mechanical systems, communication networks, and power grids failure and repair times do not follow an exponential distribution. This limitation motivates the use of a semi-Markov processes, which generalize Markov processes by allowing arbitrary sojourn time distributions.

A semi-Markov models have significant applications in reliability theory, as described by Osaki [1985] and more recently by Limnios and Oprisan [2001]. Consider a reliability system $Z$ that can be in one of $s$ states $E = \{1, \dots, s\}$ at any time $t$.

A stochastic process of successive states of the system is represented by $(Z_n)_{n \geq 0}$. The state space $E$ is partitioned into two subsets: $U$ ("up" states where the system functions) and $D$ ("down" states where the system fails), such that $E = U \cup D$ and $U \cap D = \emptyset$.

Key reliability indicators include:

1. Reliability function $R(t)$: The probability that the system operates without failure from time 0 to $t$:
$$R(t) = P(Z_s \in U, \forall s \in [0, t]).$$
2. Point-wise availability function $A(t)$: The probability the system is functional at time $t$:
$$A(t) = P(Z_t \in U).$$

3. Maintainability function $M(t)$: The probability the system transitions from a failed state $D$ to a functional state $U$ at time $t$:

$$M(t) = P(Z_t \in U \mid Z_0 \in D).$$

### 2.5.5 Performance and Performability

A semi-Markov models provide a powerful framework for evaluating system performance and performability, making them essential tools in reliability engineering, computing systems, and maintenance operations. Unlike traditional reliability analysis, which focuses solely on failure rates, these models offer a broader view by incorporating measures of system efficiency and effectiveness. This is particularly important for fault-tolerant systems, where both reliability and performance must be considered together. To meet this requirement, Meyer [1980] introduced the concept of performability, which quantifies the probability that a system maintains a desired level of performance over time. This approach helps evaluate not only whether a system is operational but also how well it functions under different conditions.

A more general way to model accumulated system performance over time is through the performance reward functional, we consider a bounded Borel function $L : E \to \mathbb{R}$ and the following continuous additive functional of the semi-Markov process $(Z_t, t \geq 0)$.

$$\Phi(t) = \int_0^t L(Z_u) du. \tag{2.7}$$

Let $\tilde{\Phi}(t, x) = \mathbb{P}\left(\Phi_t \leq x\right)$ be the distribution function of $W_t$; then the function $U(x, t) := 1 - \tilde{\Phi}(t, x)$ will be the performability function over the interval $[0, t]$. It gives capacity of the system to reach level $x$ up to time $t$.

Recent advancements have refined computational methods to accommodate semi-Markov processes more effectively, making them valuable tools for analyzing real-world systems. These developments allow for a more comprehensive evaluation of system behavior, ensuring that both reliability and performance expectations are met.

## 2.6 Numerical Methods and Simulation

In this section, we present essential numerical and simulation techniques used in the analysis of semi-Markov systems, especially when dealing with complex models where closed-form solutions are difficult or impossible to obtain. The methods covered include Markov chain approximations, Laplace transform, numerical solutions of integral equations and Monte Carlo simulation.

### 2.6.1 Markov Chain Approximation

A Markov chain approximation method seeks to approximate a controlled diffusion process of the form

$$dx(t) = b(x(t), u(t))dt + \sigma(x(t))dw,$$

where $b$ and $\sigma$ are the drift and diffusion functions, and $u(t)$ is a control process. The goal is to numerically compute or approximate the value function $V(x)$ associated with a cost functional

$$W(x,u) = \mathbb{E}\left[\int_0^\tau e^{-\beta t} k(x(t),u(t))\,dt + e^{-\beta\tau} g(x(\tau))\right],$$

where $\tau$ is the exit time from a domain and $\beta > 0$ is a discount factor (cf. Kushner and Dupuis [2001]), we have

$$V(x) = \inf_u W(x,u).$$

### 2.6.2 Laplace Transform

In semi-Markov process (SMP) modeling, solutions to key time-dependent quantities are naturally expressed in terms of Laplace transforms, as first developed in classical work by Pyke [1961a]. These quantities include state probabilities $P_{ij}(t)$, first passage distributions $G_{ij}(t)$, expected number of visits $M_{ij}(t)$, and Markov renewal functionals such as $\psi_{ij}(t)$ and $\Psi_{ij}(t)$. Because most SMPs do not admit closed-form time-domain expressions, numerical inversion of Laplace transforms becomes essential.

The Laplace transform of a function $\phi(t)$ is defined as:

$$\widetilde{\phi}(s) = \int_0^\infty e^{-st}\phi(t)\,dt,$$

and its inverse is given by the Bromwich integral:

$$\phi(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\widetilde{\phi}(s)\,ds.$$

This inversion is numerically challenging in general but becomes tractable for probability distributions with smooth, nonnegative density functions. The method employed is the EULER algorithm by Abate and Whitt [1995], which approximates the inverse using a Fourier-cosine series:

$$\phi(t) \approx \sum_{j=0}^{N} (-1)^j w_j \Re\left[\widetilde{\phi}\left(\frac{A}{2t} + \frac{j\pi i}{t}\right)\right],$$

where the weights $w_j$ depend on Euler summation coefficients, and the approximation is effective for functions with at least two continuous derivatives.

This inversion technique enables practical computation of transient behavior in SMPs, bridging the gap between classical theory and modern numerical implementation.

### 2.6.3 Numerical Solution of Integral Equations

Corradi et al. [2004] present a straightforward numerical method for solving transient behaviors in homogeneous semi-Markov processes (SMPs), focusing on computing state probabilities over time. The main idea is to express the time-dependent state probabilities $p_j(t)$, for each state $j$, through a system of convolution-type integral equations involving the semi-Markov kernel and the embedded Markov chain. The key equation has the form:

$$p_j(t) = \delta_{ij} G_i(t) + \sum_{k\in E}\int_0^t q_{kj}(t-s)p_k(s)\,ds,$$

where $\delta_{ij}$ is the Kronecker delta, $G_i(t)$ is the probability of staying in state $i$ until time $t$, and $q_{kj}(t)$ is the transition density from state $k$ to $j$. The authors propose a discretisation scheme where time is divided into intervals of size $h$, and the integrals are approximated using numerical quadrature rules, yielding a recursive algorithm to compute $p_j(t)$ step by step.

## 2.6.4 Monte Carlo Simulation

Monte Carlo methods are widely used for the numerical evaluation of a semi-Markov systems. The idea is to simulate many independent realizations of the process and use statistical estimates to approximate system performance. The following algorithm simulates a trajectory of a semi-Markov process using the competing risks construction. The output is a sequence $(J_0, S_0, \ldots, J_k, S_k)$ such that $S_k \leq t < S_{k+1}$.

### Algorithms

**Algorithm 01:** Based on the EMC.

1. Put $k = 0, S_0 = 0$, and set $j_0$ as the initial state;
2. sample random variable $J \sim P\left(j_k, \cdot\right)$ and set $j_{k+1} = J(\omega)$;
3. sample random variable $X \sim F_{j_k j_{k+1}}(\cdot)$ and set $x = X(\omega)$;
4. put $k := k + 1$ and $s_k = s_{k-1} + x$. If $s_k \geq t$ then end;
5. set $j_k := j_{k+1}$ and continue to step 2.

**Algorithm 02:** Competing Risks-Based Semi-Markov Trajectory Simulation.

1. Put $k = 0$, $S_0 = 0$ and set $J_0$ as the initial state;
2. sample random variables $X^\ell \sim A_{J_k \ell}(\cdot) := 1 - \int_0^{\cdot} Q_{J_k \ell}(du)\left[1 - H_{J_k}(u)\right]^{-1}$, for all $\ell \in E$ such that $p_{J_k \ell} > 0$;
3. put $x := \min\left\{X^\ell(\omega); \ell \in E\right\}$;
4. set $J_k := \arg\min\left\{X^\ell(\omega); \ell \in E\right\}$;
5. put $k := k + 1$ and $S_k := S_{k-1} + x$; if $S_k \geq t$ then end;
6. set $J_k := J_{k+1}$ and return to Step 2.

Monte Carlo simulations provide flexible and powerful tools for the study of a semi-Markov models, but they can be computationally expensive, particularly for large-scale systems.

In conclusion, numerical methods and simulation techniques play a crucial role in the study of semi-Markov systems. While analytical solutions are often intractable, these numerical approaches provide practical means for evaluating system performance, reliability, and other key measures. The choice of method depends on the specific application, computational constraints, and desired level of accuracy.

# Chapter 3

# Elements of Statistical Estimation

In this chapter, we start by presenting fundamental theorems and lemmas of nonparametric estimation theory, which serve as essential tools for the subsequent analysis. These results lay the theoretical foundation for establishing the properties of our estimators. We then introduce both empirical and kernel estimators for key quantities associated with semi-Markov processes in a finite state space. We study the strong consistency and the asymptotic normality of the proposed estimators, providing a solid theoretical basis for their application in statistical inference.

## 3.1 Basic Set-up of Nonparametric Estimation

**Theorem 3.1.1.** *Émile Borel [1909] **(Strong Law of Large Numbers)** Let $(X_1, X_2, \ldots)$ is an infinite sequence of i.i.d. Lebesgue integrable random variables with expected value $\mathbb{E}[X_1] = \mathbb{E}[X_2] = \ldots$, then we have*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[X_1].$$

**Theorem 3.1.2.** *Gut [1988] Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables and $(N_n)_{n\in\mathbb{N}}$ a positive integer-valued stochastic process. Suppose that*

$$Y_n \xrightarrow[n\to\infty]{a.s.} Y \text{ and } N_n \xrightarrow[n\to\infty]{a.s.} +\infty.$$

*Then,*

$$Y_{N_n} \xrightarrow[n\to\infty]{a.s.} Y.$$

**Theorem 3.1.3.** *Slutsky [1925] **(Slutsky's Theorem)** Let $X, X_n, Y_n, n \in \mathbb{N}$, be random variables or vectors. If*

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} X,$$

*and*

$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} c,$$

*with c a constant, then*

- $Y_n + X_n \xrightarrow[n\to\infty]{\mathcal{D}} c + X,$

- $Y_n X_n \xrightarrow[n\to\infty]{\mathcal{D}} cX$,

- $Y_n^{-1} X_n \xrightarrow[n\to\infty]{\mathcal{D}} c^{-1}X, \text{ for } \; c \neq 0.$

**Corollary 3.1.1.** *Kallenberg [1997] **(Elementary Operations)** Let $\xi, \xi_1, \xi_2, \dots$ and $\eta, \eta_1, \eta_2, \dots$ be random variables with $\xi_n \xrightarrow{\mathbb{P}} \xi$ and $\eta_n \xrightarrow{\mathbb{P}} \eta$. Then $a\xi_n + b\eta_n \xrightarrow{\mathbb{P}} a\xi + b\eta$ for all $a, b \in \mathbb{R}$, and $\xi_n\eta_n \xrightarrow{\mathbb{P}} \xi\eta$. Furthermore, $\xi_n/\eta_n \xrightarrow{\mathbb{P}} \xi/\eta$ whenever a.s. $\eta \neq 0$ and $\eta_n \neq 0$ for all $n$.*

**Lemma 3.1.1.** *Kallenberg [1997] **(Convergence in Probability and Distribution)** Let $\xi, \xi_1, \xi_2, \dots$ be random elements in a measurable space $(\Omega, \mathcal{F})$ into a separable metric space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. Then*

$$\xi_n \xrightarrow{\mathbb{P}} \xi \quad \Rightarrow \quad \xi_n \xrightarrow{\mathcal{D}} \xi,$$

*with equivalence when $\xi$ is a.s. a constant.*

**Theorem 3.1.4.** *Glivenko [1933] **(Glivenko-Cantelli Theorem)** Let $F_n(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\{X_k \leq x\}}$ be the empirical distribution function of the i.i.d. random sample $X_1, \dots, X_n$. Denote by $F$ the common distribution function of $X_i, i = 1, \dots, n$. Thus*

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \xrightarrow[n\to\infty]{a.s.} 0.$$

**Theorem 3.1.5.** *Nadaraya [1965] **(Strong Consistency)** Suppose that $K(x)$ is a function of bounded variation, $f(x)$ is a uniformly continuous density function, and the series $\sum_{n=1}^{\infty} e^{-\gamma n h^2}$ converges for every positive value of $\gamma$. Then*

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \longrightarrow 0,$$

*with probability one as $n \to \infty$.*

The strong law of large numbers and the central limit theorem for additive functionals of Markov renewal processes (MRPs) were established by Pyke and Schaufele [1964]. The notation used throughout this work follows that introduced by Moore and Pyke [1968].

For a real measurable function $f$, defined on $E \times E \times \mathbb{R}$, define, for each $M > 0$, the functional $W_f(M)$ as

$$W_f(M) := \sum_{n=1}^{N(M)} f\left(J_{n-1}, J_n, X_n\right).$$

Set

$$A_{ij} := \int_0^\infty f(i,j,x) dQ_{ij}(x), \quad A_i := \sum_{j=1}^{s} A_{ij},$$

$$B_{ij} := \int_0^\infty (f(i,j,x))^2 dQ_{ij}(x), \quad B_i := \sum_{j=1}^{s} B_{ij}.$$

Furthermore, we have

$$\begin{aligned} r_i &:= \sum_{d=1}^{s} A_d \frac{\mu^*_{ii}}{\mu^*_{dd}}, \\ \sigma_i^2 &:= -r_i^2 + \sum_{d=1}^{s} B_d \frac{\mu^*_{ii}}{\mu^*_{dd}} + 2\sum_{d=1}^{s}\sum_{l\neq i}\sum_{j\neq i} A_{dl} A_j \mu^*_{ii} \frac{\mu^*_{li}+\mu^*_{ij}-\mu^*_{lj}}{\mu^*_{dd}\mu^*_{jj}}. \end{aligned}$$

Finally, put

$$\begin{aligned} m_f &:= \frac{r_i}{\mu_{ii}}, \\ B_f &:= \frac{\sigma_i^2}{\mu_{ii}}. \end{aligned}$$

**Theorem 3.1.6.** *Pyke and Schaufele [1964] **(Strong Law of Large Numbers)** Consider an irreducible and aperiodic Markov reward process (MRP), we have*

$$\frac{W_f(M)}{M} \xrightarrow[M\to\infty]{a.s.} m_f.$$

**Lemma 3.1.2.** *Pyke and Schaufele [1964] **(Central Limit Theorem)** Consider an irreducible and aperiodic Markov reward process (MRP), we have*

$$M^{-1/2}\left[W_f(M) - M\cdot m_f\right] \xrightarrow[M\to\infty]{\mathcal{D}} \mathcal{N}\left(0, B_f\right).$$

Before stating the following lemma, we introduce some associated counting processes related to the semi-Markov process, defined for all $i, j \in E$ and $t \leq M$, as follows

**(i)** $N_i(M) := \sum_{l=1}^{N(M)} \mathbb{1}_{\{J_{l-1}=i\}} = \sum_{l=1}^{\infty} \mathbb{1}_{\{J_{l-1}=i, S_l\leq M\}}$, the number of visits to state $i$ up to time $M$.

**(ii)** $N_{ij}(M) := \sum_{l=1}^{N(M)} \mathbb{1}_{\{J_{l-1}=i, J_l=j\}} = \sum_{l=1}^{\infty} \mathbb{1}_{\{J_{l-1}=i, J_l=j, S_l\leq M\}}$, the number of transitions from state $i$ to state $j$ up to time $M$.

**Lemma 3.1.3.** *Limnios and Oprisan [2001] Under the previous notations, if the EMC $(J_n)_n$ is positive recurrent, then, for any $i, j \in E$ we have:*

1. $\frac{N_i(M)}{N(M)} \xrightarrow[M\to\infty]{a.s.} \nu(i)$,
2. $\frac{N_{ij}(M)}{N(M)} \xrightarrow[M\to\infty]{a.s.} \nu(i)p_{ij}$,
3. $\frac{N_i(M)}{M} \xrightarrow[M\to\infty]{a.s.} \frac{1}{\mu_{ii}}$,
4. $\frac{N_{ij}(M)}{M} \xrightarrow[M\to\infty]{a.s.} \frac{p_{ij}}{\mu_{ii}}$,
5. $\frac{N(M)}{M} \xrightarrow[M\to\infty]{a.s.} \frac{1}{\nu(i)\mu_{ii}}$.

**Theorem 3.1.7.** *Taga [1963] **(Taga's Theorem)** Let the transition distribution functions $Q_{ij}(t)$ be expressed in the form*

$$Q_{ij}(t) = p_{ij}H_i(t),$$

*where the matrix* $\boldsymbol{p} = (p_{ij})_{i,j\in E}$ *is regular, and* $H_i(t)$ *has finite means and variances which will be denoted by*

$$\tau_i = \int_0^\infty t\, dH_i(t), \quad \sigma_i^2 = \int_0^\infty (t-\tau_i)^2\, dH_i(t).$$

*Then the limiting distribution of the total sojourn time,* $S_i(t)$ *spent in state* $i$ *in* $[0,t]$*, certainly exists for any initial state so that*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{S_i(t) - \frac{\tau_i}{\mu_{ii}}t}{\sqrt{t/\mu_{ii}}} \le x\right) = \phi\left(\frac{\frac{1}{\tau_i} + \frac{1}{\mu_{ii}-\tau_i}}{\sqrt{\frac{\sigma_i^2}{\tau_i^2} + \frac{\rho_{ii}^2-\sigma_i^2}{(\mu_{ii}-\tau_i)^2}}}x\right),$$

*where* $\mu_{ii}$ *and* $\rho_{ii}^2$ *are the mean and the variance of the recurrence time of the state* $j$*, respectively, and* $\phi(u)$ *denotes the unit normal distribution function.*

## 3.2 Nonparametric Estimation of Main Related Quantities for Semi-Markov Process

In this section, we introduce nonparametric estimators, using both empirical and kernel methods, for several fundamental quantities associated with semi-Markov processes. Specifically, we consider the estimation of the semi-Markov kernel, the transition probabilities, the sojourn time distribution, the transition function, and the Markov renewal function. Furthermore, we show that these estimators are strongly consistent and asymptotically normal on a finite state space by analyzing one observed sample path.

**Definition 3.2.1.** *Let us consider a sample path of the Markov renewal process* $(J_n, S_n)_{n\in\mathbb{N}}$

$$\mathcal{Y}(M) := \left(J_0, X_1, \ldots, J_{N(M)-1}, X_{N(M)}, J_{N(M)}, u_M\right), \quad M \in \mathbb{R}_+, \tag{3.1}$$

*where* $u_M := M - S_{N(M)}$ *is the time elapsed between time* $M$ *and the last jump time* $S_{N(M)}$*, or equivalently, the censored sojourn time in the last visited state* $J_{N(M)}$*.*

### 3.2.1 Empirical Estimators

Let $\mathcal{Y}(M)$ denote the sample path of a Markov renewal processes $(J_n, S_n)_{n\in\mathbb{N}}$, as defined earlier on Equation (3.1). For all $i, j \in E$, $t > 0$ and $t \le M$, we define the empirical estimator $\widetilde{Q}_{ij}(t,M)$ of the semi-Markov kernel $Q_{ij}(t)$ (cf. Moore and Pyke [1968]; Ouhbi and Limnios [1999]), by

$$\widetilde{Q}_{ij}(t,M) = \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} \mathbb{1}_{\{J_{l-1}=i, J_l=j, X_l\le t\}}. \tag{3.2}$$

From the definition of $\widetilde{Q}_{ij}(t,M)$, we have

$$\widetilde{Q}_{ij}(t,M) = \widetilde{F}_{ij}(t,M)\cdot \widetilde{p}_{ij}(M), \tag{3.3}$$

where $\widetilde{p}_{ij}(M)$ is the empirical estimator of the transition probabilities $p_{ij}$, which is given by:

$$\widetilde{p}_{ij}(M) = \frac{N_{ij}(M)}{N_i(M)} = \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} \mathbb{1}_{\{J_{l-1}=i, J_l=j\}}, \tag{3.4}$$

and, the empirical estimator $\widetilde{F}_{ij}(t,M)$ of the conditional distribution function $F_{ij}(t)$ is defined by

$$\widetilde{F}_{ij}(t,M) = \frac{1}{N_{ij}(M)} \sum_{l=1}^{N(M)} \mathbb{1}_{\{J_{l-1}=i, J_l=j, X_l \leq t\}}. \tag{3.5}$$

Here, the conditional transition mechanism provides the probability distribution of the sojourn time in state $i$ before transitioning to state $j$, given that such a transition occurs.

Let us define the empirical estimator $\widetilde{\boldsymbol{\Psi}}(t,M) = [\widetilde{\Psi}_{ij}(t,M)]$ of the Markov renewal matrix $\boldsymbol{\Psi}(t) = [\Psi_{ij}(t)]$, by

$$\widetilde{\boldsymbol{\Psi}}(t,M) = \sum_{n=0}^{\infty} \widetilde{\mathbf{Q}}^{(n)}(t,M), \tag{3.6}$$

and, let $\widetilde{\mathbf{P}}(t,M)$ be the estimator of the transition function of the semi-Markov process, given by

$$\widetilde{\mathbf{P}}(t,M) = \widetilde{\boldsymbol{\Psi}} \star (I - diag(\widetilde{\mathbf{Q}}(t,M))). \tag{3.7}$$

### Asymptotic properties

Before establishing the asymptotic properties of the empirical estimators, we first need to define the following assumptions concerning the Markov renewal process $(J_n, S_n)_{n\in\mathbb{N}}$,

**(A.1)** The embedded Markov chain $(J_n)_{n\in\mathbb{N}}$ is irreducible and $0 < m_i < \infty$ with $m_i$ is the mean sojourn time in state $i \in E$ of the SMP.

**(A.2)** The Markov renewal process $(J_n, S_n)_{n\in\mathbb{N}}$ is aperiodic.

We recall the following theorems from Ouhbi and Limnios [1999] and Limnios and Oprisan [2001], which establish the strong uniform consistency and the asymptotic normality of the proposed estimators.

**Theorem 3.2.1.** *For any fixed $i,j \in E$ and $t \in \mathbb{R}_+, t \leq M$, under Assumptions **(A.1)-(A.2)**, we have the following results:*

*(i) The empirical estimator $\widetilde{p}_{ij}(M)$ of $p_{ij}$ is strongly consistent, i.e.*

$$\widetilde{p}_{ij}(M) \xrightarrow{a.s.} p_{ij} \quad \text{as} \quad M \to \infty.$$

*(ii) The empirical estimator $\widetilde{F}_{ij}(t,M)$ of $F_{ij}(t)$ is uniformly strong consistent, i.e.*

$$\max_{i,j\in E} \sup_{t\in[0,M)} \left| \widetilde{F}_{ij}(t,M) - F_{ij}(t) \right| \xrightarrow{a.s.} 0 \quad \text{as} \quad M \to \infty.$$

*(iii) The empirical estimator $\widetilde{Q}_{ij}(t,M)$ of $Q_{ij}(t)$ is uniformly strong consistent, i.e.*

$$\max_{i,j\in E} \sup_{t\in[0,M)} \left| \widetilde{Q}_{ij}(t,M) - Q_{ij}(t) \right| \xrightarrow{a.s.} 0 \quad \text{as} \quad M \to \infty.$$

*(iv) The empirical estimator $\widetilde{Q}_{ij}^{(n)}(t,M)$ is uniformly strong consistent, i.e.*

$$\max_{i,j\in E} \sup_{t\in[0,M)} \left| \widetilde{Q}_{ij}^{(n)}(t,M) - Q_{ij}^{(n)}(t) \right| \xrightarrow{a.s.} 0 \quad \text{as} \quad M \to \infty.$$

(v) *The empirical estimator* $\widetilde{\Psi}_{ij}(t,M)$ *of the Markov renewal function* $\Psi_{ij}(t)$ *is uniformly strong consistent, i.e.*

$$\max_{i,j\in E}\sup_{t\in[0,M)}\left|\widetilde{\Psi}_{ij}(t,M)-\Psi_{ij}(t)\right|\xrightarrow{a.s.}0 \quad \text{as} \quad M\to\infty.$$

(vi) *The empirical estimator* $\widetilde{P}_{ij}(t,M)$ *of the transition function* $P_{ij}(t)$ *is uniformly strong consistent, i.e.*

$$\max_{i,j\in E}\sup_{t\in[0,M)}\left|\widetilde{P}_{ij}(t,M)-P_{ij}(t)\right|\xrightarrow{a.s.}0 \quad \text{as} \quad M\to\infty.$$

***Proof of Theorem* 3.2.1.** *See appendix 5.4.3.* □

**Theorem 3.2.2.** *For any fixed* $i,j\in E$ *and* $t\in[0,M]$*, under Assumptions* ***(A.1)-(A.2)****, as* $M\to\infty$*, the following statements hold:*

(i) $\left(M^{1/2}\left[\widetilde{p}_{ij}(M)-p_{ij}\right], M^{1/2}\left[\widetilde{H}_i(t,M)-H_i(t)\right]\right)$ *converges in law to a bivariate normal random variable with mean zero and covariance matrix elements* $(\sigma_{mn})_{1\le m,n\le 2}$ *given by*

$$\sigma_{11}=\mu_{ii}p_{ij}\left(1-p_{ij}\right), \quad \sigma_{22}=\mu_{ii}H_i(t)\left(1-H_i(t)\right), \quad \sigma_{12}=\sigma_{21}=0.$$

(ii) $M^{1/2}\left(\widetilde{Q}_{ij}(t,M)-Q_{ij}(t)\right)$ *converges in law, as* $M$ *tends to infinity, to a zero mean normal random variable with asymptotic variance*

$$\sigma^2_{Q_{ij}}(t):=\mu_{ii}Q_{ij}(t)\left[1-Q_{ij}(t)\right].$$

(iii) $M^{1/2}\left(\widetilde{\Psi}_{ij}(t,M)-\Psi_{ij}(t)\right)$ *converges in law to a normal random variable with mean zero and asymptotic variance*

$$\sigma^2_{\Psi_{ij}}(t)=\sum_{r\in E}\sum_{k\in E}\mu_{rr}\left\{\left(\Psi_{ir}*\Psi_{kj}\right)^2*Q_{rk}-\left(\Psi_{ir}*\Psi_{kj}*Q_{rk}\right)^2\right\}(t).$$

(iv) $M^{1/2}\left(\widetilde{P}_{ij}(t,M)-P_{ij}(t)\right)$ *converges in law to a zero mean normal random variable with asymptotic variance*

$$\begin{aligned}\sigma^2_{P_{ij}}(t)=&\sum_{r\in E}\sum_{k\in E}\mu_{rr}\left[(1-H_i)*B_{irkj}-\Psi_{ij}\mathbb{1}_{\{r=j\}}\right]^2*Q_{rk}(t)\\&-\left\{\left[(1-H_i)*B_{irkj}-\Psi_{ij}\mathbb{1}_{\{r=j\}}\right]*Q_{rk}(t)\right\}^2,\end{aligned}$$

*where*

$$B_{irkj}(t)=\sum_{n=1}^{\infty}\sum_{m=1}^{n}Q_{ir}^{(m-1)}*Q_{kj}^{(n-m)}(t).$$

***Proof of Theorem* 3.2.2.** *See appendix 5.4.3.* □

## 3.2.2 Kernel Estimators

Let $\mathcal{Y}(M)$ denote the sample path of a Markov renewal processes $(J_n, S_n)_{n\in\mathbb{N}}$, as previously defined on Equation (3.1), for all $i, j \in E$, $t > 0$ and $t \leq M$, we define the kernel estimator of $Q_{ij}$ and $H_i$ respectively (cf. Ayhar et al. [2022]), by

$$\widehat{Q}_{ij}(t, M) = \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} G\left(\frac{t - X_l}{h_{ij,M}}\right) \mathbb{1}_{\{J_{l-1}=i, J_l=j\}}, \tag{3.8}$$

and

$$\widehat{H}_i(t, M) = \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} G\left(\frac{t - X_l}{h_{i,M}}\right) \mathbb{1}_{\{J_{l-1}=i\}}, \tag{3.9}$$

where $G(t) = \int_{-\infty}^{t} K(t)dt$, with $K$ is a bounded kernel function.

For fixed states $i$ and $j$, it is important to observe that the smoothing parameter of the aforementioned estimators is dependent on the sample size. Thus, we should write $h_{ij,N_{ij}(M)} = h_{ij,M}$ (resp. $h_{i,N_i(M)} = h_{i,M}$). However, for the sake of simplicity, we opt for a more concise notation.

Let us denote by $\widehat{\mathbf{\Psi}}(t, M)$ the estimators of $\mathbf{\Psi}(t)$ (cf. Mokhtari et al. [2025]), defined by

$$\widehat{\mathbf{\Psi}}(t, M) := \sum_{n=0}^{\infty} \widehat{\mathbf{Q}}^{(n)}(t, M). \tag{3.10}$$

Let $\widehat{\mathbf{P}}(t, M)$ be the estimator of the transition function of the semi-Markov process (see Mokhtari et al. [2025]), given by

$$\widehat{\mathbf{P}}(t, M) = \widehat{\mathbf{\Psi}}(\cdot, M) * (I - diag(\widehat{\mathbf{Q}}(\cdot, M).\mathbf{1}))(t). \tag{3.11}$$

### Asymptotic properties

Prior to analyzing the asymptotic properties of the kernel estimators, we first present the following assumptions which are necessary to derive the asymptotic behavior of our estimators.

### Assumptions

**(H.1)** The semi-Markov process $Z$ is regular with finite mean sojourn times $m$.

In addition, the following assumptions are required in order to establish all the asymptotic properties investigated in this work:

**(H.2)**
- i) $Q_{ij}(t)$ is continuously differentiable with respect to the Lebesgue measure, and let $q_{ij}(t)$ be the corresponding Radon-Nikodym derivative.
- ii) The first derivatives $q_{ij}(t)$ is bounded.

**(H.3)** The kernel $G$ is a distribution function, where its derivative is $K$.

**(H.4)** The kernel $K$ is a density function of bounded variation such that $\lim_{x\to\infty} |xK(x)| = 0$, $\left|\int ntK(t)G^{n-1}(t)dt\right| < \infty$, and $\left|\int t^j K^n(t)dt\right| < \infty$ for $j = 0, 1$, and $n = 1, 2$.

**(H.5)** The smoothing parameter $h_{ij,M}$ satisfies

$$\lim_{M\longrightarrow\infty} h_{ij,M} = 0 \quad \text{and} \quad \lim_{M\longrightarrow\infty} Mh_{ij,M} = \infty.$$

### Comments on the assumptions

The structural assumption **(H.1)** is the same as those classically used for the semi-Markov processes framework (see, for instance Ayhar et al. [2022], Gąmiz et al. [2011], and Dumitrescu et al. [2016]. **(H.1)** means that the counting process $\{N(t) : t \geq 0\}$ has a finite number of jumps in a finite period with probability 1. In addition, under this hypothesis we have $S_n < S_{n+1}$, for any $n \in \mathbb{N}$, and $S_n \to \infty$ as $n$ goes to infinity. Assumption **(H.2)** imposed on $Q_{ij}(t)$ is a regularity type hypothesis. Whereas, assumption **(H.2)(i)** is a constraint of the regularity type that will enable us to get strong consistency. Furthermore, the second derivative hypothesis **(H.2)(ii)** establishes more restrictive constraints when going through to state the asymptotic normality of our estimators. **(H.4)**-**(H.5)** are technical constraints. They are imposed for the sake of simplicity and the brevity of the proofs.

**Theorem 3.2.3.** *Ayhar et al. [2022] For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \leq M$, under Assumptions **(H.1)-(H.5)**, the following statement stand true.*

*I) The kernel estimator $\widehat{H}_i(t, M)$, as introduced in (3.9), is uniformly strongly consistent, i.e.,*

$$\max_{i \in E} \sup_{t \in [0,M]} \left|\dot{\widehat{H}}_i(t, M) - H_i(t)\right| \xrightarrow{a.s.} 0 \quad as \quad M \to \infty.$$

*II) The kernel estimator $\widehat{Q}_{ij}(t, M)$ proposed in (3.8) is uniformly strongly consistent, i.e.,*

$$\max_{i,j \in E} \sup_{t \in [0,M]} |\widehat{Q}_{ij}(t, M) - Q_{ij}(t)| \xrightarrow{a.s.} 0 \quad as \quad M \to \infty.$$

***Proof of Theorem 3.2.3.*** *see Ayhar et al. [2022].* □

**Theorem 3.2.4.** *Mokhtari et al. [2025] For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \leq M$, under Assumptions **(H.1)-(H.5)**, the following statement stand true.*

1. *The kernel estimator $\widehat{\mathbf{\Psi}}(t, M)$ proposed in (3.10) is uniformly strongly consistent, i.e.,*

$$\max_{i,j \in E} \sup_{t \in [0,M]} \left|\widehat{\Psi}_{ij}(t, M) - \Psi_{ij}(t)\right| \xrightarrow{a.s.} 0 \quad as \quad M \to \infty.$$

2. *The kernel estimator $\widehat{\mathbf{P}}(t, M)$ proposed in (3.11) is uniformly strongly consistent, i.e.,*

$$\max_{i,j \in E} \sup_{t \in [0,M]} \left|\widehat{P}_{ij}(t, M) - P_{ij}(t)\right| \xrightarrow{a.s.} 0 \quad as \quad M \to \infty.$$

***Proof of Theorem 3.2.4.*** *see Mokhtari et al. [2025].* □

The following results concern the asymptotic normality of the proposed estimators.

**Theorem 3.2.5.** *(see Ayhar et al. [2022] and Mokhtari et al. [2025]) For any fixed arbitrary states $i, j \in E$ and any fixed arbitrary positive $t \in \mathbb{R}_+$, $t \leq M$, under Assumptions* $(\mathbf{H.2}) - (\mathbf{H.5})$*, the following statements stand true.*

1.
$$\sqrt{M}[\widehat{H}_i(t,M) - H_i(t)] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2_{H_i}(t)) \quad as \quad M \to \infty,$$
*with the asymptotic variance*
$$\sigma^2_{H_i}(t) = \mu_{ii} H_i(t)\,[1 - H_i(t)]\,. \tag{3.12}$$

2.
$$\sqrt{M}[\widehat{Q}_{ij}(t,M) - Q_{ij}(t)] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2_{Q_{ij}}(t)) \quad as \quad M \to \infty,$$
*with the asymptotic variance*
$$\sigma^2_{Q_{ij}}(t) = \mu_{ii} Q_{ij}(t)\,[1 - Q_{ij}(t)]\,. \tag{3.13}$$

3.
$$\sqrt{M h_M}\left[\widehat{\Psi}_{ij}(t,M) - \Psi_{ij}(t)\right] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2_{\Psi_{ij}}(t)) \quad as \quad M \to \infty,$$
*where*
$$\sigma^2_{\Psi_{ij}}(t) \leq \sum_{m=1}^{s} \mu_{mm} \sum_{r=1}^{s} (\Psi_{im} * \Psi_{rj})^2 * Q_{mr}(t) \int K^2(z)dz.$$

4.
$$\sqrt{M h_M}\left[\widehat{P}_{ij}(t,M) - P_{ij}(t)\right] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2_{P_{ij}}(t)) \quad as \quad M \to \infty,$$
*where*
$$\sigma^2_{P_{ij}}(t) \leq \sum_{m=1}^{s} \mu_{mm} \sum_{u=1}^{s} [\delta_{mj}\Psi_{ij} - (1 - H_j) * \Psi_{im} * \Psi_{uj}]^2 * Q_{mu}(t) \int K^2(z)dz,$$

*where $\mu_{mm}$ is the mean recurrent time of state $m$ for the semi-Markov process $(Z_t)_{t\in\mathbb{R}_+}$.*

**Proof of Theorem 3.2.5.** The asymptotic normality of the kernel estimators relies on classical results from empirical process theory and the properties of kernel smoothing techniques. We refer the reader to Ayhar et al. [2022] and Mokhtari et al. [2025], where these results are established in detail. □

# Chapter 4

# Stationary Distribution

In recent years, results for the nonparametric estimation of the stationary distribution of a SMC have been developed. In continuous time, Limnios et al. [2005] presented an empirical estimator and also studied its asymptotic properties, such as strong convergence and asymptotic normality. One year later, Limnios [2006] proposed a nonparametric estimator for the general state-space stationary law and gave some asymptotic properties. Macci [2008] presented principles of large deviations for the estimators of Limnios et al. [2005] in the finite case. Furthermore, in discrete time, Barbu et al. [2012] studied the asymptotic behavior of an empirical estimator for a particular case of finite SMCs.

This chapter is devoted to the study of the stationary distribution of continuous-time semi-Markov processes. In Section 4.1, we present the theoretical background, including the stationary distribution of the embedded Markov chain, the mean sojourn time of the semi-Markov process, the mean first passage time of state $i$ for the Markov renewal process (MRP), and the stationary distribution of the semi-Markov process itself. Section 4.2 is dedicated to the empirical estimation of the proposed quantities, along with an analysis of their asymptotic properties. In Section 4.3, we construct kernel-based estimators for these quantities and establish their asymptotic properties, such as consistency and asymptotic normality. Confidence intervals for the stationary distribution are also derived. Finally, Section 4.4 presents an application that illustrates the practical implementation of the proposed estimation methods.

## 4.1 Stationary Distribution of Semi-Markov Process

**Definition 4.1.1.** *For a semi-Markov process $(Z_t)_{t\in\mathbb{R}_+}$, the stationary distribution $\pi = (\pi_1, \dots, \pi_s)^t$ is defined, when it exists, for every $i, j \in E$, by*

$$\pi_j = \lim_{t\to\infty} P_{ij}(t).$$

The following proposition states the explicit expression for the stationary distribution of the semi-Markov process.

**Proposition 4.1.1.** *Howard [1964] Under the assumptions* ***(A.1) - (H.1)****, the stationary distribution $\pi$ of the semi-Markov process $(Z_t)_{t\in\mathbb{R}_+}$ is given, for all $i \in E$, by*

$$\pi_i = \frac{\nu_i m_i}{\sum\limits_{k\in E} \nu_k m_k}, \tag{4.1}$$

*where $m_i$ is the mean sojourn time in a state $i$ of $Z$, which is given by*

$$m_i = \int_0^\infty (1 - H_i(t))dt. \tag{4.2}$$

Moreover, we define the mean sojourn time of $Z$, denoted $\overline{m}$, by

$$\overline{m} := \sum_{i\in E} \nu_i m_i. \tag{4.3}$$

The next proposition provides an explicit expression for the mean recurrence time in state $i$ of the Markov renewal process.

**Proposition 4.1.2.** *Limnios and Oprisan [2001] Under the assumption* ***(H.1)****, the mean recurrence times in the state $i$ of the MRP is defined by,*

$$\mu_{ii} = \frac{\overline{m}}{\nu_i}. \tag{4.4}$$

Based on Proposition 4.1.1 and Equations (4.3), we obtain the following result

$$\pi_i = \frac{\nu_i m_i}{\overline{m}}. \tag{4.5}$$

In addition to Equation (4.4), the stationary distribution can also be written as

$$\pi_i = \frac{m_i}{\mu_{ii}}. \tag{4.6}$$

**Remark 4.1.1.** *In general, the stationary distribution $\pi$ of the semi-Markov process $Z$ is not equal to the stationary distribution $\nu$ of the embedded Markov chain $(J_n)_{n\geq 0}$. However, the equality $\pi = \nu$ holds in particular cases, such as when $m_i$ is independent of $i \in E$.*

## 4.2 Empirical Estimation of the Stationary Distribution

Let $\mathcal{Y}(M)$ denote the sample path of a Markov renewal processes $(J_n, S_n)_{n\in\mathbb{N}}$, as defined earlier on Equation (3.1), for all $i, j \in E$, $t > 0$ and $t \leq M$, let consider the empirical estimator of the stationary distribution of the EMC $(J_n)_{n\in\mathbb{N}}$,

$$\widetilde{\nu}_i(M) = \frac{N_i(M)}{N(M)}. \tag{4.7}$$

**Lemma 4.2.1.** *Limnios et al. [2005] For all $i \in E$ and $M \in \mathbb{R}_+$, we have*

*(i) The empirical estimator $\widetilde{\nu}_i(M)$ of the stationary distribution $\nu_i$ of the embedded Markov chain $(J_n)_{n\geq 0}$, is strongly consistent, i.e.,*

$$\max_i |\widetilde{\nu}_i(M) - \nu_i| \xrightarrow{a.s.} 0, \quad M \to \infty.$$

*(ii) The random variables* $\sqrt{M}\,(\widetilde{\nu}_i(M) - \nu_i)$ *converge, as* $M$ *tends to infinity, to a centered normal distribution with variance*

$$\sigma^2_{\nu_i} = \nu_i\,(1-\nu_i)\,. \tag{4.8}$$

***Proof of Lemma 4.2.1.*** *See Appendix 5.4.3* ☐

For all $i \in E$ and $M \in \mathbb{R}_+$, let $\widetilde{\mu}^*_{ii}(M)$ be the estimator of the mean recurrence time $\mu^*_{ii}$ of the embedded Markov chain $(J_n)_{n\geq 0}$, defined as

$$\widetilde{\mu}^*_{ii}(M) := \frac{N(M)}{N_i(M)} = \frac{1}{\widetilde{\nu}_i(M)}. \tag{4.9}$$

By convention, let $\widetilde{\mu}^*_{ii}(M) = 0$ if $N_i(M) = 0$.

**Proposition 4.2.1.** *Limnios and Oprisan [2001] For all* $i \in E$ *and* $M \in \mathbb{R}_+$*, we have*

1. *The empirical estimator* $\widetilde{\mu}^*_{ii}(M)$*, defined by Equation (4.9), of the mean recurrence times* $\mu^*_{ii}$ *of the embedded Markov chain* $(J_n)_{n\geq 0}$*, is strongly consistent, i.e.,*

$$\max_{i\in E} |\widetilde{\mu}^*_{ii}(M) - \mu^*_{ii}| \xrightarrow{a.s.} 0 \quad as \quad M \to \infty.$$

2. *The random variables* $\sqrt{M}\,(\widetilde{\mu}^*_{ii}(M) - \mu^*_{ii})$ *converge, as* $M$ *tends to infinity, to a centered normal distribution with variance*

$$\sigma^2_{\mu^*_{ii}} = (\mu^*_{ii})^2\,(\mu^*_{ii} - 1)\,. \tag{4.10}$$

***Proof of Proposition 4.2.1.*** *See Appendix 5.4.3* ☐

For all $i \in E$, the empirical estimator of the sojourn time distribution function $\widetilde{H}_i(t,M), t \in \mathbb{R}_+$, $M \in \mathbb{R}_+$, is defined by

$$\widetilde{H}_i(t,M) = \frac{1}{N_i(M)} \sum_{l=1}^{N_i(M)} \mathbb{1}_{\{X_{il}\leq t\}}, \tag{4.11}$$

where $X_{il}$ is the $l$th sojourn time in state $i$. Based on Equation (4.11), the empirical estimator for the distribution function of the sojourn time in the state $i$ is defined by

$$\begin{aligned} \widetilde{m}_i(M) &= \int_0^\infty (1 - \widetilde{H}_i(t,M))dt \\ &= \int_0^\infty \frac{1}{N_i(M)} \sum_{l=1}^{N_i(M)} \mathbb{1}_{\{X_{il}\geq t\}} dt \\ &= \frac{1}{N_i(M)} \sum_{l=1}^{N_i(M)} \int_0^\infty \mathbb{1}_{\{X_{il}\geq t\}} dt. \end{aligned}$$

Furthermore,

$$\widetilde{m}_i(M) = \frac{1}{N_i(M)} \sum_{l=1}^{N_i(M)} X_{il}. \tag{4.12}$$

In the case where $N_i(M) = 0$, set $\widetilde{m}_i(M) = \widetilde{H}_i(t,M) = 0$.

The strong convergence, when the censoring time tends to infinity, of estimator (4.12) was shown by Limnios et al. [2005]. In the following lemma, we present the strong convergence and the asymptotic normality of this estimator.

**Lemma 4.2.2.** *Limnios et al. [2005] For any fixed $i \in E$ and $M \in \mathbb{R}_+$, we have*

1. *The estimator $\widetilde{m}_i(M)$ of the mean sojourn time in state $i$, is strongly consistent, as $M \longrightarrow \infty$, that is*
$$\max_{i\in E} |\widetilde{m}_i(M) - m_i| \xrightarrow{a.s.} 0.$$
2. *The random variable $\sqrt{M}(\widetilde{m}_i(M) - m_i)$ converges, as $M \longrightarrow \infty$, to a centered normal random variable with variance*
$$\sigma^2_{m_i} = \mu_{ii} \int_0^\infty (t - m_i)^2 dH_i(t). \tag{4.13}$$

***Proof of Lemma 4.2.2.*** *See Appendix 5.4.3* □

Now, the empirical estimator of the stationary distribution of the SMP, $\widetilde{\pi}_j(M)$ is defined by

$$\widetilde{\pi}_j(M) = \frac{\widetilde{\nu}_j(M)\widetilde{m}_j(M)}{\sum_{i\in E} \widetilde{\nu}_i(M)\widetilde{m}_i(M)}, \quad j \in E. \tag{4.14}$$

**Theorem 4.2.1.** *Limnios et al. [2005] The proposed estimator (4.14), of the stationary distribution for a semi-Markov process $\widetilde{\pi}_i(M)$ satisfies the following properties:*

1. *Strong consistency,*
$$\max_{i\in E} |\widetilde{\pi}_i(M) - \pi_i| \xrightarrow[M\to\infty]{a.s.} 0.$$
2. *Asymptotic normality,*
$$\sqrt{M}(\widetilde{\pi}_i(M) \;-\; \pi_i) \xrightarrow[M\to\infty]{\mathcal{D}} \phi \circ g,$$
*where the linear function $g$ is defined by*
$$g(x) = \left(\frac{1}{m_i} + \frac{1}{\mu_{ii} - m_i}\right) x \Big/ \sqrt{\frac{\sigma_i^2}{m_i^2} + \frac{\rho_{ii}^2 - \sigma_i^2}{(\mu_{ii} - m_i)^2}}, \qquad x \in \mathbb{R}_+,$$
*with $\rho^2_{ii}$ is the variance of the recurrence time of the state $i$, and $\phi$ is the distribution function of the standard normal random variable $\mathcal{N}(0, 1)$.*

***Proof of Theorem 4.2.1.*** *See Appendix 5.4.3.* □

## 4.3 Kernel Estimation of the Stationary Distribution

Let $\mathcal{Y}(M)$ denote the sample path of a Markov renewal process $(J_n, S_n)_{n\in\mathbb{N}}$, as defined in Equation (3.1), for all $i \in E$, $t > 0$ and $t \leq M$, the empirical estimator of the stationary distribution of the embedded Markov chain $(J_n)_{n\in\mathbb{N}}$, is given by Equation (4.7).

Assuming the integral of the corresponding survival function converges, we define the kernel estimator of the mean sojourn time in state $i$, based on Equation (4.2), as follows:

$$\begin{aligned}
\widehat{m}_i(M) &= \int_0^\infty \left(1 - \widehat{H}_i(t, M)\right) dt \\
&= \int_0^\infty t d\widehat{H}_i(t, M) \\
&= \frac{1}{N_i(M) h_{i,M}} \sum_{l=1}^{N(M)} \mathbb{1}_{\{J_{l-1}=i\}} \int_0^\infty tK\left(\frac{t - X_l}{h_{i,M}}\right) dt.
\end{aligned} \tag{4.15}$$

Based on Equations (4.3), (4.7) and (4.15), we can define the kernel estimator of the mean sojourn time of $Z$ as follows:

$$\begin{aligned}\widehat{\overline{m}}(M) &= \sum_{k\in E}\widehat{\nu}_k(M)\widehat{m}_k(M)\\ &= \frac{1}{N(M)}\sum_{k\in E}\sum_{l=1}^{N(M)}\frac{1}{h_{k,M}}\mathbb{1}_{\{J_{l-1}=k\}}\int_0^{\infty} tK\left(\frac{t-X_l}{h_{k,M}}\right)dt. \qquad (4.16)\end{aligned}$$

Since the corresponding functional is independent of time, we obtain $\widehat{\nu}=\widetilde{\nu}$.

Derived from Equations (4.4), (4.7), and (4.16), the kernel estimator of the mean first passage time to state $i$ of the Markov renewal process (MRP), is given by:

$$\begin{aligned}\widehat{\mu}_{ii}(M) &= \frac{\widehat{\overline{m}}(M)}{\widehat{\nu}_i(M)}\\ &= \frac{1}{N_i(M)}\sum_{k\in E}\sum_{l=1}^{N(M)}\frac{1}{h_{k,M}}\mathbb{1}_{\{J_{l-1}=k\}}\int_0^{\infty} tK\left(\frac{t-X_l}{h_{k,M}}\right)dt. \qquad (4.17)\end{aligned}$$

Considering Equations (4.5), (4.15), and (4.16), we deduce the kernel estimator of the stationary distribution of the semi-Markov process (SMP) $Z$ as follows:

$$\widehat{\pi}_i(M) = \frac{\widehat{m}_i(M)\widehat{\nu}_i(M)}{\widehat{\overline{m}}(M)}. \qquad (4.18)$$

### 4.3.1 Asymptotic Properties

Before analyzing the asymptotic properties of the kernel estimators, we begin by stating the assumptions required to derive their asymptotic behavior. These build upon the conditions **(H.1)–(H.5)** introduced in Section 3.2.2.

We introduce the following lemma which will be necessary to prove our results in Section 4.3.

**Lemma 4.3.1.** *Hamlat et al. [2025b] For $n=1,2$. If* ***(H.2)-(H.5)*** *hold, then as $M\longrightarrow\infty$, we have*

$$\int_0^{+\infty} G^n\left(\frac{t-x}{h_{ij,M}}\right)dQ_{ij}(x) \leqslant Q_{ij}(t).$$

**Proof of Lemma 4.3.1.** By using a change of variable and an integration by parts followed by Taylor's expansion, we have

$$\begin{aligned}\int_0^{+\infty} G^n\left(\frac{t-x}{h_{ij,M}}\right)dQ_{ij}(x) &= -\int_{-\infty}^{\frac{t}{h_{ij,M}}} G^n(z)\,dQ_{ij}(t-h_{ij,M}z)\\ &\leqslant -\int_{-\infty}^{+\infty} G^n(z)\,dQ_{ij}(t-h_{ij,M}z)\\ &\leqslant \int_{-\infty}^{+\infty} nK(z)G^{n-1}(z)Q_{ij}(t-h_{ij,M}z)dz\\ &\leqslant \int_{-\infty}^{+\infty} nK(z)G^{n-1}(z)\left[Q_{ij}(t)-h_{ij,M}zq_{ij}(t^*)\right]dz\\ &\leqslant Q_{ij}(t)\int_{-\infty}^{+\infty} nK(z)G^{n-1}(z)dz - h_{ij,M}q_{ij}(t^*)\int_{-\infty}^{+\infty} znK(z)G^{n-1}(z)dz\\ &\leqslant Q_{ij}(t),\end{aligned}$$

where $t-h_{ij,M}z\leq t^*\leq t$. □

### 4.3.2 Main results

Our first result in Section 4.3 concerns the asymptotic properties of the estimator proposed in Equation (4.15).

**Theorem 4.3.1.** *Hamlat et al. [2025b] For any fixed $i \in E$, under **(H.1)-(H.5)**, we have*

1. *The estimator $\widehat{m}_i(M)$ of $m_i$ is strongly consistent, that is*

$$\max_{i\in E} \mid \widehat{m}_i(M) - m_i \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

2. *The random variable $\sqrt{M}\,(\widehat{m}_i(M) - m_i)$ converges, as $M \longrightarrow \infty$, to a centered normal distribution with asymptotic variance*

$$\sigma^2_{m_i} \leqslant \mu_{ii} \int_0^{+\infty} H_i(t)\,(1 - H_i(t))\,dt. \tag{4.19}$$

**Proof of Theorem 4.3.1.**

1. Strong consistency, we have

$$\begin{aligned}
\max_{i\in E} \mid \widehat{m}_i(M) - m_i \mid &= \max_{i\in E} \mid \int_0^{+\infty} \widehat{\overline{H}}_i(t,M)dt - \int_0^{+\infty} \overline{H}_i(t)dt \mid \\
&= \max_{i\in E} \mid \int_0^{+\infty} \Big(H_i(t) - \widehat{H}_i(t,M)\Big)\,dt \mid \\
&\leq \int_0^{+\infty} \max_{i\in E} \mid \widehat{H}_i(t,M) - H_i(t) \mid dt.
\end{aligned}$$

For all $i \in E$ and $M \in \mathbb{R}_+$, the strong consistency of the kernel estimator $\widehat{H}_j(t,M)$ is stated in Theorem 4.1 of Ayhar et al. [2022]. Therefore,

$$\max_{i\in E} \mid \widehat{m}_i(M) - m_i \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

2. Asymptotic normality, we have

$$\begin{aligned}
\sqrt{M}\,(\widehat{m}_i(M) - m_i) &= \sqrt{M}\left(\int_0^{+\infty} \widehat{\overline{H}}_i(t,M)dt - \int_0^{+\infty} \overline{H}_i(t)dt\right) \\
&= \sqrt{M}\int_0^{+\infty} \Big(H_i(t) - \widehat{H}_i(t,M)\Big)\,dt \\
&= \sqrt{M}\int_0^{+\infty} \left(\sum_{j\in E} Q_{ij}(t) - \sum_{j\in E} \widehat{Q}_{ij}(t,M)\right)dt.
\end{aligned}$$

Then $\sqrt{M}\,[\widehat{m}_i(M) - m_i]$ has the same limit as

$$\begin{aligned}
&\sqrt{M}\int_0^{+\infty} - \sum_{j\in E} \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} \left[G\left(\frac{t - X_l}{h_{ij,M}}\right)\mathbb{1}_{\{J_{l-1}=i, J_l=j\}} - Q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}}\right]dt \\
&= \frac{1}{\sqrt{M}} \sum_{l=1}^{N(M)} \sum_{j\in E} \frac{M}{N_i(M)} \int_0^{+\infty} \left[Q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}} - G\left(\frac{t - X_l}{h_{ij,M}}\right)\mathbb{1}_{\{J_{l-1}=i, J_l=j\}}\right]dt \\
&= \frac{1}{\sqrt{M}} \sum_{l=1}^{N(M)} f(J_{l-1}, J_l, X_l).
\end{aligned}$$

Apply central limit theorem related to semi-Markov processes (see Pyke and Schaufele [1964]) to the function $W_f(t)$ such that

$$W_f(t) = \sum_{l=1}^{N(M)} f(J_{l-1}, J_l, X_l),$$

where, for any fixed $t > 0$, we define the function $f : E \times E \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned} f(d,r,x) &= \sum_{j\in E} \frac{M}{N_i(M)} \int_0^{+\infty} \left[ Q_{ij}(t)\mathbb{1}_{\{d=i\}} - G\left(\frac{t-x}{h_{ij,M}}\right)\mathbb{1}_{\{d=i,r=j\}} \right] dt \\ &= \frac{M}{N_i(M)} \int_0^{+\infty} \left[ H_i(t)\mathbb{1}_{\{d=i\}} - G\left(\frac{t-x}{h_{ir,M}}\right)\mathbb{1}_{\{d=i\}} \right] dt. \end{aligned}$$

In order to apply the Pyke and Schaufeles' CLT (cf. Pyke and Schaufele [1964]), we need to compute the quantities $A_{dr}, A_d,\ B_{dr}, B_d, r_d, m_f, \sigma_i^2(t)$ and then $\sigma_{m_i}^2$. Using Fubini's theorem and Lemma 4.3.1 with assumptions **(H.2)-(H.5)**. We have

$$\begin{aligned} A_d &= \sum_{r\in E} A_{dr} \\ &= \sum_{r\in E} \int_0^{+\infty} f\left(d,r,x\right) dQ_{dr}(x) \\ &= \sum_{r\in E} \int_0^{+\infty} \left[ \frac{M}{N_i(M)} \int_0^{+\infty} \left[ H_i(t)\mathbb{1}_{\{d=i\}} - G\left(\frac{t-x}{h_{ir,M}}\right)\mathbb{1}_{\{d=i\}} \right] dt \right] dQ_{dr}(x) \\ &= \sum_{r\in E} \frac{M}{N_i(M)} \int_0^{+\infty} \left[ H_i(t) \int_0^{+\infty} \mathbb{1}_{\{d=i\}} dQ_{dr}(x) - \int_0^{+\infty} G\left(\frac{t-x}{h_{ir,M}}\right)\mathbb{1}_{\{d=i\}} dQ_{dr}(x) \right] dt \\ &= \sum_{r\in E} \frac{M}{N_i(M)} \int_0^{+\infty} \left[ H_i(t) p_{dr}\mathbb{1}_{\{d=i\}} - \int_0^{+\infty} G\left(\frac{t-x}{h_{ir,M}}\right)\mathbb{1}_{\{d=i\}} dQ_{dr}(x) \right] dt \\ &\leqslant \sum_{r\in E} \frac{M}{N_i(M)} \mathbb{1}_{\{d=i\}} \int_0^{+\infty} \left[ H_i(t)p_{dr} - Q_{dr}(t) \right] dt. \end{aligned}$$

Then

$$A_d \leqslant \frac{M}{N_i(M)} \mathbb{1}_{\{d=i\}} \int_0^{+\infty} \left[ H_i(t) - H_d(t) \right] dt.$$

For $B_d$, by using Fubini's theorem, Jensen's inequality and Lemma 4.3.1, we have

$$\begin{aligned}
B_d &= \sum_{r\in E} B_{dr}\\
&= \sum_{r\in E}\int_0^{+\infty} [f(d,r,x)]^2 dQ_{dr}(x)\\
&= \sum_{r\in E}\int_0^{+\infty} \left[\frac{M}{N_i(M)}\int_0^{+\infty}\left[H_i(t)\mathbb{1}_{\{d=i\}} - G\left(\frac{t-x}{h_{ir,M}}\right)\mathbb{1}_{\{d=i\}}\right]dt\right]^2 dQ_{dr}(x)\\
&= \sum_{r\in E}\int_0^{+\infty} \mathbb{1}_{\{d=i\}}\left[\frac{M}{N_i(M)}\int_0^{+\infty}\left[H_i(t) - G\left(\frac{t-x}{h_{ir,M}}\right)\right]dt\right]^2 dQ_{dr}(x)\\
&\leqslant \sum_{r\in E}\int_0^{+\infty} \mathbb{1}_{\{d=i\}}\left[\left(\frac{M}{N_i(M)}\right)^2\int_0^{+\infty}\left[H_i(t) - G\left(\frac{t-x}{h_{ir,M}}\right)\right]^2 dt\right] dQ_{dr}(x)\\
&\leqslant \sum_{r\in E}\left(\frac{M}{N_i(M)}\right)^2 \mathbb{1}_{\{d=i\}}\int_0^{+\infty}\left[H_i^2(t)\int_0^{+\infty} dQ_{dr}(x) + \int_0^{+\infty} G^2\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right.\\
&\left. -2H_i(t)\int_0^{+\infty} G\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right]dt.
\end{aligned}$$

Thus

$$B_d \leqslant \sum_{r\in E}\left(\frac{M}{N_i(M)}\right)^2 \mathbb{1}_{\{d=i\}}\int_0^{+\infty}\left[H_i^2(t)p_{dr} + Q_{dr}(t) - 2H_i(t)Q_{dr}(t)\right]dt.$$

Since $\frac{N_i(M)}{M} \overset{a.s.}{\to} \frac{1}{\mu_{ii}}$ (see Limnios and Oprisan [2001]), when $M \to +\infty$, we have

$$B_d \leqslant \mu_{ii}^2 \mathbb{1}_{\{d=i\}}\int_0^{+\infty}\left[H_i^2(t) + H_d(t) - 2H_i(t)H_d(t)\right]dt.$$

Furthermore,

$$\begin{aligned}
r_i &= \sum_{d\in E} A_d\frac{\mu_{ii}^*}{\mu_{dd}^*}\\
&\leqslant \sum_{d\in E}\frac{\mu_{ii}^*}{\mu_{dd}^*}\frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\int_0^{+\infty}[H_i(t)-H_d(t)]\,dt\\
&= 0 \qquad \text{as} \qquad M\to\infty,\\
m_f &= \frac{1}{\mu_{ii}} r_i = 0 \qquad \text{as} \qquad M\to\infty.
\end{aligned}$$

Then

$$\begin{aligned}
\sigma_{m_i}^2(t) &= \frac{1}{\mu_{ii}}\sigma_i^2(t)\\
&= \frac{1}{\mu_{ii}}\sum_{d\in E} B_d\frac{\mu_{ii}^*}{\mu_{dd}^*}\\
&\leqslant \frac{\mu_{ii}^*}{\mu_{ii}}\sum_{d\in E}\frac{\mu_{ii}}{\mu_{dd}^*}\mu_{ii}\mathbb{1}_{\{d=i\}}\int_0^{+\infty}\left[H_i^2(t)+H_d(t)-2H_i(t)H_d(t)\right]dt.
\end{aligned}$$

Thus, since $\mu_{ii}^* = \frac{1}{\nu_i}$ (see Kemeny and Snell [1976]) and $\mu_{ii} = \frac{\bar{m}}{\nu_i}$ (see Limnios and Oprisan [2001]), we have

$$\sigma_{m_i}^2 \leqslant \mu_{ii}\int_0^{+\infty} H_i(t)\left(1-H_i(t)\right)dt.$$

We obtain from the CLT that $\sqrt{M}\,[\widehat{m}_i(M) - m_i]$ converges in distribution, as $M$ tends to infinity, to a normal random variable with zero mean and the variance $\sigma^2_{m_i}$ given by inequality (4.19). □

Our second result in Section 4.3 concerns the asymptotic properties of the estimator proposed in Equation (4.16).

**Theorem 4.3.2.** *Hamlat et al. [2025b] For any fixed $i \in E$, under **(H.1)-(H.5)**, we have*

1. *The estimator $\widehat{\overline{m}}(M)$ of $\overline{m}$ is consistency, that is*

$$\mid \widehat{\overline{m}}(M) - \overline{m} \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

2. *The random variable $\sqrt{M}\left(\widehat{\overline{m}}(M) - \overline{m}\right)$ converges, as $M \longrightarrow \infty$, to a centered normal distribution with asymptotic variance*

$$\sigma^2_{\overline{m}} \leqslant \overline{m} \sum_{d \in E} \nu_d \int_0^{+\infty} H_d(t)\,(1 - H_d(t))\,dt. \tag{4.20}$$

**Proof of Theorem 4.3.2.**

1. consistency, we have

$$\begin{aligned}
\mid \widehat{\overline{m}}(M) - \overline{m} \mid &= \mid \sum_{i \in E} \widehat{\nu}_i(M)\widehat{m}_i(M) - \sum_{i \in E} \nu_i m_i \mid \\
&= \mid \sum_{i \in E} \left(\widehat{\nu}_i(M)\widehat{m}_i(M) - \nu_i m_i\right) \mid \\
&\leqslant \sum_{i \in E} \mid \widehat{\nu}_i(M)\widehat{m}_i(M) - \nu_i m_i - \widehat{\nu}_i(M) m_i + \widehat{\nu}_i(M) m_i \mid \\
&\leqslant \sum_{i \in E} \mid \widehat{\nu}_i(M)\left(\widehat{m}_i(M) - m_i\right) + m_i\left(\widehat{\nu}_i(M) - \nu_i\right) \mid \\
&\leqslant \sum_{i \in E} \widehat{\nu}_i(M) \mid \widehat{m}_i(M) - m_i \mid + \sum_{i \in E} m_i \mid \widehat{\nu}_i(M) - \nu_i \mid .
\end{aligned}$$

For each $i \in E$ and and $M \in \mathbb{R}_+$, the strong consistency of the estimators $\widehat{\nu}_i(M)$ and $\widehat{m}_i(M)$ are established in Lemma 4.2.1 (1) of Limnios et al. [2005] and Theorem 4.3.1 (1), respectively. Therefore, we have

$$\mid \widehat{\overline{m}}(M) - \overline{m} \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

2. Asymptotic normality, we have

$$
\begin{aligned}
\sqrt{M}\left[\widehat{\overline{m}}(M)-\overline{m}\right] &= \sqrt{M}\left[\sum_{i\in E}\widehat{\nu}_i(M)\widehat{m}_i(M)-\sum_{i\in E}\nu_i m_i\right]\\
&= \sqrt{M}\sum_{i\in E}\left[\frac{N_i(M)}{N(M)}\int_0^\infty \widehat{\overline{H}}_i(t,M)dt-\nu_i\int_0^\infty \overline{H}_i(t)dt\right]\\
&= \sqrt{M}\sum_{i\in E}\int_0^\infty\left[\frac{N_i(M)}{N(M)}-\nu_i-\frac{N_i(M)}{N(M)}\widehat{H}_i(t,M)+\nu_i H_i(t)\right]dt\\
&= \sqrt{M}\sum_{i\in E}\int_0^\infty\left[\frac{N_i(M)}{N(M)}-\nu_i-\frac{N_i(M)}{N(M)}\sum_{j\in E}\widehat{Q}_{ij}(t,M)+\nu_i\sum_{j\in E}Q_{ij}(t)\right]dt\\
&= \sqrt{M}\sum_{i\in E}\sum_{j\in E}\int_0^\infty\left[\frac{N_i(M)}{N(M)}\mathbb{1}_{\{i=j\}}-\nu_i\mathbb{1}_{\{i=j\}}-\frac{N_i(M)}{N(M)}\widehat{Q}_{ij}(t,M)\right.\\
&\left.+\nu_i Q_{ij}(t)\right]dt.
\end{aligned}
$$

We obtain that $\sqrt{M}\left[\widehat{\overline{m}}(M)-\overline{m}\right]$ has the same limit as

$$
\begin{aligned}
&\sqrt{M}\sum_{i\in E}\sum_{j\in E}\int_0^{+\infty}\left[\frac{N_i(M)}{N(M)}\mathbb{1}_{\{i=j\}}-\nu_i\mathbb{1}_{\{i=j\}}-\frac{N_i(M)}{N(M)}\left[\frac{1}{N_i(M)}\sum_{l=1}^{N(M)}G\left(\frac{t-X_l}{h_{ij,M}}\right)\mathbb{1}_{\{J_{l-1}=i,J_l=j\}}\right]\right.\\
&\left.+\nu_i Q_{ij}(t)\right]dt\\
&= \frac{1}{\sqrt{M}}\sum_{l=1}^{N(M)}\sum_{i\in E}\sum_{j\in E}\frac{M}{N_i(M)}\left[\frac{N_i(M)}{N(M)}\int_0^{+\infty}\mathbb{1}_{\{J_{l-1}=i,i=j\}}dt-\nu_i\int_0^{+\infty}\mathbb{1}_{\{J_{l-1}=i,i=j\}}dt\right.\\
&\left.-\frac{N_i(M)}{N(M)}\mathbb{1}_{\{J_{l-1}=i,J_l=j\}}\int_0^{+\infty}G\left(\frac{t-X_l}{h_{ij,M}}\right)dt+\nu_i\mathbb{1}_{\{J_{l-1}=i\}}\int_0^{+\infty}Q_{ij}(t)dt\right]\\
&= \frac{1}{\sqrt{M}}\sum_{l=1}^{N(M)}f(J_{l-1},J_l,X_l).
\end{aligned}
$$

Apply central limit theorem related to semi-Markov processes (see Pyke and Schaufele [1964]) to the function $W_f(t)$ such that

$$
\begin{aligned}
f(d,r,x) &= \sum_{i\in E}\sum_{j\in E}\frac{M}{N_i(M)}\left[\frac{N_i(M)}{N(M)}\mathbb{1}_{\{d=i=j\}}-\nu_i\mathbb{1}_{\{d=i=j\}}-\frac{N_i(M)}{N(M)}\mathbb{1}_{\{d=i,r=j\}}\right.\\
&\left.\times G\left(\frac{t-x}{h_{ij,M}}\right)+\nu_i\mathbb{1}_{\{d=i\}}Q_{ij}(t)\right]dt\\
&= \sum_{i\in E}\frac{M}{N_i(M)}\int_0^{+\infty}\left[\mathbb{1}_{\{d=i\}}W_i-\frac{N_i(M)}{N(M)}\mathbb{1}_{\{d=i\}}G\left(\frac{t-x}{h_{ir,M}}\right)+\nu_i\mathbb{1}_{\{d=i\}}H_i(t)\right]dt\\
&= \frac{M}{N_d(M)}\int_0^{+\infty}\left[W_d-\frac{N_d(M)}{N(M)}G\left(\frac{t-x}{h_{dr,M}}\right)+\nu_d H_d(t)\right]dt,
\end{aligned}
$$

with $W_i=\frac{N_i(M)}{N(M)}-\nu_i$.
In order to apply the Pyke and Schaufeles' CLT (cf. Pyke and Schaufele [1964]), we need to compute the quantities $A_{dr}, A_d, B_{dr}, B_d, r_d, m_f, \sigma_i^2(t)$ and then $\sigma^2_{\overline{m}}$. Using Fubini's theorem and Lemma 4.3.1 with assumptions **(H.2)-(H.5)**, we have

$$
\begin{aligned}
A_d &= \sum_{r\in E} A_{dr}\\
&= \sum_{r\in E}\int_0^{+\infty} f\left(d,r,x\right) dQ_{dr}(x)\\
&= \sum_{r\in E}\int_0^{+\infty}\Bigg[\frac{M}{N_d(M)}\Bigg[\int_0^{+\infty} W_d dt - \frac{N_d(M)}{N(M)}\int_0^{+\infty} G\left(\frac{t-x}{h_{dr,M}}\right)dt\\
&\quad +\nu_d\int_0^{+\infty} H_d(t)dt\Bigg]\Bigg]\,dQ_{dr}(x)\\
&= \sum_{r\in E}\frac{M}{N_d(M)}\Bigg[\int_0^{+\infty} W_d\left(\int_0^{+\infty} dQ_{dr}(x)\right)dt - \frac{N_d(M)}{N(M)}\int_0^{+\infty}\left(\int_0^{+\infty} G\left(\frac{t-x}{h_{dr,M}}\right)dQ_{dr}(x)\right)dt\\
&\quad +\nu_d\int_0^{+\infty} H_d(t)\left(\int_0^{+\infty} dQ_{dr}(x)\right)dt\Bigg]\\
&\leqslant \sum_{r\in E}\frac{M}{N_d(M)}\int_0^{+\infty}\left[p_{dr}W_d - \frac{N_d(M)}{N(M)}Q_{dr}(t)+p_{dr}\nu_d H_d(t)\right]dt\\
&\leqslant \frac{M}{N_d(M)}\int_0^{+\infty}\left[W_d - W_d H_d(t)+\nu_d H_d(t)\right]dt.
\end{aligned}
$$

For $B_d$, by using Fubini's theorem, Jensen's inequality and Lemma 4.3.1, we have

$$
\begin{aligned}
B_d &= \sum_{r\in E} B_{dr}\\
&= \sum_{r\in E}\int_0^{+\infty}[f(d,r,x)]^2 dQ_{dr}(x)\\
&= \sum_{r\in E}\int_0^{+\infty}\left(\frac{M}{N_d(M)}\right)^2\left[\int_0^{+\infty}\left[W_d-\left(\frac{N_d(M)}{N(M)}G\left(\frac{t-x}{h_{dr,M}}\right)-\nu_d H_d(t)\right)\right]dt\right]^2 dQ_{dr}(x)
\end{aligned}
$$

Then

$$
\begin{aligned}
B_d &\leqslant \sum_{r\in E}\left(\frac{M}{N_d(M)}\right)^2\int_0^{+\infty}\Bigg[W_d^2\int_0^{+\infty} dQ_{dr}(x)+\left(\frac{N_d(M)}{N(M)}\right)^2\left(\int_0^{+\infty} G^2\left(\frac{t-x}{h_{dr,M}}\right)dQ_{dr}(x)\right)\\
&\quad +\nu_d^2 H_d^2(t)\left(\int_0^{+\infty} dQ_{dr}(x)\right) - 2\nu_d\frac{N_d(M)}{N(M)}H_d(t)\left(\int_0^{+\infty} G\left(\frac{t-x}{h_{dr,M}}\right)dQ_{dr}(x)\right)\\
&\quad -2W_d\left(\frac{N_d(M)}{N(M)}\right)\left(\int_0^{+\infty} G\left(\frac{t-x}{h_{dr,M}}\right)dQ_{dr}(x)\right)+2W_d\nu_d H_d(t)\left(\int_0^{+\infty} dQ_{dr}(x)\right)\Bigg]dt\\
&\leqslant \left(\frac{M}{N_d(M)}\right)^2\int_0^{+\infty}\Bigg[W_d^2+\left(\frac{N_d(M)}{N(M)}\right)^2 H_d(t)+\nu_d^2 H_d^2(t)-2\nu_d\frac{N_d(M)}{N(M)}H_d^2(t)\\
&\quad -2W_d\left(\frac{N_d(M)}{N(M)}\right)H_d(t)+2W_d\nu_d H_d(t)\Bigg]dt.
\end{aligned}
$$

Since $\frac{N_d(M)}{M}\overset{a.s.}{\rightarrow}\frac{1}{\mu_{dd}}$ and $W_d\overset{a.s.}{\rightarrow}0$ (see Meyn and Tweedie [1993]), when $M\rightarrow+\infty$, we get

$$
B_d \leqslant \mu_{dd}^2\nu_d^2\int_0^{+\infty} H_d(t)\left(1-H_d(t)\right)dt.
$$

Furthermore,

$$\begin{aligned}
r_i &\leqslant \sum_{d\in E} \frac{\mu^*_{ii}}{\mu^*_{dd}} \frac{M}{N_d(M)} \int_0^{+\infty} [W_d - W_d H_d(t)]\, dt \\
&= 0 \qquad \text{as} \qquad M \to \infty, \\
m_f &= \frac{1}{\mu_{ii}} r_i = 0 \qquad \text{as} \qquad M \to \infty, \\
\sigma^2_{\overline{m}} &= \frac{1}{\mu_{ii}} \sigma^2_i,
\end{aligned}$$

where

$$\begin{aligned}
\sigma^2_i &= \sum_{d\in E} B_d \frac{\mu^*_{ii}}{\mu^*_{dd}} \\
&\leqslant \mu^*_{ii} \sum_{d\in E} \frac{\mu_{dd}}{\mu^*_{dd}} \mu_{dd} \nu^2_d \int_0^{+\infty} H_d(t)\,(1 - H_d(t))\, dt.
\end{aligned}$$

Then, since $\mu^*_{ii} = \frac{1}{\nu_i}$ (see Kemeny and Snell [1976]) and $\mu_{ii} = \frac{\overline{m}}{\nu_i}$ (see Limnios and Oprisan [2001]), we have

$$\begin{aligned}
\sigma^2_{\overline{m}} &\leqslant \sum_{d\in E} \mu_{dd} \nu^2_d \int_0^{+\infty} H_d(t)\,(1 - H_d(t))\, dt \\
&\leqslant \overline{m} \sum_{d\in E} \nu_d \int_0^{+\infty} H_d(t)\,(1 - H_d(t))\, dt.
\end{aligned}$$

We obtain from the CLT that $\sqrt{M}\left[\widehat{\overline{m}}(M) - \overline{m}\right]$ converges in distribution, as $M$ tends to infinity, to a normal random variable with zero mean and the variance $\sigma^2_{\overline{m}}$ given by inequality (4.20). ☐

Our third result in Section 4.3 concerns the asymptotic properties of the estimator proposed in Equation (4.17).

**Theorem 4.3.3.** *Hamlat et al. [2025b] For any fixed $i \in E$, under **(H.1)-(H.5)**, we have*

1. *The estimator $\widehat{\mu}_{ii}(M)$ of $\mu_{ii}$ is strongly consistent, that is*

$$\max_{i\in E} \mid \widehat{\mu}_{ii}(M) - \mu_{ii} \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

2. *The random variable $\sqrt{M}\,(\widehat{\mu}_{ii}(M) - \mu_{ii})$ converges, as $M \longrightarrow \infty$, to a centered normal distribution with asymptotic variance*

$$\sigma^2_{\mu_{ii}} \leqslant \frac{1}{\nu^2_i} \sum_{d\in E} \mu_{dd} \nu^2_d \int_0^{+\infty} H_d(t)\,(1 - H_d(t))\, dt. \tag{4.21}$$

**Proof of Theorem 4.3.3.**

1. Strong consistency, we have

$$\begin{aligned}
\max_{i\in E} \mid \widehat{\mu}_{ii}(M)-\mu_{ii} \mid &= \max_{i\in E} \mid \frac{\widehat{\overline{m}}(M)}{\widehat{\nu}_i(M)} - \frac{\overline{m}}{\nu_i} \mid \\
&= \max_{i\in E} \mid \frac{\widehat{\overline{m}}(M)}{\widehat{\nu}_i(M)} - \frac{\overline{m}}{\nu_i} - \frac{\overline{m}}{\widehat{\nu}_i(M)} + \frac{\overline{m}}{\widehat{\nu}_i(M)} \mid \\
&= \max_{i\in E} \mid \frac{(\widehat{\overline{m}}(M)-\overline{m})}{\widehat{\nu}_i(M)} - \frac{\overline{m}\,(\widehat{\nu}_i(M)-\nu_i)}{\nu_i\widehat{\nu}_i(M)} \mid \\
&\leqslant \frac{1}{\widehat{\nu}_i(M)} \max_{i\in E} \mid \widehat{\overline{m}}(M)-\overline{m} \mid + \frac{\overline{m}}{\nu_i\widehat{\nu}_i(M)} \max_{i\in E} \mid \widehat{\nu}_i(M)-\nu_i \mid \\
&\leqslant \frac{1}{\widehat{\nu}_i(M)} \left[ \max_{i\in E} \mid \widehat{\overline{m}}(M)-\overline{m} \mid + \mu_{ii} \max_{i\in E} \mid \widehat{\nu}_i(M)-\nu_i \mid \right].
\end{aligned}$$

For every $i \in E$ and $M \in \mathbb{R}_+$, the strong consistency of the estimators $\widehat{\nu}_i(M)$ and $\widehat{\overline{m}}(M)$ are proven in Lemma 4.2.1 (1) of Limnios et al. [2005] and Theorem 4.3.2 (1), respectively. Consequently, we obtain

$$\max_{i\in E} \mid \widehat{\mu}_{ii}(M)-\mu_{ii} \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

2. Asymptotic normality, we have

$$\begin{aligned}
\sqrt{M}\,(\widehat{\mu}_{ii}(M)-\mu_{ii}) &= \sqrt{M}\left(\frac{\widehat{\overline{m}}(M)}{\widehat{\nu}_i(M)} - \frac{\overline{m}}{\nu_i}\right) \\
&= \sqrt{M}\left(\frac{1}{\widehat{\nu}_i(M)}\left(\widehat{\overline{m}}(M)-\overline{m}\right) - \frac{\overline{m}}{\nu_i}\frac{1}{\widehat{\nu}_i(M)}\,(\widehat{\nu}_i(M)-\nu_i)\right) \\
&= \sqrt{M}\frac{1}{\widehat{\nu}_i(M)}\left[\left(\widehat{\overline{m}}(M)-\overline{m}\right) - \mu_{ii}\,(\widehat{\nu}_i(M)-\nu_i)\right] \\
&= \sqrt{M}\frac{N(M)}{N_i(M)}\left[\sum_{k\in E}\sum_{j\in E}\int_0^\infty \left[\frac{N_k(M)}{N(M)}\mathbb{1}_{\{k=j\}} - \nu_k\mathbb{1}_{\{k=j\}}\right.\right. \\
&\left.\left. - \frac{N_k(M)}{N(M)}\widehat{Q}_{kj}(t,M) + \nu_k Q_{kj}(t)\right] dt - \mu_{ii}\left(\frac{N_i(M)}{N(M)} - \nu_i\right)\right].
\end{aligned}$$

We obtain that $\sqrt{M}\,[\widehat{\mu}_{ii}(M)-\mu_{ii}]$ has the same limit in distribution as

$$\begin{aligned}
&\sqrt{M}\frac{N(M)}{N_i(M)}\left[\sum_{k\in E}\sum_{j\in E}\int_0^\infty \left[W_k\mathbb{1}_{\{k=j\}} - \frac{N_k(M)}{N(M)}\left(\frac{1}{N_k(M)}\sum_{l=1}^{N(M)} G\left(\frac{t-X_l}{h_{kj,M}}\right)\mathbb{1}_{\{J_{l-1}=k,J_l=j\}}\right)\right.\right. \\
&\left.\left. + \nu_k Q_{kj}(t)\right] dt - \mu_{ii} W_i\right] \\
&= \frac{1}{\sqrt{M}}\sum_{l=1}^{N(M)} \frac{N(M)}{N_i(M)}\sum_{k\in E}\sum_{j\in E}\frac{M}{N_k(M)}\left[\int_0^{+\infty}\left(W_k\mathbb{1}_{\{J_{l-1}=k,k=j\}}\right.\right. \\
&\left.\left. - \frac{N_k(M)}{N(M)}G\left(\frac{t-X_l}{h_{kj,M}}\right)\mathbb{1}_{\{J_{l-1}=k,J_l=j\}} + \nu_k Q_{kj}(t)\mathbb{1}_{\{J_{l-1}=k\}}\right) dt - \mu_{ii}W_i\mathbb{1}_{\{J_{l-1}=k,k=j=i\}}\right] \\
&= \frac{1}{\sqrt{M}}\sum_{l=1}^{N(M)} f(J_{l-1},J_l,X_l).
\end{aligned}$$

Apply central limit theorem related to semi-Markov processes (see Pyke and Schaufele [1964]) to the function $W_f(t)$ such that

$$W_f(t) = \sum_{l=1}^{N(M)} f(J_{l-1}, J_l, X_l),$$

where, for any fixed $t > 0$, we define the function $f : E \times E \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned}
f(d,r,x) = {} & \frac{N(M)}{N_i(M)} \sum_{k\in E}\sum_{j\in E} \frac{M}{N_k(M)} \left[ \int_0^{+\infty} \left( W_k \mathbb{1}_{\{d=k,k=j\}} - \frac{N_k(M)}{N(M)} G\left(\frac{t-x}{h_{kj,M}}\right) \mathbb{1}_{\{d=k,r=j\}} \right.\right. \\
& \left.\left. + \nu_k Q_{kj}(t) \mathbb{1}_{\{d=k\}} \right) dt - \mu_{ii} W_i \mathbb{1}_{\{d=k,k=j=i\}} \right] \\
= {} & \frac{N(M)}{N_i(M)} \sum_{k\in E} \frac{M}{N_k(M)} \left[ \int_0^{+\infty} \left( W_k \mathbb{1}_{\{d=k\}} - \frac{N_k(M)}{N(M)} G\left(\frac{t-x}{h_{kr,M}}\right) \mathbb{1}_{\{d=k\}} \right.\right. \\
& \left.\left. + \nu_k H_k(t) \mathbb{1}_{\{d=k\}} \right) dt - \mu_{ii} W_i \mathbb{1}_{\{d=k=i\}} \right] \\
= {} & \frac{N(M)}{N_i(M)} \frac{M}{N_d(M)} \left[ \int_0^{+\infty} \left( W_d - \frac{N_d(M)}{N(M)} G\left(\frac{t-x}{h_{dr,M}}\right) + \nu_d H_d(t) \right) dt - \mu_{dd} W_d \right].
\end{aligned}$$

In order to apply the Pyke and Schaufeles' CLT (cf. Pyke and Schaufele [1964]), we need to compute the quantities $A_{dr}, A_d, B_{dr}, B_d, r_d, m_f, \sigma_i^2(t)$ and then $\sigma^2_{\mu_{ii}}$. Using Fubini's theorem and Lemma 4.3.1 with assumptions **(H.2)-(H.5)**, we have

$$\begin{aligned}
A_d = {} & \sum_{r\in E} A_{dr} \\
= {} & \sum_{r\in E} \int_0^{+\infty} f(d,r,x)\, dQ_{dr}(x) \\
= {} & \sum_{r\in E} \int_0^{+\infty} \frac{N(M)}{N_i(M)} \frac{M}{N_d(M)} \left[ \int_0^{+\infty} \left( W_d - \frac{N_d(M)}{N(M)} G\left(\frac{t-x}{h_{dr,M}}\right) + \nu_d H_d(t) \right) dt \right. \\
& \left. - \mu_{dd} W_d \right] dQ_{dr}(x) \\
= {} & \sum_{r\in E} \frac{N(M)}{N_i(M)} \frac{M}{N_d(M)} \int_0^{+\infty} \left[ \int_0^{+\infty} \left( W_d - \frac{N_d(M)}{N(M)} G\left(\frac{t-x}{h_{dr,M}}\right) + \nu_d H_d(t) \right) dt \right. \\
& \left. - \mu_{dd} W_d \right] dQ_{dr}(x).
\end{aligned}$$

Thus

$$\begin{aligned}
A_d = {} & \sum_{r\in E} \frac{N(M)}{N_i(M)} \frac{M}{N_d(M)} \left[ \int_0^{+\infty} \left( W_d \int_0^{+\infty} dQ_{dr}(x) - \frac{N_d(M)}{N(M)} \int_0^{+\infty} G\left(\frac{t-x}{h_{dr,M}}\right) dQ_{dr}(x) \right.\right. \\
& \left.\left. + \nu_d H_d(t) \int_0^{+\infty} dQ_{dr}(x) \right) dt - \mu_{dd} W_d \int_0^{+\infty} dQ_{dr}(x) \right] \\
\leqslant {} & \sum_{r\in E} \frac{N(M)}{N_i(M)} \frac{M}{N_d(M)} \left[ \int_0^{+\infty} \left( W_d p_{dr} - \frac{N_d(M)}{N(M)} Q_{dr}(t) + \nu_d H_d(t) p_{dr} \right) dt \right. \\
& \left. - \mu_{dd} W_d p_{dr} \right] \\
\leqslant {} & \frac{N(M)}{N_i(M)} \frac{M}{N_d(M)} \left[ \int_0^{+\infty} \left( W_d - W_d H_d(t) \right) dt - \mu_{dd} W_d \right].
\end{aligned}$$

For $B_d$ and by using Fubini's theorem, Jensen's inequality and Lemma 4.3.1, we have

$$
\begin{aligned}
B_d &= \sum_{r\in E} B_{dr}\\
&= \sum_{r\in E}\int_0^{+\infty}[f(d,r,x)]^2 dQ_{dr}(x)\\
&= \sum_{r\in E}\int_0^{+\infty}\Bigg[\frac{N(M)}{N_i(M)}\frac{M}{N_d(M)}\Bigg[\int_0^{+\infty}\left(W_d-\frac{N_d(M)}{N(M)}G\left(\frac{t-x}{h_{dr,M}}\right)+\nu_d H_d(t)\right)dt\\
&\quad -\mu_{dd}W_d\Bigg]\Bigg]^2 dQ_{dr}(x)\\
&= \sum_{r\in E}\left(\frac{N(M)}{N_i(M)}\right)^2\left(\frac{M}{N_d(M)}\right)^2\int_0^{+\infty}\Bigg[\int_0^{+\infty}\left(W_d-\frac{N_d(M)}{N(M)}G\left(\frac{t-x}{h_{dr,M}}\right)+\nu_d H_d(t)\right)dt\\
&\quad -\mu_{dd}W_d\Bigg]^2 dQ_{dr}(x)\\
&\leqslant \sum_{r\in E}\left(\frac{N(M)}{N_i(M)}\right)^2\left(\frac{M}{N_d(M)}\right)^2\int_0^{+\infty}\Bigg[\int_0^{+\infty}\Bigg(W_d^2+\left(\frac{N_d(M)}{N(M)}\right)^2 G^2\left(\frac{t-x}{h_{dr,M}}\right)-2W_d\\
&\quad \times\frac{N_d(M)}{N(M)}G\left(\frac{t-x}{h_{dr,M}}\right)+\nu_d^2H_d^2(t)+2\nu_d H_d(t)W_d-2\nu_d H_d(t)\frac{N_d(M)}{N(M)}G\left(\frac{t-x}{h_{dr,M}}\right)\Bigg)dt\\
&\quad +\mu_{dd}^2W_d^2-2\mu_{dd}W_d\int_0^{+\infty}\left(\nu_d H_d(t)+W_d-\frac{N_d(M)}{N(M)}G\left(\frac{t-x}{h_{dr,M}}\right)\right)dt\Bigg] dQ_{dr}(x)\\
&\leqslant \sum_{r\in E}\left(\frac{N(M)}{N_i(M)}\right)^2\left(\frac{M}{N_d(M)}\right)^2\Bigg[\int_0^{+\infty}\Bigg(W_d^2\int_0^{+\infty}dQ_{dr}(x)+\left(\frac{N_d(M)}{N(M)}\right)^2\\
&\quad \times\int_0^{+\infty}G^2\left(\frac{t-x}{h_{dr,M}}\right)dQ_{dr}(x)-2W_d\frac{N_d(M)}{N(M)}\int_0^{+\infty}G\left(\frac{t-x}{h_{dr,M}}\right)dQ_{dr}(x)\\
&\quad +\nu_d^2H_d^2(t)\int_0^{+\infty}dQ_{dr}(x)+2\nu_d H_d(t)W_d\int_0^{+\infty}dQ_{dr}(x)-2\nu_d H_d(t)\frac{N_d(M)}{N(M)}\\
&\quad \times\int_0^{+\infty}G\left(\frac{t-x}{h_{dr,M}}\right)dQ_{dr}(x)\Bigg)dt+\mu_{dd}^2W_d^2\int_0^{+\infty}dQ_{dr}(x)-2\mu_{dd}W_d\\
&\quad \times\int_0^{+\infty}\Bigg(\nu_d H_d(t)\int_0^{+\infty}dQ_{dr}(x)+W_d\int_0^{+\infty}dQ_{dr}(x)-\frac{N_d(M)}{N(M)}\\
&\quad \times\int_0^{+\infty}G\left(\frac{t-x}{h_{dr,M}}\right)dQ_{dr}(x)\Bigg)dt\Bigg].
\end{aligned}
$$

Then

$$
\begin{aligned}
B_d &\leqslant \left(\frac{N(M)}{N_i(M)}\right)^2\left(\frac{M}{N_d(M)}\right)^2\Bigg[\int_0^{+\infty}\Bigg(W_d^2+\left(\frac{N_d(M)}{N(M)}\right)^2H_d(t)-2W_d\frac{N_d(M)}{N(M)}H_d(t)\\
&\quad +\nu_d^2H_d^2(t)+2\nu_d H_d(t)W_d-2\nu_d H_d(t)\frac{N_d(M)}{N(M)}H_d(t)\Bigg)dt+\mu_{dd}^2W_d^2-2\mu_{dd}W_d\int_0^{+\infty}(\nu_d H_d(t)\\
&\quad +W_d-\frac{N_d(M)}{N(M)}H_d(t)\Bigg)dt\Bigg].
\end{aligned}
$$

Since $\frac{N_d(M)}{M}\overset{a.s.}{\to}\frac{1}{\mu_{dd}}$, $\frac{N_i(M)}{N(M)}\overset{a.s.}{\to}\nu_i$, and $W_d\overset{a.s.}{\to}0$, when $M\to+\infty$, we have

$$
B_d \leqslant \frac{1}{\nu_i^2}\mu_{dd}^2\nu_d^2\int_0^{+\infty}H_d(t)\left(1-H_d(t)\right)dt.
$$

Furthermore,

$$\begin{aligned} r_i &= \sum_{d\in E} \frac{\mu^*_{ii}}{\mu^*_{dd}} \frac{N(M)}{N_i(M)} \frac{M}{N_d(M)} \left[ \int_0^{+\infty} (W_d - W_d H_d(t))\, dt - \mu_{dd} W_d \right] \\ &= 0 \qquad \text{as} \qquad M \to \infty, \\ m_f &= \frac{1}{\mu_{ii}} r_i = 0 \qquad \text{as} \qquad M \to \infty, \\ \sigma^2_{\mu_{ii}}(t) &= \frac{1}{\mu_{ii}} \sigma^2_i(t), \end{aligned}$$

where

$$\begin{aligned} \sigma^2_i(t) &= \sum_{d\in E} B_d \frac{\mu^*_{ii}}{\mu^*_{dd}} \\ &\leqslant \mu^*_{ii} \sum_{d\in E} \frac{\mu_{dd}}{\mu^*_{dd}} \mu_{dd} \frac{1}{\nu_i^2} \nu_d^2 \int_0^{+\infty} H_d(t)\,(1 - H_d(t))\, dt. \end{aligned}$$

Then, since $\mu^*_{dd} = \frac{1}{v_d}$ (see Kemeny and Snell [1976]) and $\mu_{dd} = \frac{\bar{m}}{v_d}$ (see Limnios and Oprisan [2001]), we have

$$\sigma^2_{\mu_{ii}} \leqslant \frac{1}{\nu_i^2} \sum_{d\in E} \mu_{dd} \nu_d^2 \int_0^{+\infty} H_d(t)\,(1 - H_d(t))\, dt.$$

Hence, $\sqrt{M}\,[\widehat{\mu}_{ii}(M) - \mu_{ii}]$ converges in distribution, as $M$ tends to infinity, to a normal random variable with zero mean and the variance $\sigma^2_{\mu_{ii}}$ given by inequality (4.21). ☐

Our last result in Section 4.3 concerns the asymptotic properties of the estimator of the stationary distribution of a semi-Markov process.

**Theorem 4.3.4.** *Hamlat et al. [2025b] For any fixed $i \in E$, under **(H.1)-(H.5)**, we have*

1. *The estimator $\widehat{\pi}_i(M)$ of $\pi_i$ is strongly consistent, that is*

$$\max_{i\in E} \mid \widehat{\pi}_i(M) - \pi_i \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

2. *The random variable $\sqrt{M}\,(\widehat{\pi}_i(M) - \pi_i)$ converges, as $M \longrightarrow \infty$, to a centered normal distribution with asymptotic variance*

$$\sigma^2_{\pi_i} \leqslant \left(\frac{\nu_i}{\bar{m}}\right) \int_0^{+\infty} H_i(t)\,(1 - H_i(t))\, dt. \tag{4.22}$$

**Proof of Theorem 4.3.4.**

1. Strong consistency, we have

$$\begin{aligned} \max_{i\in E} \mid \widehat{\pi}_i(M) - \pi_i \mid &= \max_{i\in E} \mid \frac{\widehat{m}_i(M)}{\widehat{\mu}_{ii}(M)}_i (M) - \frac{m_i}{\mu_{ii}} \mid \\ &= \max_{i\in E} \mid \frac{\widehat{m}_i(M)}{\widehat{\mu}_{ii}(M)}_i (M) - \frac{m_i}{\mu_{ii}} - \frac{m_i}{\widehat{\mu}_{ii}(M)} - \frac{m_i}{\widehat{\mu}_{ii}(M)} \mid \\ &\leqslant \frac{1}{\widehat{\mu}_{ii}(M)} \max_{i\in E} \mid \widehat{m}_i(M) - m_i \mid + \frac{m_i}{\mu_{ii}\widehat{\mu}_{ii}(M)} \max_{i\in E} \mid \widehat{\mu}_{ii}(M) - \mu_{ii} \mid . \end{aligned}$$

For all $i \in E$ and $M \in \mathbb{R}_+$, the strong consistency of the kernel estimators $\widehat{m}_i(M)$ and $\widehat{\mu}_{ii}(M)$ are established in Theorem 4.3.1 (1) and Theorem 4.3.3 (1), respectively. Then, we obtain

$$\max_{i\in E} \mid \widehat{\pi}_i(M) - \pi_i \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

2. Asymptotic normality, from Theorem 4.3.2 (1), we have the converge in probability of $\widehat{\overline{m}}(M)$ to $\overline{m}$. So, we have

$$\begin{aligned}
\sqrt{M}\left(\widehat{m}_i(M)\nu_i(M) - m_i\nu_i\right) &= \sqrt{M}\left(\left(\widehat{\nu}_i(M) - \nu_i\right)\left(\widehat{m}_i(M) - m_i\right) + \nu_i\widehat{m}_i(M)\right. \\
&\left. +\widehat{\nu}_i(M)m_i - \nu_i m_i - \nu_i m_i\right) \\
&= \sqrt{M}\left(\left(\widehat{\nu}_i(M) - \nu_i\right)\left(\widehat{m}_i(M) - m_i\right) + \nu_i\left(\widehat{m}_i(M) - m_i\right)\right. \\
&\left. +m_i\left(\widehat{\nu}_i(M) - \nu_i\right)\right).
\end{aligned}$$

For every $i \in E$, $\sqrt{M}[\widehat{m}_i(M) - m_i]$ converges in distribution to a normal random variable is stated in Theorem 4.3.1, as $M \to \infty$, and from Lemma 4.2.1 (1), we have $\mid \widehat{\nu}_i(M) - \nu_i \mid \xrightarrow{P} 0$ as $M \to \infty$.
So, using Slutsky's Theorem we obtain that $\sqrt{M}[\left(\widehat{m}_i(M) - m_i\right)\left(\widehat{\nu}_i(M) - \nu_i\right)] \xrightarrow{D} 0$ as $M \to \infty$.
Thus, $\sqrt{M}\,[\widehat{\pi}_i(M) - \pi_i]$ has the same limit in distribution as

$$\begin{aligned}
&\sqrt{M}\left[\frac{\nu_i}{\overline{m}}\left(\widehat{m}_i(M) - m_i\right) + \frac{m_i}{\overline{m}}\left(\widehat{\nu}_i(M) - \nu_i\right)\right] \\
&= \sqrt{M}\left[\sum_{l=1}^{N(M)}\sum_{j\in E}\frac{1}{N_i(M)}\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(Q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}} - G\left(\frac{t - X_l}{h_{ij,M}}\right)\mathbb{1}_{\{J_{l-1}=i,J_l=j\}}\right)dt\right. \\
&\left. +\frac{m_i}{\overline{m}}\left(\widehat{\nu}_i(M) - \nu_i\right)\right] \\
&= \frac{1}{\sqrt{M}}\sum_{l=1}^{N(M)}\sum_{j\in E}\frac{M}{N_i(M)}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(Q_{ij}(t)\mathbb{1}_{\{J_{l-1}=i\}} - G\left(\frac{t - X_l}{h_{ij,M}}\right)\mathbb{1}_{\{J_{l-1}=i,J_l=j\}}\right)dt\right. \\
&\left. +\frac{m_i}{\overline{m}}W_i\mathbb{1}_{\{J_{l-1}=i,i=j\}}\right] \\
&= \frac{1}{\sqrt{M}}\sum_{l=1}^{N(M)} f(J_{l-1}, J_l, X_l).
\end{aligned}$$

Apply central limit theorem related to semi-Markov processes (see Pyke and Schaufele [1964]) to the function $W_f(t)$ such that

$$W_f(t) = \sum_{l=1}^{N(M)} f(J_{l-1}, J_l, X_l),$$

where, for any fixed $t > 0$, we define the function $f : E \times E \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned}
f(d, r, x) &= \sum_{j\in E}\frac{M}{N_i(M)}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(Q_{ij}(t)\mathbb{1}_{\{d=i\}} - G\left(\frac{t - x}{h_{ij,M}}\right)\mathbb{1}_{\{d=i,r=j\}}\right)dt + \frac{m_i}{\overline{m}}W_i\mathbb{1}_{\{d=i=j\}}\right] \\
&= \frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(H_i(t) - G\left(\frac{t - x}{h_{ir,M}}\right)\right)dt + \frac{m_i}{\overline{m}}W_i\right].
\end{aligned}$$

In order to apply the Pyke and Schaufeles' CLT (cf. Pyke and Schaufele [1964]), we need to compute the quantities $A_{dr}, A_d, B_{dr}, B_d, r_d, m_f, \sigma_i^2(t)$ and then $\sigma^2_{\pi_i}$. Using Fubini's theorem and Lemma 4.3.1 with assumptions **(H.2)-(H.5)**, we have

$$
\begin{aligned}
A_d &= \sum_{r\in E}\int_0^{+\infty} f\left(d,r,x\right)dQ_{dr}(x)\\
&= \sum_{r\in E}\int_0^{+\infty} \frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(H_i(t)-G\left(\frac{t-x}{h_{ir,M}}\right)\right)dt+\frac{m_i}{\overline{m}}W_i\right]dQ_{dr}(x)\\
&= \sum_{r\in E}\int_0^{+\infty} \frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}H_i(t)dt-\frac{\nu_i}{\overline{m}}\int_0^{+\infty}G\left(\frac{t-x}{h_{ir,M}}\right)dt+\frac{m_i}{\overline{m}}W_i\right]dQ_{dr}(x)\\
&= \sum_{r\in E}\frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}H_i(t)\left(\int_0^{+\infty}dQ_{dr}(x)\right)dt-\frac{\nu_i}{\overline{m}}\right.\\
&\times\left.\int_0^{+\infty}\left(\int_0^{+\infty}G\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right)dt+\frac{m_i}{\overline{m}}W_i\int_0^{+\infty}dQ_{dr}(x)\right]\\
&\leqslant \sum_{r\in E}\frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(H_i(t)p_{dr}-Q_{dr}(t)\right)dt+\frac{m_i}{\overline{m}}W_ip_{dr}\right]\\
&\leqslant \frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(H_i(t)-H_d(t)\right)dt+\frac{m_i}{\overline{m}}W_i\right].
\end{aligned}
$$

For $B_d$ and by using Jensen's inequality, Fubini's theorem and Lemma 4.3.1, we have

$$
\begin{aligned}
B_d &= \sum_{r\in E}B_{dr}\\
&= \sum_{r\in E}\int_0^{+\infty}[f(d,r,x)]^2dQ_{dr}(x)\\
&= \sum_{r\in E}\int_0^{+\infty}\left[\frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(H_i(t)-G\left(\frac{t-x}{h_{ir,M}}\right)\right)dt+\frac{m_i}{\overline{m}}W_i\right]\right]^2dQ_{dr}(x)\\
&\leqslant \sum_{r\in E}\int_0^{+\infty}\left(\frac{M}{N_i(M)}\right)^2\mathbb{1}_{\{d=i\}}\left[\left(\frac{\nu_i}{\overline{m}}\right)^2\int_0^{+\infty}\left(H_i(t)-G\left(\frac{t-x}{h_{ir,M}}\right)\right)^2dt+\left(\frac{m_i}{\overline{m}}\right)^2W_i^2\right.\\
&\left.+2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(H_i(t)-G\left(\frac{t-x}{h_{ir,M}}\right)\right)dt\right]dQ_{dr}(x)\\
&\leqslant \sum_{r\in E}\left(\frac{M}{N_i(M)}\right)^2\mathbb{1}_{\{d=i\}}\left[\left(\frac{\nu_i}{\overline{m}}\right)^2\int_0^{+\infty}\left(H_i^2(t)\int_0^{+\infty}dQ_{dr}(x)+\int_0^{+\infty}G^2\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right.\right.\\
&\left.-2H_i(t)\int_0^{+\infty}G\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right)dt+\left(\frac{m_i}{\overline{m}}\right)^2W_i^2\int_0^{+\infty}dQ_{dr}(x)+2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}\\
&\left.\times\int_0^{+\infty}H_i(t)\left(\int_0^{+\infty}dQ_{dr}(x)\right)dt-2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(\int_0^{+\infty}G\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right)dt\right].
\end{aligned}
$$

Thus

$$\begin{aligned}
B_d \leqslant{} & \sum_{r\in E}\left(\frac{M}{N_i(M)}\right)^2 \mathbb{1}_{\{d=i\}}\left[\left(\frac{\nu_i}{\overline{m}}\right)^2\int_0^{+\infty}\left(H_i^2(t)\int_0^{+\infty} dQ_{dr}(x)+\int_0^{+\infty} G^2\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right.\right.\\
& \left.-2H_i(t)\int_0^{+\infty} G\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right)dt+\left(\frac{m_i}{\overline{m}}\right)^2 W_i^2\int_0^{+\infty} dQ_{dr}(x)+2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}\\
& \left.\times\int_0^{+\infty} H_i(t)\left(\int_0^{+\infty} dQ_{dr}(x)\right)dt-2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(\int_0^{+\infty} G\left(\frac{t-x}{h_{ir,M}}\right)dQ_{dr}(x)\right)dt\right]\\
\leqslant{} & \sum_{r\in E}\left(\frac{M}{N_i(M)}\right)^2 \mathbb{1}_{\{d=i\}}\left[\left(\frac{\nu_i}{\overline{m}}\right)^2\int_0^{+\infty}\left(H_i^2(t)p_{dr}+Q_{dr}(t)-2H_i(t)Q_{dr}(t)\right)dt\right.\\
& \left.+\left(\frac{m_i}{\overline{m}}\right)^2 W_i^2 p_{dr}+2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}p_{dr}\int_0^{+\infty} H_i(t)dt-2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}\int_0^{+\infty} Q_{dr}(t)dt\right]\\
\leqslant{} & \left(\frac{M}{N_i(M)}\right)^2 \mathbb{1}_{\{d=i\}}\left[\left(\frac{\nu_i}{\overline{m}}\right)^2\int_0^{+\infty}\left(H_i^2(t)+H_d(t)-2H_i(t)H_d(t)\right)dt\right.\\
& \left.+\left(\frac{m_i}{\overline{m}}\right)^2 W_i^2+2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}\int_0^{+\infty} H_i(t)dt-2\frac{m_i}{\overline{m}}W_i\frac{\nu_i}{\overline{m}}\int_0^{+\infty} H_d(t)dt\right].
\end{aligned}$$

Since $\frac{N_d(M)}{M}\overset{a.s.}{\to}\frac{1}{\mu_{dd}}$ and $W_d\overset{a.s.}{\to}0$, when $M\to+\infty$, we have

$$B_d \leqslant \mu_{ii}^2\mathbb{1}_{\{d=i\}}\left(\frac{\nu_i}{\overline{m}}\right)^2\int_0^{+\infty}\left[H_i^2(t)+H_d(t)-2H_i(t)H_d(t)\right]dt.$$

Furthermore,

$$\begin{aligned}
r_i &= \sum_{d\in E}\frac{\mu_{ii}^*}{\mu_{dd}^*}\frac{M}{N_i(M)}\mathbb{1}_{\{d=i\}}\left[\frac{\nu_i}{\overline{m}}\int_0^{+\infty}\left(H_i(t)-H_d(t)\right)dt+\frac{m_i}{\overline{m}}W_i\right]\\
&= 0 \qquad \text{as} \qquad M\to\infty,\\
m_f &= \frac{1}{\mu_{ii}}r_i = 0 \qquad \text{as} \qquad M\to\infty.
\end{aligned}$$

Thus

$$\begin{aligned}
\sigma_{\pi_i}^2(t) &= \frac{1}{\mu_{ii}}\sigma_i^2(t)\\
&= \frac{1}{\mu_{ii}}\sum_{d\in E}B_d\frac{\mu_{ii}^*}{\mu_{dd}^*}\\
&\leqslant \frac{\mu_{ii}^*}{\mu_{ii}}\sum_{d\in E}\frac{1}{\mu_{dd}^*}\mu_{ii}^2\mathbb{1}_{\{d=i\}}\left(\frac{\nu_i}{\overline{m}}\right)^2\int_0^{+\infty}\left[H_i^2(t)+H_d(t)-2H_i(t)H_d(t)\right]dt.
\end{aligned}$$

Then, since $\mu_{ii}^*=\frac{1}{\nu_i}$ (see Kemeny and Snell [1976]) and $\mu_{ii}=\frac{\overline{m}}{\nu_i}$ (see Limnios and Oprisan [2001]), we have

$$\begin{aligned}
\sigma_{\pi_i}^2 &\leqslant \mu_{ii}\left(\frac{\nu_i}{\overline{m}}\right)^2\int_0^{+\infty}H_i(t)\left(1-H_i(t)\right)dt\\
&\leqslant \left(\frac{\nu_i}{\overline{m}}\right)\int_0^{+\infty}H_i(t)\left(1-H_i(t)\right)dt.
\end{aligned}$$

Hence, $\sqrt{M}\left[\widehat{\pi}_i(M)-\pi_i\right]$ converges in distribution, as $M$ tends to infinity, to a normal random variable with zero mean and the variance $\sigma_{\pi_i}^2$ given by inequality (4.22). □

### 4.3.3 Confidence Interval

The main purpose of the confidence interval is to supplement the estimate at a point with information about the uncertainty in this estimate. It is considered as a direct application of the Central Limit Theorem. In order to provide a confidence interval for the stationary distribution $\pi_i$, we need first to propose a consistent estimator of the variance $\sigma_\pi^2$. A natural consistent estimator of this variance, denoted by $\widehat{\sigma}^2_{\pi_i}(M)$, is obtained by estimating the parameters involved in this quantity such as $\nu_i, m_i, \overline{m}$ and $\mu_{ii}$.
Indeed, from the strong consistency of $\widehat{\nu}_i(M), \widehat{m}_i(M), \widehat{\overline{m}}(M)$ and $\widehat{\mu}_{ii}(M)$, (see the proof of Lemma 4.2.1 (1), and Theorems 4.3.1, 4.3.2 and 4.3.3), we deduce the strong consistency of $\widehat{\sigma}^2_{\pi_i}(M)$.
Consequently, from Theorem 4.3.4, we get that

$$\sqrt{M}\left[\widehat{\pi}_i(M)-\pi_i\right] \xrightarrow{D} N\left(0, \widehat{\sigma}^2_{\pi_i}(M)\right).$$

Then

$$\frac{\sqrt{M}}{\widehat{\sigma}_{\pi_i}(M)}\left[\widehat{\pi}_i(M)-\pi_i\right] \xrightarrow{D} N\left(0,1\right).$$

Therefore, hence for $\alpha \in (0,1)$, an asymptotic $100(1-\alpha)\%$ confidence interval for $\widehat{\pi}_i(M)$ that can be straightforwardly computed is

$$\mathbf{CI} = [\mathbf{LCL}, \mathbf{UCL}] = \left[\widehat{\pi}_i(M) - z_{\frac{\alpha}{2}}\frac{\widehat{\sigma}_{\pi_i}(M)}{\sqrt{M}}, \widehat{\pi}_i(M) + z_{\frac{\alpha}{2}}\frac{\widehat{\sigma}_{\pi_i}(M)}{\sqrt{M}}\right],$$

where

- $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.
- **LCL** (Lower Confidence Limit): The lower bound of the confidence interval.
- **UCL** (Upper Confidence Limit): The upper bound of the confidence interval.

## 4.4 Application

In this section, we carry out a simulation study to evaluate the finite sample performance of the estimation procedure described in the previous sections. we consider a system whose behavior is described by a three state semi-Markov process in the following way. The state space of the system is given by $E = \{1, 2, 3\}$, and the transition probabilities of the embedded Markov chain are given by the matrix $p$,

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0.8 & 0 & 0.2 \\ 1 & 0 & 0 \end{pmatrix}.$$

Where the system is defined by the initial distribution $\alpha = (1/3, 1/3, 1/3)$.
It should be noticed that the sojourn times in the states of this process are Weibull distributed,

and the semi-Markov $\mathbf{Q}(t)$ is defined by

$$\boldsymbol{Q}(t)=\begin{pmatrix} 0 & 1-\exp\left[-\left(\frac{x}{\beta_1}\right)^{\lambda}\right] & 0 \\ 0.8\left(1-\exp\left[-\left(\frac{x}{\beta_2}\right)^{\lambda}\right]\right) & 0 & 0.2\left(1-\exp\left[-\left(\frac{x}{\beta_3}\right)^{\lambda}\right]\right) \\ 1-\exp\left[-\left(\frac{x}{\beta_4}\right)^{\lambda}\right] & 0 & 0 \end{pmatrix}.$$

The parameters of these distributions are: $\lambda=2, \beta_1=1.2^{(-1/2)}, \beta_2=0.5^{(-1/2)}, \beta_3=0.9^{(-1/2)}$ and $\beta_4=1.4^{(-1/2)}$.

Recall that the stationary distribution of the EMC $(J_n)_{n\in\mathbb{N}}$ is the row vector $\boldsymbol{\nu}=(\nu_i)_{i\in E}$ such that

$$\boldsymbol{\nu}=\boldsymbol{\nu}p,$$

Therefore, the competition leads to

$$\nu_1=0.4545, \quad \nu_2=0.4545, \text{ and } \quad \nu_3=0.0910.$$

By applying Equation (4.2), we obtain the mean sojourn time in a state $i$ of the SMP,

$$m_1=2.0948, \quad m_2=1.8863, \text{ and } \quad m_3=2.1590.$$

Utilizing Equation (4.4), we derive the mean recurrence times for state $i$ in the MRP, as follows:

$$\mu_{11}=4.4129, \quad \mu_{22}=4.4129, \text{ and } \quad \mu_{33}=22.0644.$$

Using Equation (4.5), we obtain the stationary distribution of the SMP,

$$\pi_1=0.4747, \quad \pi_2=0.4275, \text{ and } \quad \pi_3=0.0978.$$

In this numerical example we have chosen the cumulative distribution function $G(u)=\int_{-\infty}^{u}\frac{3}{4}\left(1-z^2\right)\mathbb{1}_{[-1,1]}(z)dz$. In addition, the bandwidth $h_T$ has been obtained by the "PBbw" method, which computes the plug-in bandwidth of the Polansky and Baker method, cf. Polansky and Baker [2000].

We realize one sample trajectory of this system for a number of jumps equal to 500 (with the total time $M=2000$).

Using Equations (4.15), (4.16), and (4.18), we obtain the estimators $\widehat{\nu}_i(M)$, $\widehat{m}_i(M)$, and $\widehat{\mu}_{ii}(M)$, respectively, by

$$\begin{array}{llll} \widehat{\nu}_1(M)=0.4547, & \widehat{\nu}_2(M)=0.4547, & \text{and} & \widehat{\nu}_3(M)=0.0906, \\ \widehat{m}_1(M)=2.1944, & \widehat{m}_2(M)=1.9188, & \text{and} & \widehat{m}_3(M)=2.1644, \\ \widehat{\mu}_{11}(M)=4.5442, & \widehat{\mu}_{22}(M)=4.5442, & \text{and} & \widehat{\mu}_{33}(M)=22.0824. \end{array}$$

| | State 1 | State 2 | State 3 |
|---|---|---|---|
| $\pi_i$ | 0.4747 | 0.4275 | 0.0978 |
| Kernel estimator of $\pi_i$ | 0.4827 | 0.4225 | 0.0948 |
| 80% C.I. of $\widehat{\pi}_i(M)$ | ( 0.4683, 0.4974) | (0.4101, 0.4344) | (0.0883, 0.1014) |
| 90% C.I. of $\widehat{\pi}_i(M)$ | (0.4642, 0.5016) | (0.4066, 0.4380) | (0.0864, 0.1033) |
| 95% C.I. of $\widehat{\pi}_i(M)$ | (0.4606, 0.5051) | (0.4036, 0.4410) | (0.0848, 0.1049) |
| 98% C.I. of $\widehat{\pi}_i(M)$ | (0.4565, 0.5093) | (0.4001, 0.4445) | (0.0829, 0.1067) |

Table 4.1: Confidence interval of the kernel estimator of the stationary distribution.

Table 4.1 gives a comparison between the true value of the stationary distribution for the SMP with its kernel estimator, for a number of jumps equal to 500.

Furthermore, by leveraging the asymptotic properties of these estimators, we construct confidence intervals (CI) for the true values of $\pi_i$ at confidence levels of $80\%, 90\%, 95\%$, and $98\%$. These intervals give a range of plausible values for the true stationary distribution and allow us to assess the precision of the estimators:

- The 80% CI provides a narrower range but offers less confidence that the true value is within this interval.
- As we increase the confidence level to $90\%, 95\%$, and $98\%$, the intervals become wider, reflecting greater certainty that the true value lies within the range, though at the cost of reduced precision.

Figure 4.1 shows the estimated stationary distribution across series simulations together with the true stationary distribution for states $1, 2$, and 3. We analyze simulations conducted for fixed values of $N(M)$ ranging from 1 to 500, expressed as $N(M) = 1, 2, \ldots, 500$. The black lines reflect the variability in the estimator across different simulations (Ker_stat), while the red line represents the true stationary distribution (True value) as a reference.

Furthermore, Figure 4.1 demonstrates how closely the kernel estimator approximates the true stationary distribution for each state. These results highlight the accuracy of the estimator and its capacity to capture the underlying stationary behavior across the three states.

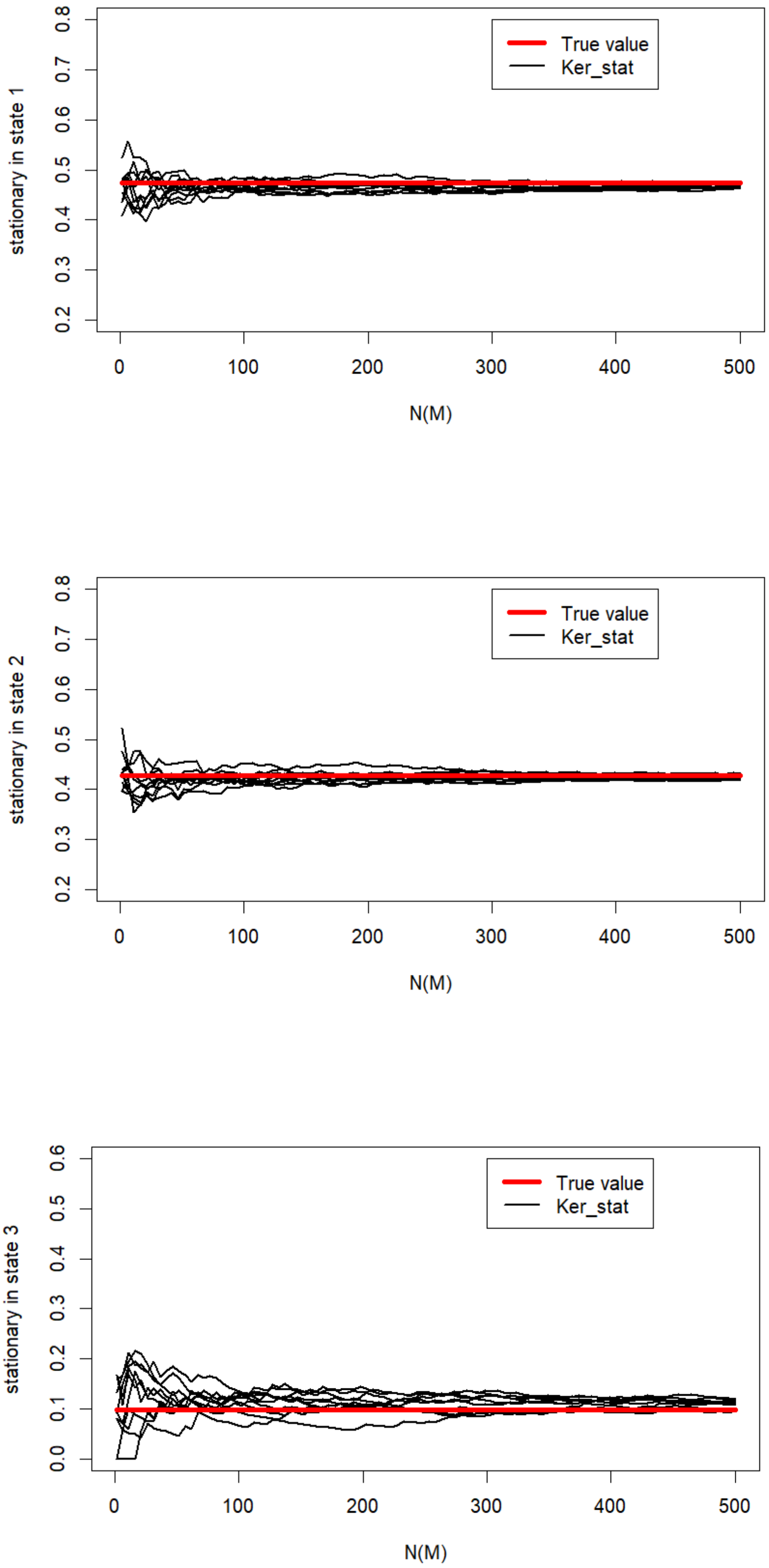


Figure 4.1: Comparison of the true value of the stationary distribution from simulated series and the kernel estimator for states 1, 2, and 3.

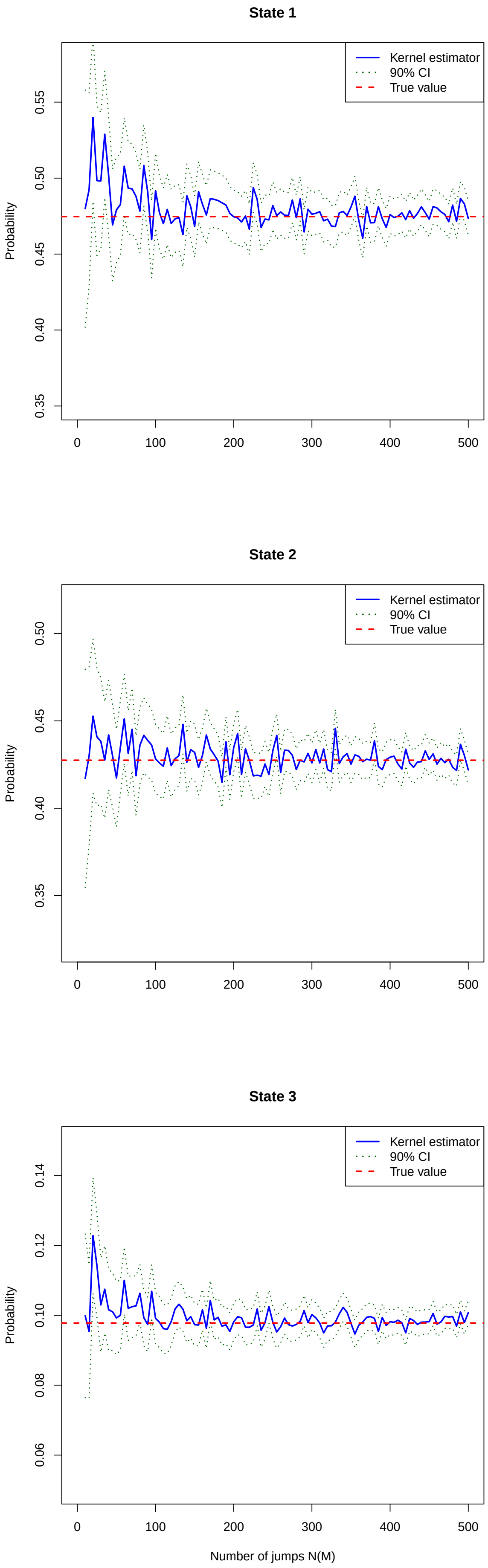
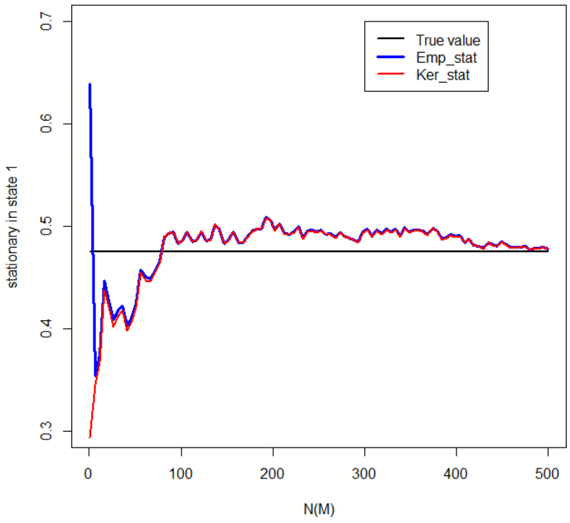
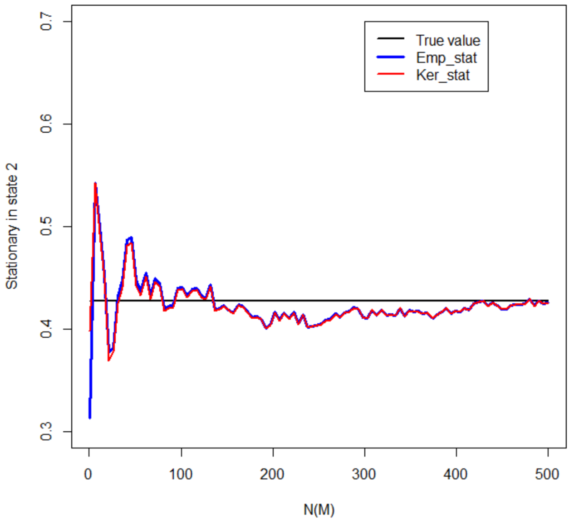


Figure 4.2: Comparison of the kernel estimator with 90% confidence interval and the true value of the stationary distribution for states 1, 2, and 3.

Figure 4.2 illustrates the comparison between the kernel estimator (blue line: Ker_stat) and the true stationary distribution (black line: True value) for states 1, 2, and 3, along with the 90% confidence interval bounds, we analyze simulations conducted for fixed values of $N(M)$. The confidence interval is indicated by the green line for the lower confidence limit (90% LCL) and the red line for the upper confidence limit (95% UCL). This interval provides a range within which the true stationary distribution is expected to lie with 90% confidence. The consistent inclusion of the true values within these bounds across all states confirms that the kernel estimator reliably approximates the stationary distribution.





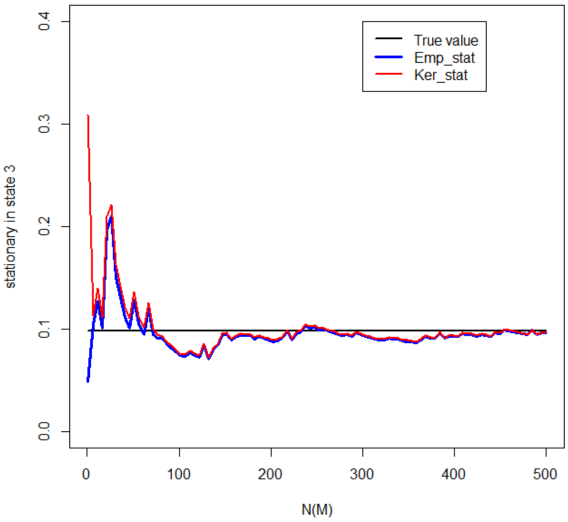


Figure 4.3: Comparison of the kernel and the empirical estimators with the true stationary distribution for states 1, 2, and 3.

Figure 4.3 shows the comparison between the kernel estimator (Ker_stat) and the empirical estimator (Emp_stat) introduced in Limnios et al. [2005] states 1, 2, and 3, alongside the true stationary distribution (True value). The figure illustrates how both estimators converge toward the true stationary distribution as the sample size increases. Overall, both the empirical and kernel estimators provide results that are close to the true value, indicating better estimation performance.

One way of illustrating the accuracy of the estimator is by providing the Mean Square Error (MSE). We have carried out $N = 200$ repetitions of the experiment, and we have taken 100 points of discretisation, we analyze simulations conducted with fixed values of $N(M)$ ranging from 1 to 500.

$$\mathrm{MSE}\left(\widehat{\pi}_i\right) = \mathbb{E}\left[\left(\widehat{\pi}_i(M) - \pi_i\right)^2\right].$$

Table 4.2: MSEs for the both methods, kernel and empirical estimation.

| | State 1 | State 2 | State 3 |
|---|---|---|---|
| Empirical estimation | 0.00262 | 0.00241 | 0.00146 |
| Kernel estimation | 0.00038 | 0.00021 | 0.00083 |

Table 4.2 presents the MSEs values for the both methods. It can be seen that the kernel method generates better result than the empirical one. Therefore, the kernel estimator of the stationary distribution converges to the true value and achieves the minimal mean square error among the considered estimators.

# Chapter 5

# Performance and Performability Modeled by Homogeneous Semi-Markov Processes

There are many applications and real cases for which the continuous time description is the most natural one. Discretizing the time in these situations is in fact only an approximation for the model under study. Many of these applications fit into the semi Markov description. In other side, the discrete time case can be obtained from the continuous one, by considering countable measure for discrete time points. This is why we present, a systematic modeling of performance measurements in the framework of continuous-time semi-Markov processes with a finite-state space. More precisely, the mean performance is estimated by using the nonparametric kernel method. As it is mentioned in Ciardo et al. [1990], a performance level, or a reward rate is associated with each state. The resulting semi-Markov reward process is then able to capture not only failure and repair of system components, but degradable performance as well. therefore, the development of performance model is needed when we are interested in the level of productivity of a system.

## 5.1 Performance and Performability of Semi-Markov Processes

The performance process at time $t > 0$, denoted $\Phi(t)$ is the real-valued integral functional of a homogeneous semi-Markov process $(Z_t)_{t\in\mathbb{R}_+}$ defined by

$$\Phi(t) = \int_0^t L\left(Z_u\right) du = \sum_{j\in E} L(j) \int_0^t \mathbb{1}_{\{z_u=j\}} du, \quad t \geq 0, \tag{5.1}$$

where $L$ is a real-valued function defined on $E$.

The function describing the marginal distribution of $\Phi(t)$ at time $t$ is referred to as performability. Specifically, it is given by

$$\mathcal{G}(t, V) := \mathbb{P}(\Phi(t) \in V), \quad t \geq 0, \quad V \in \mathcal{B}(\mathbb{R}_+). \tag{5.2}$$

Furthermore, If $(J_n, S_n)_{n\in\mathbb{N}}$ is the MRP defining the process $\Phi(t)$ at time $t$, then

$$\Phi(t) = L(J_0)X_1 + \ldots + L(J_{n-1})X_n + L(J_n)(t - S_n) \quad \text{for} \quad t \in [S_n, S_{n+1}), \tag{5.3}$$

Moreover, it is important to generate the trajectories of the performance process $\Phi(t)$, using the definition of the counting process, we obtain an equivalent form of formula (5.3)

$$\Phi(t) = \sum_{k=1}^{N(t)} L(J_{k-1})X_k + \left(t - S_{N(t)}\right) L\left(J_{N(t)}\right) \tag{5.4}$$

More general results and functional limit theorems concerning performance measures of semi-Markov systems are available (cf. Limnios and Oprisan [2001]), with further extensions to broader settings (see Korolyuk and Limnios [2005] and Korolyuk and Limnios [2004]).

**Example 5.1.** Consider a semi-Markov process $(Z_t)_{t\geqslant 0}$ with a state space $E = \{0, 1, 2\}$, driven by a Markov renewal process $(J_n, S_n)_{n\in\mathbb{N}}$. The real-reward function $L(x)$ is given by $L(x) = 0.6x$ for $x \geqslant 0$ and the time interval is $0 \leqslant t \leqslant 10$. The sequence of state transitions and associated sojourn times is a follows:

$$\begin{aligned}&\{(J_0 = 2, X_0 = 0), (J_1 = 1, X_1 = 2.5), (J_2 = 2, X_2 = 2)\\ &(J_3 = 0, X_3 = 1.5), (J_4 = 1, X_4 = 2.6), \ldots\}\end{aligned} \tag{5.5}$$

From the given trajectory of the stochastic process $\Phi(t)$ at time $t$, we can express it as follows:

$$\Phi(t) = \begin{cases} 1.2t & \text{for } t \in [0, 2.5), \\ 3 + 0.6(t - 2.5) & \text{for } t \in [2.5, 4.5), \\ 4.2 + 1.2(t - 4.5) & \text{for } t \in [4.5, 6), \\ 6 & \text{for } t \in [6, 8.6), \\ 6 + 0.6(t - 8.6) & \text{for } t \in [8.6, S_5). \end{cases}$$

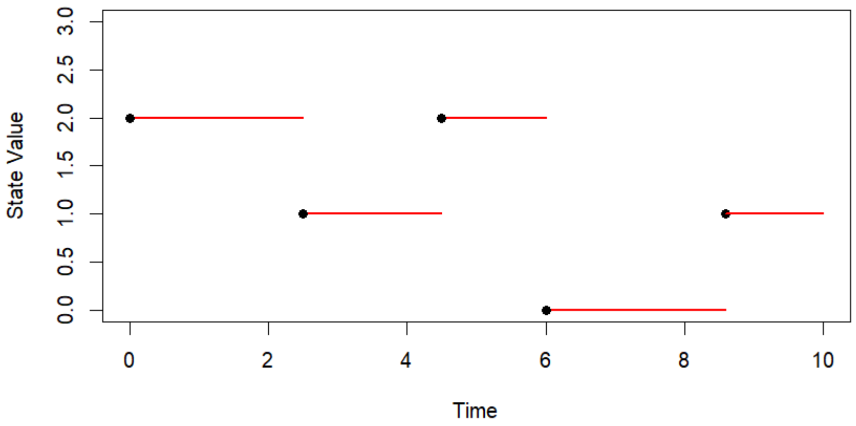


Figure 5.1: Trajectory of the semi-Markov process.

Figure 5.1 presents a sample path of the semi-Markov process $(Z_t)_{t\geq 0}$ corresponding to the Markov renewal process realization specified in expression (5.5).

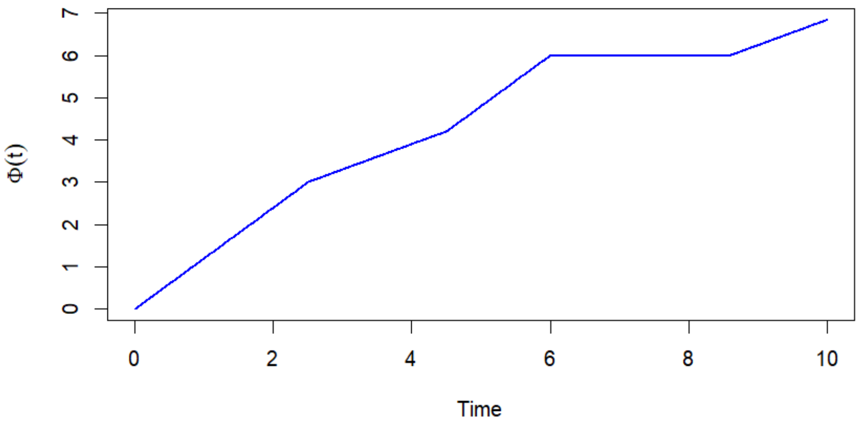


Figure 5.2: A trajectory of the integral functional of semi-Markov process.

Figure 5.2 illustrates the temporal evolution of the performance function $\Phi(t)$ associated with a semi-Markov process $(Z_t)_{t\geq 0}$, over the interval $[0, 10]$. The function $\Phi(t)$ represents the cumulative reward up to time $t$, obtained by integrating a real-valued reward function along the trajectory of the process.

**Definition 5.1.1.** *The mean performance at time* $t > 0$*, denoted by* $\overline{\Phi}(t)$*, is defined by*

$$\overline{\Phi}(t) := \mathbb{E}[\Phi(t)] = \sum_{i\in E} \alpha_i \overline{\Phi}_i(t) = \sum_{i\in E}\sum_{j\in E} \alpha_i L(j) \int_0^t P_{ij}(u)du, \tag{5.6}$$

*where* $\overline{\Phi}_i(t) := \mathbb{E}_i[\Phi(t)] = \sum_{j\in E} L(j) \int_0^t P_{ij}(u)du$ *and the row vector* $\boldsymbol{\alpha} = [\alpha_i : i \in E]$ *where* $\alpha_i = P\left(J_0 = i\right)$ *defines an initial distribution of* $(J_n)_{n\in\mathbb{N}}$.

Prior to stating the main result, we introduce an intermediate result that provides the strong law of large numbers for the performance and mean performance processes.

**Lemma 5.1.1.** *Limnios et al. [2006] For any state* $i$*, we have the following results:*

$$(i) \qquad \frac{\Phi(t)}{t} \overset{\mathbf{P}_i-a.s.}{\longrightarrow} \pi L := \sum_{j\in E} \pi(j)L(j), \quad t \to +\infty,$$

$$(ii) \qquad \frac{\overline{\Phi}_i(t)}{t} \overset{\mathbf{P}_i-a.s.}{\longrightarrow} \pi L, \quad t \to +\infty.$$

## 5.2 Empirical Estimation of SMP's Performance

Let $\mathcal{Y}(M)$ represent the trajectory of the Markov renewal process $(J_n, S_n)_{n\in\mathbb{N}}$, as previously defined in Equation (3.1). This holds for all $i, j \in E$, $t > 0$, and $t \leq M$, we define the empirical estimator $\widetilde{\overline{\Phi}}_i(t, M)$ for the mean performance $\overline{\Phi}_i(t)$ at a fixed time $t$, by

$$\widetilde{\overline{\Phi}}_i(t, M) = \sum_{j\in E} L(j) \int_0^t \widetilde{P}_{ij}(s, M)ds. \tag{5.7}$$

This study focuses on estimating $\Phi(t)$ from a sample path observed over $[0, M]$, and to investigate the consistency and asymptotic normality of the empirical estimator (5.7) as $M$ tends to infinity.

**Theorem 5.2.1.** *Limnios et al. [2006] For any fixed* $0 \leq t \leq M$ *and* $i \in E$*, under* ***(H.1)*** *–* ***(H.2)****, the estimator* $\widetilde{\overline{\Phi}}_i(t, M)$ *of* $\overline{\Phi}_i(t)$ *is strongly uniformly consistent, that is*

$$\max_{i\in E} \sup_{t\in[0,M]} \left| \widetilde{\overline{\Phi}}_i(t, M) - \overline{\Phi}_i(t) \right| \xrightarrow{a.s.} 0, \quad M \to \infty.$$

***Proof of Theorem 5.2.1.*** *See Appendix 5.4.3.* □

Next, we state a result related to the asymptotic normality of the empirical estimator (5.7).

**Theorem 5.2.2.** *Limnios et al. [2006] For any fixed* $t$*,* $t \in [0, \infty)$*,* $\sqrt{M}(\widetilde{\overline{\Phi}}(t, M) - \overline{\Phi}(t))$ *converges in distribution to a centered normal random variable with variance*

$$\begin{aligned}
\sigma^2(t) = \sum_{i=1}^{s}\sum_{j=1}^{s} \mu_{ii} \Big\{ & (B_{ij})^2 * Q_{ij} - (B_{ij} * Q_{ij})^2 \\
& + \int_0^\infty \left[ \int_0^\infty L(j)(x \wedge (t-u))dA_i(u) \right]^2 dQ_{ij}(x) \\
& - \left[ \int_0^\infty \int_0^\infty L(j)(x \wedge (t-u))dA_i(u)dQ_{ij}(x) \right]^2 \\
& + 2 \int_0^\infty B_{ij}(t-x) \int_0^\infty L(j)(x \wedge (t-u))dA_i(u)dQ_{ij}(x) \\
& -2 \left(B_{ij} * Q_{ij}\right)(t) \times \left(A_i * \left(L(j)(x \wedge \cdot)\right)\right)(t) \Big\}
\end{aligned}$$

*where for* $t \in \mathbb{R}_+$*:*

$$A_i(t) = \sum_{k=1}^{s} \alpha_k L(i) \Psi_{ki}(t), \quad B_{ki}(t) = \sum_{i=1}^{s}\sum_{j\in U} \alpha_i \left(\Psi_{ik} * \Psi_{ij} * I_j\right)(t), \quad \text{and} \quad a \wedge b := \min\{a, b\}.$$

***Proof of Theorem 5.2.2.*** *See Appendix 5.4.3.* □

## 5.3 Kernel Estimation of SMP's Performance

Let $\mathcal{Y}(M)$ denote the sample path of a Markov renewal processes $(J_n, S_n)_{n\in\mathbb{N}}$, as defined earlier on Equation (3.1). For all $i, j \in E$, $t > 0$, and $t \leq M$, based on Equations (5.6), (3.10) and (3.11), we define an nonparametric kernel estimator of the mean performance $\widehat{\overline{\Phi}}(t, M)$ by the following expression:

$$\widehat{\overline{\Phi}}(t, M) = \sum_{i\in E}\sum_{j\in E} \alpha_i L(j) \int_0^t \widehat{P}_{ij}(u, M)du. \tag{5.8}$$

### 5.3.1 Asymptotic Properties

Before investigating the asymptotic properties of the kernel estimators, we first present the necessary assumptions to derive their asymptotic behavior as defined in (5.8). Taking into account the assumptions **(H.1)-(H.5)** presented in Section 3.2.2, we proceed.

Now we are ready to show that the kernel estimator proposed in (5.8) have good asymptotic properties.

### 5.3.2 Main Results

Our first result in Section 5.3 concerns the strong uniform consistency of the kernel estimator proposed in Equation (5.8).

**Theorem 5.3.1.** *Hamlat et al. [2025a] For any fixed $t \geq 0$, $t \leq M$ and $i \in E$, under **(H.1)-(H.6)**, the kernel estimator $\widehat{\overline{\Phi}}(t, M)$ of $\overline{\Phi}(t)$ is strongly uniformly consistent, that is*

$$\max_{i\in E}\sup_{t\in[0,M]}\left|\widehat{\overline{\Phi}}_i(t,M)-\overline{\Phi}_i(t)\right| \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

**Proof of Theorem 5.3.1.** We have

$$\begin{aligned}
\max_{i\in E}\sup_{t\in[0,M]}\left|\widehat{\overline{\Phi}}_i(t,M)-\overline{\Phi}_i(t)\right| &= \max_{i\in E}\sup_{t\in[0,M]} \mid \sum_{j\in E} L(j)\int_0^t \widehat{P}_{ij}(u,M)du \\
&\quad - \sum_{j\in E} L(j)\int_0^t P_{ij}(u)du \mid \\
&\leq \sum_{j\in E} L(j)t \max_{i\in E}\sup_{t\in[0,M]}\left|\left(\widehat{\Psi}_{ij}(\cdot,M) * \widehat{\overline{H}}_j(\cdot,M)\right)(t)\right. \\
&\quad \left. - \left(\Psi_{ij} * \overline{H}_j\right)(t)\right| \\
&\leq \sum_{j\in E} L(j)t \max_{i\in E}\sup_{t\in[0,M]}\left|\widehat{\Psi}_{ij}(t,M)-\Psi_{ij}(t)\right| \\
&\quad + \sum_{j\in E} L(j)t \max_{i\in E}\sup_{t\in[0,M]}\left|\widehat{\Psi}_{ij}(t,M)-\Psi_{ij}(t)\right| * \widehat{H}_j(t,M) \\
&\quad + \sum_{j\in E} L(j)t \max_{i\in E}\sup_{t\in[0,M]}\left|\widehat{H}_j(t,M)-H_j(t)\right| * \Psi_{ij}(t).
\end{aligned}$$

Thus

$$\begin{aligned}
\max_{i\in E}\sup_{t\in[0,M]} \mid \widehat{\overline{\Phi}}_i(t,M)-\overline{\Phi}_i(t) \mid &\leq \sum_{j\in E} L(j)t \sum_{n=0}^{\infty} \max_{i\in E}\sup_{t\in[0,M]}\left|\widehat{Q}_{ij}^{(n)}(t,M)-Q_{ij}^{(n)}(t)\right| \\
&\quad + \sum_{j\in E} L(j)t \sum_{n=0}^{\infty} \max_{i\in E}\sup_{t\in[0,M]}\left|\widehat{Q}_{ij}^{(n)}(t,M)-Q_{ij}^{(n)}(t)\right| * \widehat{H}_j(t,M) \\
&\quad + \sum_{j\in E} L(j)t \max_{i\in E}\sup_{t\in[0,M]}\left|\widehat{H}_j(t,M)-H_j(t)\right| * \Psi_{ij}(t).
\end{aligned}$$

For all $i, j \in E$, $n \in \mathbb{N}^*$ and $M \in \mathbb{R}_+$, based on a straightforward adaptation of the proof for the strong uniform consistency from Theorem 3.2.1 (iv), we get that the estimator $\widehat{Q}_{ij}^{(n)}(t,M)$ ) is uniformly strong consistent in $[0, M]$. In addition, the uniform strong consistency of the kernel estimator $\widehat{H}_j(t, M)$ is stated in Theorem 4.1 of Ayhar et al. [2022]. Then,

$$\max_{i\in E}\ \sup_{t\in[0,M]} \mid \widehat{\overline{\Phi}}_i(t,M) - \overline{\Phi}_i(t) \mid \xrightarrow{a.s.} 0, \quad as \quad M \longrightarrow \infty.$$

□

Now, we introduce the following lemma that will be necessary to prove our second result in Section 5.3.

**Lemma 5.3.1.** *Hamlat et al. [2025a] For $n = 1, 2$. If **(H.3)-(H.5)** hold, we have*

$$\frac{1}{h_{dr,M}} \int_0^{+\infty} K^n\left(\frac{v-x}{h_{dr,M}}\right) dQ_{dr}(x) \leqslant q_{dr}(v) \int_{-\infty}^{+\infty} K^n(z)\, dz + O(h_{dr,M}).$$

**Proof of Lemma 5.3.1.** By using a change of variable and an integration by parts followed by Taylor's expansion, we have

$$\begin{aligned}
\frac{1}{h_{dr,M}} \int_0^{+\infty} K^n\left(\frac{v-x}{h_{dr,M}}\right) dQ_{dr}(x) &= \int_{-\infty}^{\frac{v}{h_{dr,M}}} K^n(z)\, q_{dr}(v - h_{dr,M} z) dz \\
&= \int_{-\infty}^{\frac{v}{h_{dr,M}}} K^n(z) \left[q_{dr}(v) - h_{dr,M} z q_{dr}^{'}(v^*)\right] dz \\
&= q_{dr}(v) \int_{-\infty}^{\frac{v}{h_{dr,M}}} K^n(z)\, dz - h_{dr,M} q_{dr}^{'}(v^*) \int_{-\infty}^{\frac{v}{h_{dr,M}}} z K^n(z)\, dz \\
&\leqslant q_{dr}(v) \int_{-\infty}^{+\infty} K^n(z)\, dz - h_{dr,M} q_{dr}^{'}(v^*) \int_{-\infty}^{+\infty} z K^n(z)\, dz \\
&\leqslant q_{dr}(v) \int_{-\infty}^{+\infty} K^n(z)\, dz + O(h_{dr,M}),
\end{aligned}$$

where $v - h_{dr,M} z \leq v^* \leq v$. □

Next, we present results regarding the asymptotic normality of the proposed estimator (5.8).

**Theorem 5.3.2.** *Hamlat et al. [2025a] For any fixed $0 \leq t \leq M$, if **(H.1)-(H.6)** hold, we have*

$$\sqrt{M h_M}\left[\widehat{\overline{\Phi}}(t, M) - \overline{\Phi}(t)\right] \xrightarrow{D} N\left(0, \sigma^2_{\overline{\Phi}}(t)\right), \quad as \quad M \longrightarrow \infty,$$

*with $h_M = \min\limits_{i,j\in E} \{h_{ij,M}\}$ and the asymptotic variance*

$$\sigma^2_{\overline{\Phi}}(t) \leqslant \sum_{i\in E} \sum_{j\in E} \mu_{ii} \int_0^t \left[(R_{ij} - D_i)^2 * \left(Q_{ij}(\cdot) \int_{-\infty}^{+\infty} K^2(z)\, dz\right)\right](u) du, \tag{5.9}$$

*where*

$$R_{ij} = \sum_{d\in E} \sum_{r\in E} \alpha_d L(r) \left(\Psi_{di} * \Psi_{jr} * \overline{H}_r\right), \tag{5.10}$$

*and*

$$D_i = \sum_{d\in E} \sum_{r\in E} \alpha_d L(r) \mathbf{1}_{\{i=r\}} \Psi_{dr}. \tag{5.11}$$

**Proof of Theorem 5.3.2.** We have

$$\begin{aligned}\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right] &= \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t \hat{P}_{dr}(u,M)du-\int_0^t P_{dr}(u)du\right]\\ &= \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t\left[\left(\widehat{\Psi}_{dr}(\cdot,M)*\left(I-\widehat{H}_r(\cdot,M)\right)\right)(u)\right.\right.\\ &\left.\left.-\left(\Psi_{dr}*(I-H_r)\right)(u)\right]du\right].\end{aligned}$$

Note that the last right side can be written as follows:

$$\begin{aligned}&\sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t\left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right)*\left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du\right.\\ &\left.+\int_0^t\left(\Psi_{dr}(\cdot)*\left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du+\int_0^t\left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right)*\overline{H}_r(\cdot)\right)(u)du\right].\end{aligned}$$

For every $t\geq 0, t\leq M$, and for every $d,r\in E$, $\sqrt{M}[\widehat{H}_r(t,M)-H_r(t)]$ converges in distribution to a normal random variable (see Theorem 4.3 in Ayhar et al. [2022]), as $M\to\infty$, and $|\widehat{\Psi}_{dr}(t,M)-\Psi_{dr}(t)|\xrightarrow{P} 0$ as $M\to\infty$ (see Theorem 3.2.1 (v)).
So, using Slutsky's Theorem we obtain that $\sqrt{Mh_M}[(\widehat{\Psi}-\Psi)*(\widehat{H}-H)]_{dr}(t)\xrightarrow{P} 0$ as $M\to\infty$. Thus, applying Slutsky's Theorem we obtain that $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right]$ has the same limit in distribution as

$$\begin{aligned}&\sqrt{Mh_M}\sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\left[\int_0^t\left(\Psi_{dr}(\cdot)*\left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du\right.\\ &\left.+\int_0^t\left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right)*\overline{H}_r(\cdot)\right)(u)du\right].\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\sqrt{Mh_M}\left[\widehat{\mathbf{\Psi}}(\cdot,M)-\mathbf{\Psi}(\cdot)\right]_{dr}(t) &= \sqrt{Mh_M}\left[\widehat{\mathbf{\Psi}}(\cdot,M)-(\widehat{\mathbf{\Psi}}(\cdot,M)\star\mathbf{\Psi}(\cdot))+(\widehat{\mathbf{\Psi}}(\cdot,M)\right.\\ &\left.\star\mathbf{\Psi}(\cdot))-\mathbf{\Psi}(\cdot)\right]_{dr}(t)\\ &= \sqrt{Mh_M}\left[\widehat{\mathbf{\Psi}}(\cdot,M)\star(\mathbf{I}-\mathbf{\Psi}(\cdot))+(\widehat{\mathbf{\Psi}}(\cdot,M)-\mathbf{I})\star\mathbf{\Psi}(\cdot)\right]_{dr}(t)\\ &= \sqrt{Mh_M}\left[-\widehat{\mathbf{\Psi}}(\cdot,M)\star\mathbf{Q}(\cdot)\star\mathbf{\Psi}(\cdot)+\widehat{\mathbf{\Psi}}(\cdot,M)\star\widehat{\mathbf{Q}}(\cdot,M)\star\mathbf{\Psi}(\cdot)\right]_{dr}(t)\\ &= \sqrt{Mh_M}\left[\widehat{\mathbf{\Psi}}(\cdot,M)\star\Delta\mathbf{Q}\star\mathbf{\Psi}(\cdot)-\mathbf{\Psi}(\cdot)\star\Delta\mathbf{Q}\star\mathbf{\Psi}(\cdot)+\mathbf{\Psi}(\cdot)\right.\\ &\left.\star\Delta\mathbf{Q}\star\mathbf{\Psi}(\cdot)\right]_{dr}(t)\\ &= \sqrt{Mh_M}\left[\left(\widehat{\mathbf{\Psi}}(\cdot,M)-\mathbf{\Psi}(\cdot)\right)\star\Delta\mathbf{Q}\star\mathbf{\Psi}(\cdot)\right]_{dr}(t)\\ &+\sqrt{Mh_M}\left[\mathbf{\Psi}(\cdot)\star\Delta\mathbf{Q}\star\mathbf{\Psi}(\cdot)\right]_{dr}(t),\end{aligned}$$

For every $t\geq 0, t\leq M$, and for every $d,r\in E$, $\sqrt{M}[\Delta Q_{dr}](t)$ converges in distribution to a normal random variable (see Corollary 4.4 (ii) in Ayhar et al. [2022]), as $M\to\infty$, and $\widehat{\Psi}_{dr}(t,M)-\Psi_{dr}(t)\xrightarrow{P} 0$ as $M\to\infty$.
Then, using Slutsky's Theorem we obtain that $\sqrt{Mh_M}[(\widehat{\mathbf{\Psi}}-\mathbf{\Psi})\star\Delta\mathbf{Q}\star\mathbf{\Psi}]_{dr}(t)\xrightarrow{P} 0$ as $M\to\infty$. Thus, applying again Slutsky's Theorem we get that $\sqrt{Mh_M}\left[\widehat{\mathbf{\Psi}}(\cdot,M)-\mathbf{\Psi}(\cdot)\right]_{dr}(t)$ has the

same limit in distribution as $\sqrt{Mh_M}[\boldsymbol{\Psi}(\cdot)\star\Delta\mathbf{Q}\star\boldsymbol{\Psi}(\cdot)]_{dr}(t)$.
Where $\Delta\mathbf{Q} = (\widehat{\mathbf{Q}} - \mathbf{Q})$, which is written as follows:

$$\Delta Q_{dr}(\cdot) = \frac{1}{N_d(M)} \sum_{l=1}^{N(M)} \left[ G\left(\frac{\cdot - X_l}{h_{dr,M}}\right) \mathbb{1}_{\{J_{l-1}=d, J_l=r\}} - Q_{dr}(\cdot)\mathbb{1}_{\{J_{l-1}=d\}} \right].$$

Furthermore,

$$\Psi_{dr} * \left(\widehat{\overline{H}}_r - \overline{H}_r\right) = -\sum_{k\in E} \Psi_{dr} * \Delta Q_{rk} = -\sum_{k\in E}\sum_{m\in E} \mathbb{1}_{\{m=r\}} \Psi_{dr} * \Delta Q_{mk}.$$

Then $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M) - \overline{\Phi}(t)\right]$ has the same limit as

$$\begin{aligned}
&\sqrt{Mh_M} \sum_{d\in E}\sum_{r\in E} \alpha_d L(r) \left[ -\int_0^t \sum_{m\in E}\sum_{k\in E} \left(\mathbb{1}_{\{m=r\}}\Psi_{dr} * \Delta Q_{mk}\right)(u)du \right.\\
&\left. + \int_0^t \sum_{m\in E}\sum_{k\in E} \left(\overline{H}_r * \Psi_{dm} * \Psi_{kr} * \Delta Q_{mk}\right)(u)du \right]\\
&= \frac{1}{\sqrt{M}} \sum_{d\in E}\sum_{r\in E} \alpha_d L(r) \sum_{l=1}^{N(M)} \sum_{m\in E}\sum_{k\in E} \frac{M}{N_m(M)} \sqrt{h_M} \left[ \int_0^t \left[ \left(\overline{H}_r * \Psi_{dm} * \Psi_{kr} - \mathbb{1}_{\{m=r\}}\Psi_{dr}\right) \right.\right.\\
&\left.\left. * \left( G\left(\frac{. - X_l}{h_{mk,M}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=k\}} - Q_{mk}(.)\mathbb{1}_{\{J_{l-1}=m\}} \right)\right](u)du \right]\\
&= \frac{1}{\sqrt{M}} \sum_{d\in E}\sum_{r\in E} \alpha_d L(r) \sum_{l=1}^{N(M)} \sum_{m\in E}\sum_{k\in E} \frac{M}{N_m(M)} \sqrt{h_M} \left[ \int_0^t \left[ \left(\overline{H}_r * \Psi_{dm} * \Psi_{kr}\right) \right.\right.\\
&\left.\left. * \left( G\left(\frac{. - X_l}{h_{mk,M}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=k\}} - Q_{mk}(.)\mathbb{1}_{\{J_{l-1}=m\}} \right)\right](u)du \right]\\
&- \frac{1}{\sqrt{M}} \sum_{d\in E}\sum_{r\in E} \alpha_d L(r) \sum_{l=1}^{N(M)} \sum_{m\in E}\sum_{k\in E} \frac{M}{N_m(M)} \sqrt{h_M} \left[ \int_0^t \left[ \left(\mathbb{1}_{\{m=r\}}\Psi_{dr}\right) \right.\right.\\
&\left.\left. * \left( G\left(\frac{. - X_l}{h_{mk,M}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=k\}} - Q_{mk}(.)\mathbb{1}_{\{J_{l-1}=m\}} \right)\right](u)du \right].
\end{aligned}$$

By using (5.10) and (5.11), this formula is equal to

$$\begin{aligned}
&\frac{1}{\sqrt{M}} \sum_{l=1}^{N(M)} \sum_{m\in E}\sum_{k\in E} \frac{M}{N_m(M)} \sqrt{h_M} \left[ \int_0^t \left[ R_{mk} * \left( G\left(\frac{. - X_l}{h_{mk,M}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=k\}} - Q_{mk}(.)\mathbb{1}_{\{J_{l-1}=m\}} \right)\right](u)du \right.\\
&\left. - \int_0^t \left[ D_m * \left( G\left(\frac{. - X_l}{h_{mk,M}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=k\}} - Q_{mk}(.)\mathbb{1}_{\{J_{l-1}=m\}} \right)\right](u)du \right]\\
&= \frac{1}{\sqrt{M}} \sum_{l=1}^{N(M)} \sum_{m\in E}\sum_{k\in E} \frac{M}{N_m(M)} \sqrt{h_M} \left[ \int_0^t \left[ (R_{mk} - D_m) * \left( G\left(\frac{. - X_l}{h_{mk,M}}\right) \mathbb{1}_{\{J_{l-1}=m, J_l=k\}} \right.\right.\right.\\
&\left.\left.\left. - Q_{mk}(.)\mathbb{1}_{\{J_{l-1}=m\}} \right)\right](u)du \right]\\
&= \frac{1}{\sqrt{M}} \sum_{l=1}^{N(M)} f(J_{l-1}, J_l, X_l).
\end{aligned}$$

Apply central limit theorem related to semi-Markov processes (see Pyke and Schaufele [1964]) to the function $W_f(t)$ such that

$$\begin{aligned}
W_f(t) &= \sum_{l=1}^{N(M)} f(J_{l-1}, J_l, X_l) \\
&= \sum_{l=1}^{N(M)} \sum_{m\in E}\sum_{k\in E} \frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t \left[(R_{mk}-D_m) * \left(G\left(\frac{.-X_l}{h_{mk,M}}\right)\mathbb{1}_{\{J_{l-1}=m, J_l=k\}}\right.\right.\right. \\
&\quad \left.\left.\left.-Q_{mk}(.)\mathbb{1}_{\{J_{l-1}=m\}}\right)\right](u)du\right],
\end{aligned}$$

where, for any fixed $t>0$, we have defined the function $f : E\times E\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}
f(i,j,x) &= \sum_{m\in E}\sum_{k\in E}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-x}{h_{mk,M}}\right)\mathbb{1}_{\{i=m,j=k\}}\right.\right.\right. \\
&\quad \left.\left.\left.-Q_{mk}(.)\mathbb{1}_{\{i=m\}}\right)\right](u)du\right].
\end{aligned}$$

In order to apply the Pyke and Schaufeles' CLT, we need to compute the quantities $A_{dr}, A_d, B_{dr}, B_d, r_d, m_f, \sigma_d^2$ and then $\sigma_{\Phi}^2(t)$, using Fubini's theorem and Lemma 5.3.1 with assumptions **(H.3)-(H.5)**. We have

$$\begin{aligned}
A_i &= \sum_{j\in E} A_{ij} \\
&= \sum_{j\in E}\int_0^{+\infty} f(i,j,x)\,dQ_{ij}(x) \\
&= \sum_{j\in E}\sum_{k\in E}\sum_{m\in E}\int_0^{+\infty}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-x}{h_{mk,M}}\right)\mathbb{1}_{\{i=m,j=k\}}\right.\right.\right. \\
&\quad \left.\left.\left.-Q_{mk}(.)\mathbb{1}_{\{i=m\}}\right)\right](u)du\right] dQ_{ij}(x) \\
&= \sum_{j\in E}\frac{M}{N_i(M)}\sqrt{h_M}\int_0^{+\infty}\left[\int_0^t\left[(R_{ij}-D_i)*G\left(\frac{.-x}{h_{ij,M}}\right)\right.\right. \\
&\quad \left.\left.-\sum_{k\in E}(R_{ik}-D_i)*Q_{ik}(.)\right](u)du\right]dQ_{ij}(x) \\
&= \sum_{j\in E}\frac{M}{N_i(M)}\sqrt{h_M}\left[\int_0^t\int_0^u (R_{ij}-D_i)(u-v)\left(\frac{1}{h_{ij,M}}\int_0^{+\infty}K\left(\frac{v-x}{h_{ij,M}}\right)dQ_{ij}(x)\right)dvdu\right. \\
&\quad \left.-\sum_{k\in E}\int_0^t\int_0^u(R_{ik}-D_i)(u-v)q_{ik}(v)\int_0^{+\infty}dQ_{ij}(x)dvdu\right] \\
&= \sum_{j\in E}\frac{M}{N_i(M)}\sqrt{h_M}\left[\int_0^t\int_0^u (R_{ij}-D_i)(u-v)\left(\frac{1}{h_{ij,M}}\int_0^{+\infty}K\left(\frac{v-x}{h_{ij,M}}\right)dQ_{ij}(x)\right)dvdu\right. \\
&\quad \left.-\sum_{k\in E}\int_0^t\int_0^u(R_{ik}-D_i)(u-v)q_{ik}(v)p_{ij}dvdu\right].
\end{aligned}$$

Thus

$$
\begin{aligned}
A_i &= \sum_{j \in E} \frac{M}{N_i(M)} \sqrt{h_M} \left[ \int_0^t \int_0^u (R_{ij} - D_i)(u-v) \left( \frac{1}{h_{ij,M}} \int_0^{+\infty} K\left( \frac{v-x}{h_{ij,M}} \right) dQ_{ij}(x) \right) dvdu \right. \\
&\left. - \sum_{k \in E} \int_0^t \int_0^u (R_{ik} - D_i)(u-v) q_{ik}(v) p_{ij} dvdu \right] \\
&\leqslant \sum_{j \in E} \frac{M}{N_i(M)} \sqrt{h_M} \int_0^t \left[ (R_{ij} - D_i) * (Q_{ij}(\cdot) + O(h_{ij,M})) - \sum_{k \in E} (R_{ik} - D_i) * Q_{ik}(\cdot) \right] (u) du \\
&\leqslant \sum_{j \in E} \frac{M}{N_i(M)} \sqrt{h_M} \int_0^t \left[ (R_{ij} - D_i) * \left( O\left( h_{ij,M} \right) \right) \right] (u) du.
\end{aligned}
$$

For $B_d$ and by using Jensen's inequality and Lemma 5.1.1, we have

$$
\begin{aligned}
B_i &= \sum_{j \in E} B_{ij} \\
&= \sum_{j \in E} \int_0^{+\infty} [f(i,j,x)]^2 dQ_{ij}(x) \\
&= \sum_{j \in E} \int_0^{+\infty} \left[ \frac{M}{N_i(M)} \sqrt{h_M} \left[ \int_0^t \left[ (R_{ij} - D_i) * G\left( \frac{.-x}{h_{ij,M}} \right) \right. \right. \right. \\
&\left. \left. \left. - \sum_{k \in E} (R_{ik} - D_i) * Q_{ik}(.) \right] (u) du \right] \right]^2 dQ_{ij}(x) \\
&= \sum_{j \in E} \int_0^{+\infty} \left( \frac{M}{N_i(M)} \right)^2 h_M \left[ \int_0^t \left[ (R_{ij} - D_i) * G\left( \frac{.-x}{h_{ij,M}} \right) \right. \right. \\
&\left. \left. - \sum_{k \in E} (R_{ik} - D_i) * Q_{ik}(.) \right] (u) du \right]^2 dQ_{ij}(x) \\
&\leqslant \sum_{j \in E} \int_0^{+\infty} \left( \frac{M}{N_i(M)} \right)^2 h_M \left[ \int_0^t \left[ \int_0^u (R_{ij} - D_i)(u-v) \left( \frac{1}{h_{ij,M}} K\left( \frac{v-x}{h_{ij,M}} \right) \right) dv \right. \right. \\
&\left. \left. - \sum_{k \in E} \int_0^u (R_{ik} - D_i)(u-v) q_{ik}(v) dv \right]^2 du \right] dQ_{ij}(x) \\
&\leqslant \sum_{j \in E} \left( \frac{M}{N_i(M)} \right)^2 h_M \int_0^{+\infty} \left[ \int_0^t \int_0^u \left[ (R_{ij} - D_i)^2 (u-v) \left( \frac{1}{h_{ij,M}^2} K^2\left( \frac{v-x}{h_{ij,M}} \right) \right) \right. \right. \\
&+ \sum_{k \in E} (R_{ik} - D_i)^2 (u-v) q_{ik}^2(v) \\
&\left. \left. -2 \sum_{k \in E} (R_{ik} - D_i)(R_{ij} - D_i)(u-v) q_{ik}(v) \frac{1}{h_{ij,M}} K\left( \frac{v-x}{h_{ij,M}} \right) \right] dvdu \right] dQ_{ij}(x).
\end{aligned}
$$

Then

$$\begin{aligned}B_i \leqslant \sum_{j\in E}\left(\frac{M}{N_i(M)}\right)^2 \int_0^t\int_0^u \Bigg[&(R_{ij}-D_i)^2\,(u-v)\left(q_{ij}(v)\int_{-\infty}^{+\infty}K^2(z)dz + O(h_{ij,M})\right)\\ &+\sum_{k\in E}(R_{ik}-D_i)^2\,(u-v)h_M q_{ik}^2(v)\\ &-2\sum_{k\in E}(R_{ik}-D_i)\,(R_{ij}-D_i)\,(u-v)q_{ik}(v)h_M\,(q_{ij}(v)+O(h_{ij,M}))\Bigg]dvdu.\end{aligned}$$

Since $\frac{N_d(M)}{M}\overset{a.s.}{\to}\frac{1}{\mu_{dd}}$ (see Limnios and Oprisan [2001]), when $M\to+\infty$ and from the assumption **(H.6)**, the second and the third term in the last inequality converge to zero, we get

$$B_i \leqslant \sum_{j\in E}\mu_{ii}^2\int_0^t\left[(R_{ij}-D_i)^2 * \left(Q_{ij}(\cdot)\int_{-\infty}^{+\infty}K^2(z)\,dz\right)\right](u)du.$$

Furthermore,

$$\begin{aligned}r_d &= \sum_{i\in E}A_i\frac{\mu_{dd}^*}{\mu_{ii}^*} = 0 \qquad \text{as} \qquad M\to\infty,\\ m_f &= \frac{1}{\mu_{dd}}r_d = 0 \qquad \text{as} \qquad M\to\infty,\\ \sigma_{\overline{\Phi}}^2(t) &= \frac{1}{\mu_{dd}}\sigma_d^2(t),\end{aligned}$$

where

$$\begin{aligned}\sigma_d^2(t) &= \sum_{i\in E}B_i\frac{\mu_{dd}^*}{\mu_{ii}^*}\\ &\leqslant \mu_{dd}^*\sum_{i\in E}\sum_{j\in E}\frac{\mu_{ii}^2}{\mu_{ii}^*}\int_0^t\left[(R_{ij}-D_i)^2 * \left(Q_{ij}(\cdot)\int_{-\infty}^{+\infty}K^2(z)\,dz\right)\right](u)du.\end{aligned}$$

Then, since $\mu_{ii}^* = \dfrac{1}{\nu_i}$ (see Kemeny and Snell [1976]) and $\mu_{ii} = \dfrac{\bar{m}}{\nu_i}$ (see Limnios and Oprisan [2001]), we have

$$\sigma_{\overline{\Phi}}^2(t) \leqslant \sum_{i\in E}\sum_{j\in E}\mu_{ii}\int_0^t\left[(R_{ij}-D_i)^2 * \left(Q_{ij}(\cdot)\int_{-\infty}^{+\infty}K^2(z)\,dz\right)\right](u)du.$$

We obtain from the CLT that $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right]$ converges in distribution, as $M$ tends to infinity, to a normal random variable with zero mean and the variance $\sigma_{\overline{\Phi}}^2(t)$ given by inequality (5.9). □

## 5.4 Applications

The cumulative operational time is considered as one of the key performance indicators in reliability analysis. In this section, we introduce a nonparametric kernel estimator for the expected cumulative operational time in semi-Markov systems. We then establish its asymptotic properties, including uniform strong consistency and asymptotic normality. To illustrate these results, we apply them to a three-state continuous-time semi-Markov process.

### 5.4.1 The cumulative Operational Time

In reliability studies, the state space $E$ is typically divided into two disjoint subsets. One subset, denoted by $U$, consists of up states (operational states), while the other, denoted by $D$, consists of down states (non-operational states). Key events, such as component failures or full repairs, are often represented by transitions between these states. Assuming the system is repairable, the process alternates between the subsets $U$ and $D$.

The cumulative operational time is defined by

$$\mathcal{W}(t) = \int_0^t \mathbb{1}_{\{Z_u \in U\}} du.$$

It is the total spent by the semi-Markov process $Z$ in the set of up states $U$ during the time interval $[0, t]$.

Based on proposition 4.1.1 and under the assumptions **(H.1)-(H.2)**, we have the following result:

$$\lim_{t\to+\infty} \frac{\mathcal{W}(t)}{t} = \frac{\displaystyle\sum_{j\in U} \nu_j m_j}{\displaystyle\sum_{k\in E} \nu_k m_k}.$$

The quantity that we want to study is the expected cumulative operational time of a semi-Markov system, denoted by $\overline{\mathcal{W}}(t) := \mathbb{E}[W(t)]$. Which is given by

$$\begin{aligned}
\overline{\mathcal{W}}(t) &= \mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z_u \in U\}} du\right] \\
&= \sum_{j\in U} \int_0^t \mathbb{P}\left(Z_u = j\right) du \\
&= \sum_{i=1}^{s} \sum_{j\in U} \int_0^t \alpha_i P_{ij}(u) du,
\end{aligned}$$

and can be written as

$$\overline{\mathcal{W}}(t) = \sum_{i\in E} \sum_{j\in U} \alpha_i \int_0^t \left(\Psi_{ij} * \overline{H}_j\right)(u) du. \tag{5.12}$$

From equation (5.12), the kernel estimator of $\overline{\mathcal{W}}(t)$ is given by the following expression:

$$\widehat{\overline{\mathcal{W}}}(t, M) = \sum_{i\in E} \sum_{j\in U} \alpha_i \int_0^t \left(\widehat{\Psi}_{ij}(\cdot, M) * \widehat{\overline{H}}_j(\cdot, M)\right)(u) du.$$

The expected cumulative operational time is an important indicator in the maintenance studies since it allows us to derive the average system availability given by

$$\overline{\mathcal{A}}(t) = \frac{1}{t}\overline{\mathcal{W}}(t) = \frac{1}{t} \sum_{i\in E} \sum_{j\in U} \int_0^t P_{ij}(u) du.$$

In the sequel we will estimate $\overline{\mathcal{W}}(t)$ from a sample path truncated to the time interval $[0, M]$ of this process, and to study its uniform strong consistency and asymptotic normality when $M$ goes to infinity.

The nonparametric kernel estimator $\widehat{\overline{\mathcal{W}}}(t,M)$ of $\overline{\mathcal{W}}(t)$ is given by

$$\widehat{\overline{\mathcal{W}}}(t,M)=\sum_{i\in E}\sum_{j\in U}\alpha_i\int_0^t\widehat{P}_{ij}(u,M)du. \tag{5.13}$$

We are, now in position to state the asymptotic properties of the proposed estimator (5.13).

**Corollary 5.4.1.** *Hamlat et al. [2025a] For any fixed $t\geq 0$ , $t\leq M$, under the same assumptions of Theorem 5.3.1, the estimator of the expected operational time, $\widehat{\overline{\mathcal{W}}}(t,M)$ is strongly consistent, that is*

$$\sup_{t\in[0,M]}\mid\widehat{\overline{\mathcal{W}}}(t,M)-\overline{\mathcal{W}}(t)\mid\xrightarrow{a.s.}0\quad as\quad M\to\infty.$$

**Proof of Corollary 5.4.1.** This corollary is a special case of Theorem 5.3.1, and therefore, its proof is omitted. □

The next result establishes the asymptotic normality of the proposed estimator (5.13).

**Corollary 5.4.2.** *Hamlat et al. [2025a] For any fixed $t\in\mathbb{R}_+$, $t\leq M$, we have $\sqrt{Mh_M}\left[\widehat{\overline{\mathcal{W}}}(t,M)-\overline{\mathcal{W}}(t)\right]$ converges in law to a zero mean normal random variable with the asymptotic variance*

$$\sigma^2_{\overline{\mathcal{W}}}(t)\leqslant\sum_{r\in U}\mu_{mm}\int_0^t\left[(Y_{dr}-C_d)^2*\left(Q_{dr}(\cdot)\int_{-\infty}^{+\infty}K^2\left(z\right)dz\right)\right](u)du,$$

*where*

$$Y_{mk}=\sum_{i\in E}\sum_{j\in U}\alpha_i\left(\overline{H}_j*\Psi_{im}*\Psi_{kj}\right)\quad and\quad C_m=\sum_{i\in E}\sum_{j\in U}\alpha_i\mathbb{1}_{\{m=j\}}\Psi_{ij}.$$

**Proof of Corollary 5.4.2.** The proof of this result is based on the same principles as in the proof of Theorem 5.3.2. □

## 5.4.2 Confidence Intervals

The main purpose of the confidence interval is to supplement the point estimate with information about the uncertainty of this estimate. It is considered a direct application of the central limit theorem (CLT). To provide a confidence interval for the expected cumulative operational time $\overline{\mathcal{W}}(t)$, we first need to propose a consistent estimator for the variance $\sigma^2_{\overline{\mathcal{W}}}(t)$. A natural consistent estimator for this variance, denoted by $\widehat{\sigma}^2_{\overline{\mathcal{W}}}(t,M)$, is obtained by estimating the parameters involved in this quantity, such as $Q_{mk}(t)$, $\overline{H}_j(t)$, and $\Psi_{im}(t)$.
Indeed, from the strong consistency of $\widehat{Q}_{mk}(t,M),\widehat{\overline{H}}_j(t,M)$ and $\widehat{\Psi}_{im}(t,M)$, (see the proof of Theorems 5.3.1 and 5.3.2 as well as Theorem 4.1 and Theorem 4.2 (v) in Ayhar et al. [2022]), we deduce the strong consistency of $\widehat{\sigma}^2_{\overline{\mathcal{W}}}(t,M)$.
Consequently, from Corollary 5.4.2, we get that

$$\sqrt{Mh_M}\left[\widehat{\overline{\mathcal{W}}}(t,M)-\overline{\mathcal{W}}(t)\right]\xrightarrow{D}N\left(0,\widehat{\sigma}^2_{\overline{\mathcal{W}}}(t,M)\right).$$

Then

$$\frac{\sqrt{Mh_M}}{\widehat{\sigma}_{\overline{\mathcal{W}}}(t,M)}\left[\widehat{\overline{\mathcal{W}}}(t,M)-\overline{\mathcal{W}}(t)\right]\xrightarrow{D}N\left(0,1\right).$$

Therefore, hence for $\alpha \in (0,1)$, an asymptotic $100(1-\alpha)\%$ confidence interval for $\widehat{\overline{\mathcal{W}}}(t,M)$ that can be straightforwardly computed is

$$I = \left( \widehat{\overline{\mathcal{W}}}(t,M) \pm z_{\frac{\alpha}{2}} \frac{\widehat{\sigma}_{\overline{\mathcal{W}}}(t,M)}{\sqrt{M h_M}} \right),$$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.

### 5.4.3 Numerical Application

Let us consider a three state semi-Markov system as illustrated in Figure 5.3. States 1 and 2 are up states and state 3 is a down state. We have two exponential and two Weibull distribution functions as conditional transitions, for all $x \geq 0$, say $F_{12}(x) = 1 - \exp(-\lambda_1 x)$, $F_{31}(x) = 1 - \exp(-\lambda_2 x)$, $F_{23}(x) = 1 - \exp\left[-\left(\frac{x}{\alpha_1}\right)^{\beta_2}\right]$, $F_{21}(x) = 1 - \exp\left[-\left(\frac{x}{\alpha_1}\right)^{\beta_1}\right]$. The parameters of these distributions are : $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\alpha_1 = 0.3$, $\beta_1 = 2$, $\alpha_2 = 0.1$, $\beta_2 = 2$.

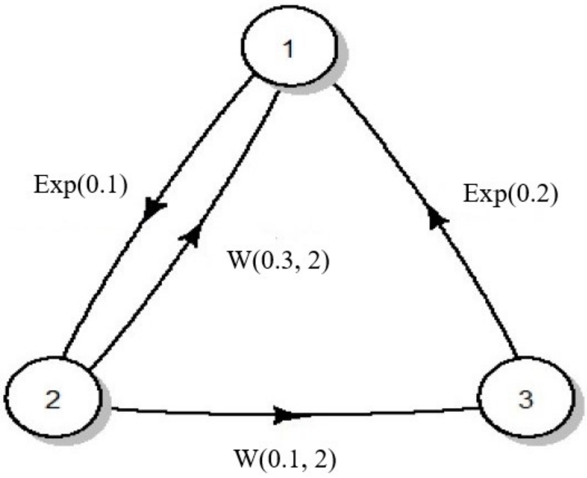


Figure 5.3: A three state semi-Markov system.

The transition probability matrix of the embedded Markov chain $(J_n)$ is:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.95 & 0 & 0.05 \\ 1 & 0 & 0 \end{pmatrix}$$

where the system is defined by the initial distribution $\alpha = (1/3, 1/3, 1/3)$.

To construct the kernel estimator for the mean performance of a continuous-time semi-Markov process. the smoothed function $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{4}\left(1 - u^2\right)$ for $|u| \leq 1$ and the cumulative distribution function $G(u) = \int_{-\infty}^{u} \frac{3}{4}\left(1 - z^2\right) \mathbf{1}_{[-1,1]}(z) \mathrm{d}z$. The bandwidth $h_T$ has been obtained by the "PBbw" method, which computes the plug-in bandwidth of the Polansky and Baker method, (cf. Polansky and Baker [2000]). We have considered that the observation period is the interval $[0, T]$ with $T = 20000$.

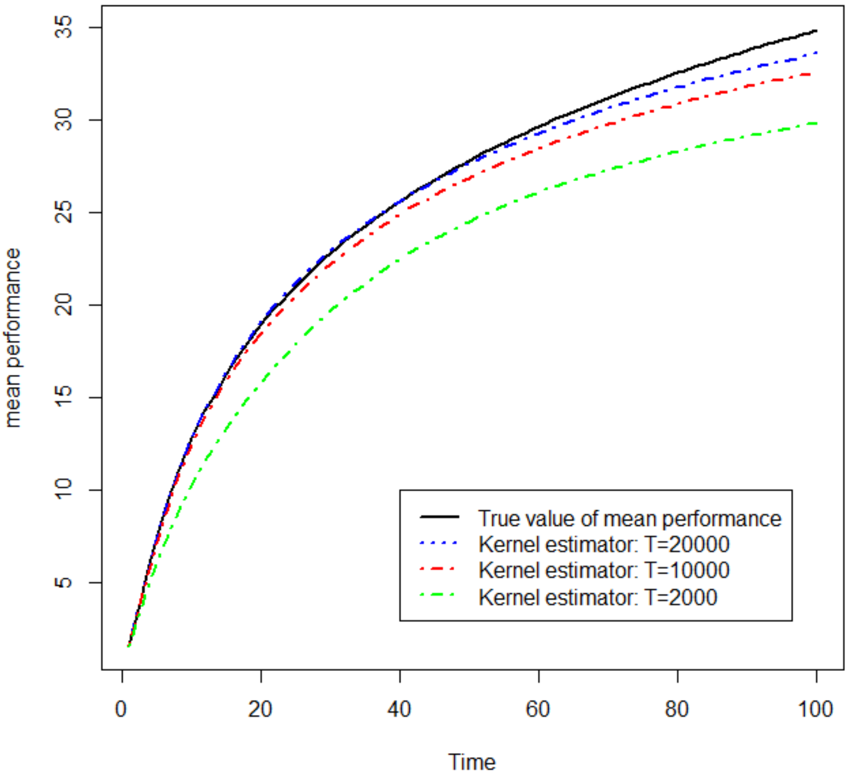


Figure 5.4: Comparison between the kernel estimator of the mean performance for different sample sizes and the true value.

Figure 5.4 gives a comparison between the kernel estimator of the mean performance for different sample sizes ($M = 2000, M = 10000$ and $M = 20000$ ). We observe that this estimator converges to the true value of the mean performance as $M$ increases.

One way of illustrating the accuracy of the estimator is by providing the Mean square error (MSE). We have carried out N = 100 repetitions of the experiment, and we have taken 100 points of discretisation.

$$\mathrm{MSE}\left(\widehat{\mathcal{W}}\right) = \mathbb{E}\left[\left(\widehat{\mathcal{W}}(t,M) - \overline{\mathcal{W}}(t)\right)^2\right].$$

Table 5.1: MSEs for the both methods, kernel and empirical estimation.

| | Kernel estimation | Empirical estimation |
|---|---|---|
| $\mathrm{MSE}\left(\widehat{\mathcal{W}}\right)$ | 0.4985763 | 0.8522993 |

The results presented in the table 5.1 give the MSEs values for the both methods. It can be seen that the kernel method generates better result than the empirical one.

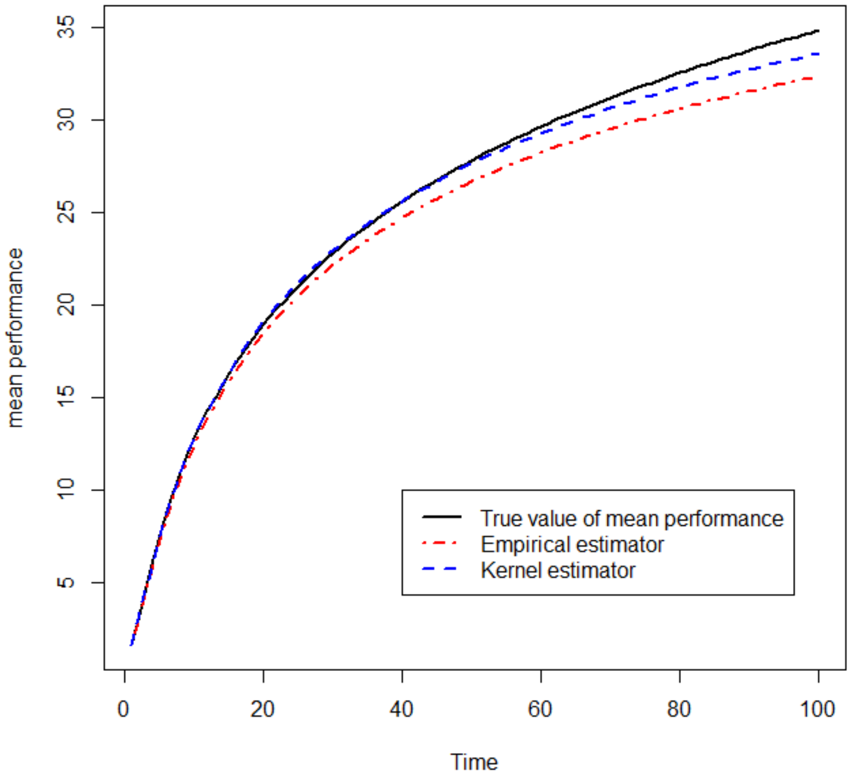


Figure 5.5: Comparison between the true values of the mean performance and their estimators the empirical and the kernel.

Figure 5.5 gives a comparison between the empirical and the kernel estimators of the mean performance. We remark easily that, our method provides better results than the empirical one.

# Conclusion and Prospects

## General conclusion

The main objective of this thesis was to develop nonparametric estimators for several fundamental characteristics of continuous-time semi-Markov processes, relying on kernel smoothing techniques and long-term observations of the system. From the very outset of this study, our objective was to develop flexible and robust nonparametric estimators of capturing the complexity inherent in semi-Markov dynamics, while remaining free from restrictive parametric assumptions. Within this framework, we have established theoretical results, proposed practical estimation procedures, and validated their performance through numerical illustrations.

**(i) Estimation of the stationary distribution and related characteristics.** In the first part of the thesis, we introduced a nonparametric estimator, based on the Parzen-Rosenblatt kernel method, for the stationary distribution of a continuous-time semi-Markov process. We also considered associated quantities such as the mean sojourn time in a current state, the mean sojourn time, and the mean recurrence time. We proved strong consistency and asymptotic normality of the proposed estimators. Compared to classical empirical estimators, which are discontinuous by nature, the kernel-based approach yields smoother and more realistic results when the underlying distributions are continuous. A numerical application to a three-state semi-Markov process have illustrated both the accuracy of our estimators and the practical relevance of our theoretical findings. Since the stationary distribution governs the long-term behavior of the system and plays a crucial role in the computation of reliability indicators (availability, failure rate, mean time to failure, mean time to repair, etc.), these contributions have direct implications for reliability analysis and stochastic modeling.

**(ii) Estimation of the mean performance.** In the second contribution of this thesis, we focused on the mean performance of semi-Markov systems. Using the same kernel strategy, we constructed an estimator for this quantity and established its strong consistency and asymptotic normality. Our results highlight once again the advantages of smoothing techniques in providing accurate, continuous, and stable estimates, especially when contrasted with empirical estimators that often suffer from abrupt discontinuities.

**(iii) Estimation of the cumulative operational time.** In the third part of this thesis, a special case of our study was devoted to the cumulative operational time of semi-Markov systems, a quantity of major importance for maintenance optimization, cost minimization, and system performance evaluation. For this indicator, we conclude the uniform strong consistency and the asymptotic normality, constructed confidence intervals, and conducted simulation studies. These simulations clearly demonstrated the superiority of the kernel estimator compared to standard empirical approaches.

Overall, the contributions of this thesis provide a coherent theoretical and practical framework for the nonparametric analysis of continuous-time semi-Markov processes. The proposed kernel estimators exhibit solid asymptotic properties and practical effectiveness, thus offering valuable alternatives to classical empirical methods. The results obtained here not only advance the nonparametric theory of semi-Markov systems but also create new opportunities for their application in reliability engineering, survival analysis, and other fields involving complex stochastic dynamics.

# Perspectives

Although this thesis provides answers to several important questions concerning nonparametric estimation for continuous-time semi-Markov processes, numerous research directions remain to be explored. We outline below some promising perspectives that naturally extend our work.

**Bandwidth selection and convergence rates.** While our estimators were shown to be consistent and asymptotically normal under mild assumptions, establishing optimal convergence rates requires stronger regularity conditions. Because kernel estimators rely on a delicate bias variance trade-off, identifying optimal or data-driven bandwidth selection procedures is an important avenue for improving both theoretical and practical performance.

**Advanced smoothing techniques.** We focused primarily on classical kernel estimators, but other smoothing approaches such as local linear methods, recursive estimators, or basis expansions may offer additional benefits, including reduced boundary effects and improved mini-max properties. Extending our results to these techniques would broaden the methodological scope of nonparametric semi-Markov analysis.

**Censoring mechanisms.** In many real-world settings, and particularly in survival analysis, the observed sojourn times may be censored. Adapting our estimation methodology to incorporate various censoring structures would not only increase its practical relevance but also raise significant theoretical questions concerning identifiability and the asymptotic behavior of the resulting estimators.

**Hidden semi-Markov models.** An important research direction involves semi-Markov processes with an unobservable underlying chain. In such *hidden semi-Markov models* (HSMM), only an observable process is available, creating new methodological challenges. Developing nonparametric kernel estimators within this framework would considerably enlarge the range of potential applications.

**Inhomogeneous and multivariate semi-Markov systems.** All results presented in this thesis are based on homogeneous semi-Markov processes. Extending them to time-inhomogeneous or multivariate settings would generalize the theoretical framework and provide useful tools for real-world systems whose transition mechanisms evolve over time.

In conclusion, the methods and results developed in this thesis constitute a first step toward a comprehensive nonparametric theory for continuous-time semi-Markov processes. We hope that this work will inspire future investigations and contribute to a deeper understanding of these models, which remain essential tools in reliability engineering, applied probability, and many fields of modern science.

# Appendix

## Mathematical Background

**Definition 5.4.1.** ***(Stopping time or Markov time)*** *A random variable $T$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, is called a stopping time with respect to the sequence $(X_n)_{n\in\mathbb{N}}$ if the occurrence of the event $\{T = n\}$ is determined by the past of the chain up to time $n$, $(X_k; k \leq n)$. More precisely, let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n), n \geq 0$, be the $\sigma$-algebra generated by $X_0, \ldots, X_n$, i.e., the information known at time $n$. The random variable $T$ is called a stopping time if, for every $n \in \mathbb{N}, \{T = n\} \in \mathcal{F}_n$.*

**Definition 5.4.2.** ***(Strong Markov property)*** *A Markov chain $(X_n)_{n\in\mathbb{N}}$ is said to have the strong Markov property if, for any stopping time $T$, for any integer $m \in \mathbb{N}$ and state $j \in E$ we have*

$$\mathbb{P}\left(X_{m+T} = j \mid X_k, k \leq T\right) = \mathbb{P}_{X_T}\left(X_m = j\right) \qquad a.s.$$

**Proposition 5.4.1.** *Any Markov chain has the strong Markov property.*

For double series $(u(m,n))_{m,n\in\mathbb{N}}$ we have the following result:

**Proposition 5.4.2.** ***(Dominated convergence theorem for sequences)***. *Let $u(m,n)_{m,n\in\mathbb{N}}$ be a double sequence such that $\lim_{m\to\infty} u(m,n)$ exists for each $n$ and that $|u(m,n)| \leq v(n)$, with $\sum_{n=0}^{\infty} v(n) < \infty$. Then:*

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} u(m,n) = \sum_{n=0}^{\infty} \lim_{m\to\infty} u(m,n) < \infty.$$

**Theorem 5.4.1.** *Let $(X_n)_{n\in\mathbb{N}}$ be a martingale with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$, such that $\sup\limits_{n\in\mathbb{N}} \mathbb{E}\left(|X_n|\right) < \infty$. Then, there exists a random variable $X_\infty$, integrable, such that*

$$X_n \xrightarrow[n\to\infty]{a.s.} X_\infty$$

The following result is the Lindeberg-Lévy Central limit theorem for martingales (see Billingsley [1995]).

**Theorem 5.4.2.** ***(Central limit theorem for martingales)***. *Let $(X_n)_{n\in\mathbb{N}^*}$, be a martingale with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ and define the process $Y_n := X_n - X_{n-1}, n \in \mathbb{N}^*$, (with $Y_1 := X_1$ ), called a difference martingale. If*

1. $\frac{1}{n} \sum\limits_{k=1}^{n} \mathbb{E}\left[Y_k^2 \mid \mathcal{F}_{k-1}\right] \xrightarrow[n\to\infty]{\mathbb{P}} \sigma^2 > 0$;

*2.* $\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[Y_k^2 \mathbb{1}_{\{|Y_k|>\epsilon\sqrt{n}\}}\right] \xrightarrow[n\to\infty]{} 0$, *for all* $\epsilon > 0$, *then*

$$\frac{X_n}{n} \xrightarrow[n\to\infty]{a.s.} 0$$

*and*

$$\frac{1}{\sqrt{n}}X_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_k \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \sigma^2\right)$$

# Proofs

**Proof of Theorem 1.1.1.** Suppose $X \sim \mathbb{E}(\lambda)$, then

$$\mathbb{P}(X > s+t \mid X > s) = \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t)$$

On the other hand, suppose that $X$ has the memoryless property whenever $\mathbb{P}(X > s) > 0$. Then $h(t) = \mathbb{P}(X > t)$ satisfies

$$h(s+t) = h(s)h(t) \quad \text{for all } s, t \geq 0$$

We assumed $X > 0$ so that $h(1/n) > 0$ for some $n$. Then, by induction

$$h(1) = h\left(\frac{1}{n} + \ldots + \frac{1}{n}\right) = h\left(\frac{1}{n}\right)^n > 0$$

so $h(1) = e^{-\lambda}$ for some $0 \leq \lambda < \infty$. Furthermore, $h(r) = e^{-\lambda r}$ for all rationals $r > 0$. For real $t > 0$, choose rationals $r, s > 0$ with $r \leq t \leq s$. Since $h$ is decreasing,

$$e^{-\lambda r} = h(r) \geq h(t) \geq h(s) = e^{-\lambda s}$$

and, since we can choose $r$ and $s$ arbitrarily close to $t$, this forces $h(t) = e^{-\lambda t}$, so $X \sim Exp(\lambda)$. □

**Proof of Lemma 1.4.1.** We can compute $p_{ij}(s+t)$ by considering all possible states the chain could be at time $s$. We then apply condition and uncondition, using the Markov property to simplify the conditioning process, i.e.,

$$\begin{aligned}
p_{ij}(s+t) &= \mathbb{P}(J_{s+t} = j \mid J_0 = i) \\
&= \sum_{u\in E} \mathbb{P}(J_{s+t} = j, J_s = u \mid J_0 = i) \\
&= \sum_{u\in E} \mathbb{P}(J_s = u \mid J_0 = i)\mathbb{P}(J_{s+t} = j \mid J_s = u, J_0 = i) \\
&= \sum_{u\in E} \mathbb{P}(J_s = u \mid J_0 = i)\mathbb{P}(J_{s+t} = j \mid J_s = u) \quad \text{(Markov property)} \\
&= \sum_{u\in E} p_{iu}(s)p_{uj}(t) \quad \text{(stationary transition probabilities).}
\end{aligned}$$

□

**Proof of Lemma 1.4.2.** The argument going forward in time is straightforward: By Lemma 1.4.1, if $p_{ij}(s) > 0$, then

$$p_{ij}(s+t) = \sum_k p_{ik}(s)p_{kj}(t) \geq p_{ij}(s)p_{jj}(t) \geq p_{ij}(s)e^{B_{jj}t} > 0 \quad \text{for all} \quad t > 0,$$

because $p_{jj}(t)$ is bounded below by the probability of no transition at all from state $j$ in time $t$, which is $e^{B_{jj}t}$, and $B_{jj}$ is the total rate of leaving state $j$. □

**Proof of Theorem 1.4.2.** First we obtain

$$\pi \mathbf{p}(t) = \pi e^{\mathbf{A}\cdot t} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \pi \mathbf{A}^n = \pi \mathbf{I}_E + \sum_{n=1}^{\infty} \frac{t^n}{n!} \pi \mathbf{A}^n = \pi + 0 = \pi,$$

For every $t \geq 0$, where 0 signifies the zero measure on $E$. Consequently, Theorem 1.4.1 implies

$$\begin{aligned}\mathbb{P}\left(J_{t_1} = j_1, \ldots, J_{t_n} = j_n\right) &= \sum_{i \in E} \alpha_i p_{ij_1}\left(t_1\right) p_{j_1 j_2}\left(t_2 - t_1\right) \ldots p_{j_{n-1} j_n}\left(t_n - t_{n-1}\right) \\ &= \alpha_{j_1} p_{j_1 j_2}\left(t_2 - t_1\right) \ldots p_{j_{n-1} j_n}\left(t_n - t_{n-1}\right) \\ &= \sum_{i \in E} \alpha_i p_{ij_1}\left(t_1 + s\right) p_{j_1 j_2}\left(t_2 - t_1\right) \ldots p_{j_{n-1} j_n}\left(t_n - t_{n-1}\right) \\ &= \mathbb{P}\left(J_{t_1+s} = j_1, \ldots, J_{t_n+s} = j_n\right)\end{aligned}$$

For any times $t_1 < \ldots < t_n$ with $n \in \mathbb{N}$, and states $j_1, \ldots, j_n \in E$, the process $J$ is stationary. Conversely, if $\nu$ is a stationary distribution, it follows that $\pi \mathbf{p}(t) = \pi e^{\mathbf{A}\cdot t} = \pi$ for all $t \geq 0$. Consequently, this implies that $\sum_{n=1}^{\infty} \frac{t^n}{n!} \pi \mathbf{A}^n = \mathbf{0}$, leading to the result $\nu \mathbf{A} = \mathbf{0}$ due to the uniqueness of the zero power series. □

**Proof of Proposition 2.2.1.** From the definition of conditional probabilities, we have

$$\begin{aligned}F_{ij}(t) &= \mathbb{P}(X_{n+1} \leq t | J_n = i, J_{n+1} = j) \\ &= \frac{\mathbb{P}(X_{n+1} \leq t, J_n = i, J_{n+1} = j)}{\mathbb{P}(J_n = i, J_{n+1} = j)} \\ &= \frac{\mathbb{P}(X_{n+1} \leq t, J_n = i, J_{n+1} = j)}{\mathbb{P}(J_n = i)} \frac{\mathbb{P}(J_n = i)}{\mathbb{P}(J_n = i, J_{n+1} = j)} \\ &= \frac{\mathbb{P}(J_{n+1} = j, X_{n+1} \leq t | J_n = i)}{\mathbb{P}(J_{n+1} = j | J_n = i)} \\ &= \frac{Q_{ij}(t)}{p_{ij}}.\end{aligned}$$

□

**Proof of Theorem 3.2.1.** For any fixed $t \in \mathbb{R}_+, t \leq M$, we have

(i) The consistency of $p_{ij}(M)$ is a direct consequence of Lemma 3.1.3 for more details (see Limnios and Oprisan [2001]).

(ii) The strong consistency of the empirical estimator $\widetilde{F}_{ij}(t, M)$ can be directly obtained by using the SLLN and Glivenko Cantelli theorem.

Let us denote by $\left\{n_1, n_2, \ldots, n_{N_{ij}(M)}\right\}$ the transition times from state $i$ to state $j$, up to time $M$. Note that we have

$$\widetilde{F}_{ij}(t, M) = \frac{1}{N_{ij}(M)} \sum_{l=1}^{N_{ij}(M)} \mathbf{1}_{\{X_{n_l} \leq t\}}.$$

For any $l \in \{1, 2, \ldots, N_{ij}(M)\}$ we have

$$\mathbb{E}\left[\mathbf{1}_{\{X_{n_l} \leq t\}}\right] = \mathbb{P}\left(X_{n_l} \leq t\right) = F_{ij}(t).$$

Since $N_{ij}(M) \xrightarrow[M\to\infty]{\text{a.s.}} \infty$, applying the SLLN to the sequences of i.i.d. random variables $\left\{\mathbf{1}_{\{X_{n_l}\leq k\}}\right\}_{l\in\{1,2,\ldots,N_{ij}(M)\}}$, we have

$$\widetilde{F}_{ij}(t,M) = \frac{1}{N_{ij}(M)}\sum_{l=1}^{N_{ij}(M)} \mathbb{1}_{\{X_{n_l}\leq t\}} \xrightarrow[M\to\infty]{a.s.} \mathbb{E}\left[\mathbb{1}_{\{X_{n_l}\leq t\}}\right] = F_{ij}(t).$$

In order to obtain uniform consistency, from the Glivenko-Cantelli theorem (cf. Billingsley [1995]) we have

$$\max_{0\leq t\leq m}\left|\frac{1}{m}\sum_{l=1}^{m}\mathbf{1}_{\{X_{n_l}\leq t\}} - F_{ij}(t)\right| \xrightarrow[m\to\infty]{\text{a.s.}} 0.$$

Let us define $\xi_m := \max\limits_{0\leq t\leq m}\left|\frac{1}{m}\sum\limits_{l=1}^{m}\mathbb{1}_{\{X_{nl}\leq t\}} - F_{ij}(t)\right|$. The previous convergence tells us that $\xi_m \xrightarrow[m\to\infty]{\text{a.s.}} 0$. As $N(M) \xrightarrow[m\to\infty]{\text{a.s.}} \infty$, applying Theorem 5.4.2 we obtain $\xi_{N(M)} \xrightarrow[M\to\infty]{\text{a.s.}} 0$, which reads

$$\max_{0\leq t\leq M}\left|\widetilde{F}_{ij}(t,M) - F_{ij}(t)\right| \xrightarrow[M\to\infty]{a.s.} 0.$$

As the state space $E$ is finite, we take the maximum with respect to $i,j\in E$ and the desired result for $\widetilde{F}_{ij}(t,M)$ follows.

(iii) From the definition of $Q_{ij}(t)$ and that of $\widetilde{Q}_{ij}(t,M)$, we obtain that

$$\begin{aligned} Q_{ij}(t) &= \mathbb{P}\left(J_{n+1}=j, X_{n+1}\leq t \mid J_n = i\right) = p_{ij}F_{ij}(t),\\ \widetilde{Q}_{ij}(t,M) &= \widetilde{p}_{ij}(M)\widetilde{F}_{ij}(t,M). \end{aligned}$$

Now, we have

$$\begin{aligned} &\max_{i,j\in E}\max_{0\leq t\leq M}\left|\widetilde{Q}_{ij}(t,M) - Q_{ij}(t)\right|\\ &= \max_{i,j\in E}\max_{0\leq t\leq M} \mid \widetilde{p}_{ij}(M)\widetilde{F}_{ij}(t,M) - \widetilde{p}_{ij}(M)F_{ij}(t)\\ &+ \widetilde{p}_{ij}(M)F_{ij}(t) - p_{ij}F_{ij}(t) \mid\\ &\leq \max_{i,j\in E}\left|\widetilde{p}_{ij}(M) - p_{ij}\right| + \max_{i,j\in E}\max_{0\leq t\leq M}\left|\widetilde{F}_{ij}(t,M) - F_{ij}(t)\right|, \end{aligned}$$

and from the consistency of $\widehat{p}_{ij}(M)$ and $\widehat{F}_{ij}(k,M)$, we have the following result,

$$\max_{i,j\in E}\sup_{t\in[0,M)}\left|\widetilde{Q}_{ij}(t,M) - Q_{ij}(t)\right| \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad M\to\infty.$$

(iv) For $m=1$, the result holds by (iii) in Theorem 3.2.1. For $m\geq 2$, we observe that

$$\begin{aligned} \max_{i,j\in E}\sup_{t\in[0,M]}\left|\widetilde{Q}_{ij}^{(m)}(t,M) - Q_{ij}^{(m)}(t)\right| \leq{} & \max_{i,j\in E}\sup_{t\in[0,M]}\left|\widetilde{Q}_{ij}^{(m-1)}(t,M) - Q_{ij}^{(m-1)}(t)\right|\\ &+ s\cdot\max_{i,j\in E}\sup_{t\in[0,M]}\left|\widetilde{Q}_{ij}(t,M) - Q_{ij}(t)\right|. \end{aligned}$$

Thus, the result follows from the principle of mathematical induction. For more details, see Ouhbi and Limnios [1997a].

(v) Let consider a constant $k_0 > 0$ such that $\max\limits_{i\in E} \sum\limits_{j=1}^{s} Q_{ij}^{(k_0)}(t) < 1$. Based on $(iv)$ in Theorem 3.2.1, we obtain the uniformly strong consistent for all $n \geqslant 1$. Set $\epsilon = 1 - \max\limits_{i\in E} \sum_{j=1}^{s} Q_{ij}^{(k_0)}(t)$. For all $\omega \in \Omega$, there exists $T_0(\omega)$, such that for all $M \geqslant M_0(\omega)$

$$\begin{aligned}\max_{i\in E} \sum_{j=1}^{s} \widetilde{Q}_{ij}^{(k_0)}(t,M) \leqslant \max_{i\in E} &\left| \sum_{j=1}^{s} \left[ \widetilde{Q}_{ij}^{(k_0)}(t,M) - Q_{ij}^{(k_0)}(t) \right] \right| + \\ &+ \max_{i\in E} \sum_{j=1}^{s} Q_{ij}^{(k_0)}(t) \leqslant 1 - \frac{\epsilon}{2}.\end{aligned}$$

Moreover, for all $m \geqslant k_0$, there exists $(q,r) \in \mathbb{N}^* \times \mathbb{N}$ such that $m = qk_0 + r$ where $0 \leqslant r < k_0$ and we see that,

$$\begin{aligned}\max_{i,j\in E} \widetilde{Q}_{ij}^{(m)}(t,M) &= \max_{i,j\in E} \sum_{n=1}^{s} \widetilde{Q}_{in}^{(r)} * \widetilde{Q}_{nj}^{(qk_0)}(t,M) \\ &\leqslant \max_{i,j\in E} \sum_{n=1}^{s} \widetilde{Q}_{in}^{(r)}(t,M) \cdot \widetilde{Q}_{nj}^{(qk_0)}(t,M) \\ &\leqslant \max_{i,j\in E} \widetilde{Q}_{ij}^{(qk_0)}(t,M).\end{aligned}$$

Therefore, we have

$$\widetilde{\Psi}_{ij}(t,M) \leq \sum_{q=0}^{\infty} \sum_{r=0}^{k_0-1} \left(1 - \frac{\epsilon}{2}\right)^q = k_0 \sum_{q=0}^{\infty} \left(1 - \frac{\epsilon}{2}\right)^q < \infty.$$

Thus by the Lebesgue's dominated convergence theorem, we get

$$\widetilde{\Psi}_{ij}(t,M) \xrightarrow{\text{a.s.}} \Psi_{ij}(t) \quad \text{as} \quad M \longrightarrow \infty.$$

To prove that the estimator of the Markov renewal matrix is uniformly strongly consistent on compact $[0,M]$, for $M \in \mathbb{R}^+$, observe that $\Psi_{ij}(t)$ is monotone and continuous. So, for each fixed $t$ and $i,j \in E$, we deduce that the convergence is uniform on $[0,M]$. Therefore,

$$\max_{i,j\in E} \sup_{t\in[0,M]} \left| \widetilde{\Psi}_{ij}(t,M) - \Psi_{ij}(t) \right| \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad M \longrightarrow \infty.$$

(vi) Let us define the matrices $A(t) = (I - diag(Q(t)\cdot \mathbf{1}))$ and $\widetilde{A}(t,M) = \left(I - diag(\widetilde{Q}(t,M)\cdot \mathbf{1})\right)$. For any fixed $i,j \in E$, we have

$$\widetilde{P}_{ij}(t,M) - P_{ij}(t) = \left(\widetilde{\Psi}(\cdot,M) * \widetilde{A}(\cdot,M)\right)_{ij}(t) - (\Psi * A)_{ij}(t).$$

Accordingly, we obtain the following inequality

$$\begin{aligned}\max_{i,j\in E} \sup_{t\in[0,M]} \left| \widetilde{P}_{ij}(t,M) - P_{ij}(t) \right| &\leq \max_{i,j\in E} \sup_{t\in[0,M]} \left| (\widetilde{\Psi} * \widetilde{A})_{ij}(t,M) - (\widetilde{\Psi} * A)_{ij}(t,M) \right| \\ &\quad + \max_{i,j\in E} \sup_{t\in[0,M]} \left| (\widetilde{\Psi} * A)_{ij}(t,M) - (\Psi * A)_{ij}(t) \right| \\ &\leq \max_{i,j\in E} \sup_{t\in[0,M]} \left| \widetilde{\Psi}_{ij}(t,M) - \Psi_{ij}(t) \right| \\ &\quad + \max_{i,j\in E} \sup_{t\in[0,M]} \left| \left(\widetilde{\Psi}_{ij}(t,M) - \Psi_{ij}(t)\right) \cdot \mathrm{diag}(\widetilde{Q}(t,M)\cdot \mathbf{1}) \right| \\ &\quad + \max_{i,j\in E} \sup_{t\in[0,M]} \left| \mathrm{diag}((\widetilde{Q} - Q)\cdot \mathbf{1})_{jj}(t,M) \right| \cdot \Psi_{ij}(M).\end{aligned}$$

Since $\Psi_{ij}(t)$ is finite (see Ouhbi and Limnios [1999]). Moreover, by (iii) and (v) in Theorem 3.2.1, which establish the uniform strong consistency of the estimators of both the semi-Markov kernel and the Markov renewal function on the interval $[0, M]$, it follows that both $\operatorname{diag}((\widetilde{Q} - Q) \cdot \mathbf{1})_{jj}(t, M)$ and the difference $\widetilde{\Psi}_{ij}(t, M) - \Psi_{ij}(t)$ converge almost surely to zero as $M \to \infty$. $\square$

**Proof of Theorem 3.2.2.** For any fixed $t \in \mathbb{R}_+, t \leq M$, we have

(i) Let $\omega_1$ and $\omega_2$ be arbitrary constants. To prove the asymptotic joint normality it suffices to show that

$$\omega_1 M^{1/2} \left[\widetilde{p}_{ij}(M) - p_{ij}\right] + \omega_2 M^{1/2} \left[\widetilde{H}_i(t, M) - H_i(t)\right] \tag{5.14}$$

converges in distribution to a normal random variable for all $\omega_1$ and $\omega_2$. We rewrite Equation (5.14) as the product of $\left[M/N_i(M)\right] M^{-1/2}$, and using Theorem 5.4.2, we consider that the function $f$ as follows

$$f(J_{l-1}, J_l, X_l) = \left\{\omega_1 \left[\mathbb{1}_{\{J_l = j\}} - p_{ij}\right] + \omega_2 \left[\epsilon(t - X_l) - H_i(t)\right]\right\} \mathbb{1}_{\{J_{l-1} = i\}}.$$

For this function

$$A_r = \omega_1 \mathbb{1}_{r=i} \left[p_{rj} - p_{ij}\right] + \omega_2 \mathbb{1}_{r=i} \left[H_r(t) - H_i(t)\right] = 0,$$

and

$$B_r = \left\{\omega_1^2 \left[p_{rj} + p_{ij}^2 - 2 p_{rj} p_{ij}\right] + \omega_2^2 \left[H_r(t) + H_i^2(t) - 2 H_r(t) H_i(t)\right]\right\} \mathbb{1}_{r=i}.$$

Hence, based on Theorem 5.4.2, we conclude that $r_i = 0$, and the third sum in (5.14) is zero. Consequently, the asymptotic variance is defined by

$$\sigma_i^2 = \sum_{r=1}^{s} B_r \frac{\mu_{ii}^*}{\mu_{rr}^*} = \omega_1^2 p_{ij} \left[1 - p_{ij}\right] + \omega_2 H_i(t) \left[1 - H_i(t)\right].$$

(ii) We have

$$M^{1/2} \left[\widetilde{Q}_{ij}(t, M) - Q_{ij}(t)\right] = \frac{M}{N_i(M)} M^{-1/2} \sum_{l=1}^{N(t)} \left(\mathbb{1}_{\{J_l = j, X_l \leq t\}} - Q_{ij}(t)\right) \mathbb{1}_{\{J_{l-1} = i\}}.$$

Consider the function

$$f(J_{l-1}, J_l, X_l) = \left(\mathbb{1}_{\{J_l = j, X_l \leq t\}} - Q_{ij}(t)\right) \mathbb{1}_{\{J_{l-1} = i\}}.$$

By the Pyke and Schaufele CLT (see Theorem 5.4.2), and since $N_i(M)/M$ converges to $1/\mu_{ii}$ (a.s.), we get the desired result.

(iii) By the Markov renewal equation, we see that

$$\begin{aligned}
M^{1/2} \left[\widetilde{\Psi}_{ij}(t, M) - \Psi_{ij}(t)\right] &= M^{1/2} \left[\widetilde{\Psi}_{ij}(t) - (\widetilde{\Psi} * \Psi)_{ij}(t) + (\widetilde{\Psi} * \Psi)_{ij}(t) - \Psi_{ij}(t)\right] \\
&= M^{1/2} \left[[\widetilde{\Psi} * (I - \Psi)]_{ij}(t) + [(\widetilde{\Psi} - I) * \Psi]_{ij}(t)\right] \\
&= M^{1/2} \left[-(\widetilde{\Psi} * Q * \Psi)_{ij}(t) + (\widetilde{\Psi} * \widetilde{Q} * \Psi)_{ij}(t)\right] \\
&= M^{1/2} [\widetilde{\Psi} * \Delta Q * \Psi]_{ij}(t) \\
&= M^{1/2} [\widetilde{\Psi} * \Delta Q * \Psi * \Delta Q * \Psi]_{ij}(t) \\
&\quad + M^{1/2} [\Psi * \Delta Q * \Psi]_{ij}(t).
\end{aligned} \tag{5.15}$$

Since for all $i, k, l, r, v, w \in E$,

$$\sup_{s \leqslant t} \left[\widetilde{\Psi}_{ik}(\cdot, M) * \Psi_{lr}(\cdot) * \Psi_{vw}(\cdot)\right](s) \leqslant \left[\sum_{m=1}^{\infty} \beta_{ij}^{m}(t)\right] \cdot \Psi_{lr}(t) \cdot \Psi_{vw}(t) < +\infty,$$

we conclude by (iv), that the first term of 5.15 converges in probability to zero as $M$ tends towards infinity.

The last term can be written as follows:

$$\begin{aligned}
&M^{1/2}[\mathbf{\Psi} * \Delta\mathbf{Q} * \mathbf{\Psi}]_{ij}(t) \\
=&M^{1/2} \sum_{l=1}^{s} \sum_{r=1}^{s} \left(\Psi_{il} * (\widetilde{\mathbf{Q}}(\cdot, M) - \mathbf{Q})_{lr} * \Psi_{rj}\right)(t) \\
=&M^{1/2} \sum_{l=1}^{s} \sum_{r=1}^{s} \left(\Psi_{il} * \widetilde{Q}_{lr}(\cdot, M) * \Psi_{rj}\right)(t) - \sqrt{M} \sum_{l=1}^{s} \sum_{r=1}^{s} \left(\Psi_{il} * Q_{lr} * \Psi_{rj}\right)(t) \\
=&\frac{1}{M^{1/2}} \sum_{n=1}^{N(M)} \sum_{l=1}^{s} \frac{M}{N_l(M)} \sum_{r=1}^{s} \left[\left(\Psi_{il} * \mathbb{1}_{\{J_{n-1}=l, J_n=r, X_n=\cdot\}} * \Psi_{rj}\right)(t)\right. \\
&\left. - \left(\Psi_{il} * Q_{lr} \mathbb{1}_{\{J_{n-1}=l\}} * \Psi_{rj}\right)(t)\right].
\end{aligned}$$

Since $N_l(M)/M \xrightarrow[M \to \infty]{\text{a.s.}} 1/\mu_{ll}$, using Slutsky's Theorem we obtain that $M^{1/2}\left[\widetilde{\Psi}_{ij}(t, M) - \Psi_{ij}(t)\right]$ has the same limit in distribution as

$$\begin{aligned}
&\frac{1}{M^{1/2}} \sum_{n=1}^{N(M)} \sum_{l=1}^{s} \mu_{ll} \sum_{r=1}^{s} \left[\left(\Psi_{il} * \mathbb{1}_{\{J_{n-1}=l, J_n=r, X_n=\cdot\}} * \Psi_{rj}\right)(t)\right. \\
&\left. - \left(\Psi_{il} * Q_{lr} \mathbb{1}_{\{J_{n-1}=l\}} * \Psi_{rj}\right)(t)\right] \\
=&\sqrt{\frac{N(M)}{M}} \frac{1}{\sqrt{N(M)}} \sum_{n=1}^{N(M)} f(J_{n-1}, J_n, X_n)
\end{aligned}$$

where the random variables $f(J_{n-1}, J_n, X_n)$ are defined by

$$\begin{aligned}
f(J_{n-1}, J_n, X_n) :=& \sum_{l=1}^{s} \mu_{ll} \sum_{r=1}^{s} \left[\left(\Psi_{il} * \mathbb{1}_{\{J_{n-1}=l, J_n=r, X_n=\cdot\}} * \Psi_{rj}\right)(t)\right. \\
&\left. - \left(\Psi_{il} * Q_{lr} \mathbb{1}_{\{J_{n-1}=l\}} * \Psi_{rj}\right)(k)\right].
\end{aligned}$$

By Lemma 3.1.2, we deduce that the second term of 5.15 converges in law to a normal random variable with mean zero and variance $\sigma_{ij}^2(t)$.

(iv) Let define $\Delta P_{ij} = \left(\widetilde{P}_{ij}(t, M) - P_{ij}(t)\right)$, $\Delta Q_{ij} = \left(\widetilde{Q}_{ij}(t, M) - Q_{ij}(t)\right)$, from 3.7, we have

$$\begin{aligned}
M^{1/2}\Delta P_{ij} =& M^{1/2}\left[\widetilde{\Psi}_{ij} * (I - diag(\widetilde{Q}\mathbf{1}))_{jj} - \Psi_{ij} * (I - diag(Q\mathbf{1}))_{jj}\right] \\
=& M^{1/2}\left[\left(\widetilde{\Psi}_{ij} - \Psi_{ij}\right) * (I - diag(Q\mathbf{1}))_{jj}\right. \\
&- \Psi_{ij} * diag([\widetilde{Q} - Q]\mathbf{1})_{jj} \\
&\left. - \left(\widetilde{\Psi}_{ij} - \Psi_{ij}\right) * diag([\widetilde{Q} - Q]\mathbf{1})_{jj}\right].
\end{aligned} \tag{5.16}$$

Based on Lemma 2 in Ouhbi and Limnios [1997a], $\sqrt{M}\left(\widetilde{\Psi}_{ij}-\Psi_{ij}\right) * \operatorname{diag}\left([\widetilde{\mathbf{Q}}-\mathbf{Q}]\mathbf{1}\right)_{jj}$ converges in probability to zero as $M \to \infty$. Consequently, $M^{1/2}\Delta P_{ij}(t,M)$ has the same limit in distribution as

$$M^{1/2}\left[\left(\widetilde{\Psi}_{ij}-\Psi_{ij}\right) * (I - diag(Q\mathbf{1}))_{jj} - \Psi_{ij} * diag([\widetilde{Q}-Q]\mathbf{1})_{jj}\right].$$

From Theorem 3 in Ouhbi and Limnios [1997a], it has the same limit as

$$\begin{aligned}
&M^{1/2}\left[\left(1-\sum_{m=1}^{s} Q_{jm}\right) * \left(\sum_{n=1}^{s}\sum_{k=1}^{s} B_{inkj} * \Delta Q_{nk}\right) - \Psi_{ij} * \left(\sum_{k=1}^{s}\Delta Q_{jk}\right)\right] \\
&= \sum_{n=1}^{s}\sum_{k=1}^{s} M^{1/2}\left[\left(1-\sum_{m=1}^{s} Q_{jm}\right) * B_{inkj} * \Delta Q_{nk}\right] - \sum_{k=1}^{s} M^{1/2}\Psi_{ij} * \Delta Q_{jk}.
\end{aligned}$$

Let $f$ be a real function defined on $\mathbf{E}\times\mathbf{E}\times\mathbb{R}_+$ by

$$\begin{aligned}
f(r,m,x) = &\left[\left(1-\sum_{l=1}^{s} Q_{jl}\right) * B_{inkj} - \boldsymbol{\Psi}_{ij}1_{\{n=j\}}\right] \times \\
&\times \mathbf{1}_{\{r=n\}}\left(\mathbf{1}_{\{m=k,x\leqslant t\}} - Q_{nk}\right).
\end{aligned}$$

Therefore, $M^{1/2}\Delta P_{ij}(t,M) = M^{1/2}W_f(t)$, where

$$W_f(t) = \sum_{n=1}^{s}\sum_{k=1}^{s} \frac{\sum_{l=1}^{N_n} f\left(J_{l-1}, J_l, X_l\right)}{N_n}.$$

We obtain the desired findings using Pyke and Schaufele's (Pyke and Schaufele [1964]) central limit theorem 5.4.2.

□

**Proof of Lemma 4.2.1.** For all $i \in E$, we have:

(i) From the law of large numbers for Markov chains (cf. Meyn and Tweedie [1993]), we obtain

$$\frac{N_i(M)}{N(M)} \xrightarrow{\text{a.s.}} \nu_i \quad \text{as} \quad M \to \infty.$$

(ii) Hence, the consistency of the estimator is established. The asymptotic normality of the proposed estimator can be derived using the central limit theorem for Markov chains and Anscombe's theorem.

From the above lemma and the boundedness of $\widetilde{\nu}_i(M)$, it also follows that

$$\mathbb{E}[\widetilde{\nu}_i(M)] \to \nu_i.$$

It is worth noting that if the Markov chain is stationary, i.e., with initial distribution $\mu = \nu$, then the proposed empirical estimator coincides with the maximum likelihood estimator of the stationary distribution. Furthermore, the same asymptotic properties hold. For more details, see Limnios et al. [2005].

□

**Proof of Proposition 4.2.1.** We have

1. The strong consistency of the empirical estimator $\widetilde{\mu}^*_{ii}(M)$ has been established (see Limnios and Oprisan [2001]).

2. To demonstrate asymptotic normality, we use the delta method, since the mean recurrence time in state $i \in E$ can be expressed as a function of the stationary distribution in the same state. Based on $\mu^*_{ii} = \frac{1}{\nu_i}$. Consequently, we have

$$\sqrt{M}\,(\widetilde{\mu}^*_{ii}(M) - \mu^*_{ii}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{1}{\nu_i^4}\sigma^2_{\nu_i}\right) \quad \text{as} \quad M \to \infty,$$

and Proposition 4.2.1 is proven.

□

**Proof of Lemma 4.2.2.**

1. For any $i \in E$, we have

$$\widetilde{m}_i(M) = \frac{1}{N_i(M)} \sum_{l=1}^{N_i(M)} X_{il} = \sum_{j \in E} \frac{N_{ij}(M)}{N_i(M)} \frac{1}{N_{ij}(M)} \sum_{l=1}^{N_{ij}(M)} X_{ijl},$$

where $X_{ijl}$ is the $l$th sojourn time in state $i$ before going to state $j$. We have that

$$\frac{1}{N_{ij}(M)} \sum_{l=1}^{N_{ij}(M)} X_{ijl} \underset{M\to\infty}{\xrightarrow{a.s.}} \mathbb{E}[X_{ij}].$$

We get that the estimator $\widetilde{m}_i(M)$ converges almost surely, as $M \longrightarrow \infty$, to

$$\begin{aligned} \sum_{j\in E} p_{ij}\mathbb{E}[X_{ij}] &= \sum_{j\in E} p_{ij} \int_0^\infty t dF_{ij}(t) \\ &= \sum_{j\in E} p_{ij} \int_0^\infty \frac{t}{p_{ij}} dQ_{ij}(t) \\ &= m_i. \end{aligned}$$

Hence, $\widetilde{m}_i(M)$ is strongly consistent, as $M$ tends to infinity.

2. To prove the normality of the sojourn time estimator, we apply Anscombe's central limit Theorem 3.1.2,

$$\sqrt{N_i(M)} \left( \frac{1}{N_i(M)} \sum_{l=1}^{N_i(M)} X_{il} - m_i \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_i^2), \quad \text{as} \quad M \longrightarrow \infty,$$

where $\sigma_i^2 := var(X_{i,1}) := \int_0^\infty (t - m_i)^2 dH_i(t)$.

From Lemma 3.1.3, we have for any $i \in E$, $N_i(M)/M \underset{M\to\infty}{\xrightarrow{a.s.}} 1/\mu_{ii}$, and hence

$$\sqrt{M} \left( \frac{1}{N_i(M)} \sum_{l=1}^{N_i(M)} X_{il} - m_i \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_i^2), \quad \text{as} \quad M \longrightarrow \infty.$$

□

**Proof of Theorem 4.2.1.**

1. Using the results of Lemma 4.2.1 and Lemma 4.2.2, we obtain
$$\max_{i\in E}|\widetilde{\pi}_i(M)-\pi_i| \xrightarrow[M\to\infty]{a.s.} 0.$$
2. For any fixed arbitrary state $i$, we can assume that the initially visited state $J_0$ is $i$. Let $S_i(M)$ denote the total time spent by the SMP in state $i$ up to time $M$, without considering the last censored time $u_M := M - S_{N(M)}$, i.e.,
$$S_i(M)=\sum_{l=1}^{N_i(M)} X_{il} = \widetilde{m}_i(M)N_i(M).$$
Similarly, let $S_i^*(M)$ denote the total time spent by the SMP in state $i$ up to time $M$, considering the last censored time, i.e.,
$$S_i^*(M)=S_i(M)+u_M\mathbb{1}_{\{J_N(M)=i\}}.$$
From Equations (4.7) and (4.12), we obtain
$$\frac{\widetilde{\nu}_j(M)}{\sum_{i\in E}\widetilde{\nu}_i(M)\widetilde{m}_i(M)}=\frac{N_j(M)}{\sum_{i\in E}\sum_{l=1}^{N_i(M)} X_{il}}=\frac{N_j(M)}{S_{N(M)}}.$$
Consequently, we have
$$\widetilde{\pi}_j(M)=\frac{N_j(M)\widetilde{m}_j(M)}{S_{N(M)}}=\frac{N_j(M)\widetilde{m}_j(M)}{M-u_M}=\frac{S_j(M)}{M-u_M}.$$
Furthermore,
$$\begin{aligned}\sqrt{M}\,[\widetilde{\pi}_j(M)-\pi_j] &= \sqrt{M}\left[\frac{S_j(M)}{M-u_M}-\pi_j\right]=\sqrt{M}\left[\frac{S_j(M)/M}{1-u_M/M}-\pi_j\right]\\ &= \sqrt{M}\left[\frac{S_j(M)}{M}-\pi_j\right]+\frac{S_j(M)}{M}\frac{u_M}{\sqrt{M}}(O_p(1)).\end{aligned}$$
Since
$$\frac{S_j(M)}{M}=\widetilde{m}_j(M)\frac{N_j(M)}{M}\xrightarrow[M\to\infty]{a.s.}\frac{m_j}{\mu_{jj}}<\infty.$$
Thus
$$\sqrt{M}\,[\widetilde{\pi}_j(M)-\pi_j]=\sqrt{M}\left[\frac{S_j^*(M)}{M}-\pi_j\right]-\frac{u_M}{\sqrt{M}}\mathbb{1}_{\{J_N(M)=j\}}+\frac{S_j(M)}{M}\frac{u_M}{\sqrt{M}}(O_p(1)).$$
Hence, $u_M\mathbb{1}_{\{J_N(M)=j\}}<\infty$ a.s., and therefore $\frac{u_M}{\sqrt{M}}\mathbb{1}_{\{J_N(M)=j\}}\xrightarrow[M\to\infty]{a.s.} 0.$
By using Slutsky's theorem, $\sqrt{M}\,[\widetilde{\pi}_j(M)-\pi_j]$ and $\sqrt{M}\left[\frac{S_j^*(M)}{M}-\pi_j\right]$ have the same limit in distribution. Finally, by applying Taga's Theorem 3.1.7, we obtain the desired result.

□

**Proof of Theorem 5.2.1.** Based on Theorem 7 of Ouhbi and Limnios [1999] (see also Limnios and Ouhbi [2003]), we have
$$\sup_{t\in[0,M]}\left|\widetilde{P}_{ij}(t,M)-P_{ij}(t)\right|\xrightarrow{a.s.}0,\quad M\to\infty.$$

According to the inequality

$$\int_0^t \left|\widetilde{P}_{ij}(s,M) - P_{ij}(s)\right| ds \leq t \sup_{s\in[0,t]} \left|\widetilde{P}_{ij}(s,M) - P_{ij}(s)\right|.$$

Using relation (5.4) and the finiteness of $E$, the desired result follows.

□

**Proof of Theorem 5.2.2.** Let $I_j(t) = \int_0^t \overline{H}_j(u)du$ and $\widetilde{I}_j(t,M) = \int_0^t \widetilde{\overline{H}}_j(u,M)du$. Clearly,

$$\begin{aligned}
\sqrt{M}(\widetilde{\overline{\Phi}}(t,M) - \overline{\Phi}(t)) &= \sum_{i=1}^{s}\sum_{j\in E} \alpha_i L(j)\sqrt{M}\left[\left(\widetilde{I}_j * \widetilde{\Psi}_{ij}\right)(t,M) - \left(I_j * \Psi_{ij}\right)(t)\right] \\
&= \sum_{i=1}^{s}\sum_{j\in E} \alpha_i L(j)\sqrt{M}\left[\left(\widetilde{I}_j - I_j\right) * \left(\widetilde{\Psi}_{ij} - \Psi_{ij}\right)(t)\right. \\
&\quad \left. + \left(\widetilde{I}_j - I_j\right) * \Psi_{ij}\right)(t) + I_j * \left(\widetilde{\Psi}_{ij} - \Psi_{ij}\right)(t)\Big]
\end{aligned} \tag{5.17}$$

From Meyer [1980], the first term on the right-hand side of the first term of (5.17) converges to zero. Then, $\sqrt{M}(\widetilde{\overline{\Phi}}(t,M) - \overline{\Phi}(t))$ has the same limit in distribution as

$$\begin{aligned}
&\sum_{i=1}^{s}\sum_{j\in U} \alpha_i L(j)\sqrt{M}\left[\frac{1}{N_j(M)}\sum_{l=1}^{N_j(M)} \left(\left(L(j)\left(X_{jl}\wedge t\right) - I_j\right) * \Psi_{ij}\right)(t)\right. \\
&+ \left.\left(\sum_{k=1}^{s}\sum_{r=1}^{s} I_j * \Psi_{rk} * \Psi_{rj}\right) * \left(\widetilde{Q}_{kr} - Q_{kr}\right)(t)\right] \\
&= \sum_{i=1}^{s}\sum_{j\in U} \alpha_i L(j)\sqrt{M}\left[\frac{1}{N_j(M)}\sum_{l=1}^{N_j(M)} \left(\left(X_{jl}\wedge t - I_j\right) * \Psi_{ij}\right)(t)\right. \\
&+ \left.\left(\sum_{k=1}^{s}\sum_{r=1}^{s} I_j * \Psi_{ik} * \Psi_{rj}\right) * \left(\frac{1}{N_k(M)}\sum_{l=1}^{N_k(M)} \left(\mathbb{1}_{\{J_{l+1}=r, X_l\leq\cdot\}} - Q_{kr}\right)(t)\right)\mathbb{1}_{\{J_l=k\}}\right],
\end{aligned}$$

equivalently,

$$\begin{aligned}
&\sum_{k=1}^{s}\sum_{r=1}^{s}\frac{\sqrt{M}}{N_k(M)}\sum_{l=1}^{N_k(M)}\left[\mathbb{1}_{\{J_l=k, k\in U, J_{l+1}=r\}}\left(\left(X_l\wedge t - I_k\right) * A_k\right)(t)\right. \\
&+ \left.\left(B_{kr} * \left(\mathbb{1}_{\{J_{l+1}=r, X_l\leq\cdot\}} - Q_{kr}\right)(t)\right)\mathbb{1}_{\{J_l=k\}}\right].
\end{aligned}$$

Since $\frac{M}{N_k(M)} \to \mu_{kk}$, when $M$ goes to infinity, we can consider the function

$$\begin{aligned}
f\left(J_l, J_{l+1}, X_n\right) = {}& \mu_{kk}A_k * \left(\left(X_n\wedge t\right) - I_k\right)\mathbb{1}_{\{J_l=k, k\in U, J_{l+1}=r\}} \\
&+ \mu_{kk}B_{kk} * \left(\mathbb{1}_{\{J_{l+1}=r, X_l\leq\cdot\}} - Q_{kj}\right)(t)\mathbb{1}_{\{J_l=k\}}.
\end{aligned}$$

By the central limit theorem for semi-Markov processes (see Pyke and Schaufele [1964]), applied to this function, we derive the desired result.

□

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