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Digital control systems

Presented by:

Dr. BENSAAD ABDELLAH

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Preface

During the past two decades, the control theory has experienced rapid expansion due to the challenges of the stringent requirements posed by modern systems. Control systems represent one of the most important fields of applied science and engineering because control is today typically related to the development of all forms of technology. Their performance analysis is mainly carried out with microcontrollers or microprocessors, and several procedures, described by analogue systems, can be controlled using digital systems.

By definition, computers are digital systems. Hence, all input data streams are in a digital form, that is, digital signals. However, various signals in the natural world are continuous and must be transformed into digital signals before being processed by computer systems. The basic sampling mode of operation is necessary to transform an analogue signal into a discrete one. Several methods have emerged to study and analyze digital control systems. For a thorough understanding of these methods, we need to have adequate knowledge of control theory, state-space methods, and conventional frequency-domain methods. In order to emphasize the fundamental ideas involved and steer free of excessively mathematical arguments, the theoretical background materials for the study and analysis of control systems are provided.

The present textbook is dedicated to digital control systems and state variable methods. It aims to provide the reader with theoretical and applied scientific knowledge regarding the broader field of digital control systems. It is designed for Master of Electronics students in digital control systems to prepare them for advanced control methods studies developed during the past two decades.

Contents

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Parameters:

A	State matrix
B	Input matrix,
C	Output matrix,
C_A	Controllability matrix,
D	feedforward matrix,
$\phi(k)$	State-transition matrix,
K	Gain matrix,
T	Transformation matrix,
T_w	Pulse width,
λ	Eigenvalues,
$\alpha_c(z)$	Closed-loop system characteristic polynomial,
$\alpha(z)$	Characteristic equation of the plant,

Variables:

f_s	Sampling frequency,
f_c	frequency band limited,
f_h	Highest frequency (Nyquist frequency),
$f_h(t)$	ZOH impulse response,
$h(t)$	Impulse response,
W	Setpoint,
X	Measure,
$x(n)$	Discrete input signal,

$x_a(t)$	Continuous-time signal
Y	Output,
$y(t)$	Continuous-time output signal,
$y(n)$	Discrete output signal,
$\varepsilon(t)$	System error,
T_s	Sampling period,

Transformations:

p	Laplace transform,
z	z-transform,
w	Bilinear transform,

Abreviations:

A/D	A/D
D/A	D/A
ZOH	ZOH
FOH	FOH
CCS	CCS
FTBF	FTBF
FTBO	FTBO
SISO	SISO
MIMO	MIMO
2-DOF	2-DOF

Chapter I: Sampled-Data Control Systems

I.1 Introduction

Before the advent of the digital computer, most mechanical and electrical control systems had previously been approximated as continuous time (or analogue) systems. Nowadays, control systems are invariably based on digital computers, which are inherently digital systems because they receive and generate electrical impulses rather than the continuous electrical signals on which binary arithmetic is based. Digital computers as part of control systems have several advantages, such as the simplicity and universality of implementing complex control laws by programming the computer, relative insensitivity to noise, and low-cost [4].

The use of digital processors in systems that operate in the continuous time domain requires that their input signals be discrete, which requires sampling of the signals used by the controller. This sampling may be an inherent property of the system. A system that inputs a continuous-time signal and outputs a digital signal is called an analog-to-digital converter (A/D converter). A system that converts a digital input to an analogue output is called a digital-to-analogue converter (D/A converter). Since digital computers only process digital input signals and produce digital outputs, when using digital computers in feedback control systems to control analogue systems, we must have both A/D and D/A converters, as shown in Fig I.1. The A/D conversion is based on two distinct processes sampling and holding while D/A conversion is a simple and continuous-time process. The sampling rate must not be lower than the bandwidth of the analogue signal. Otherwise, a distorted digital output will be produced. The A/D converter is therefore seen as a low-pass filter, with a cutoff frequency that is equivalent to the sampling rate. The cutoff frequency (sampling rate) should be much greater than the bandwidth of the signal we wish to pass through the filter in order to minimize signal distortion [2].

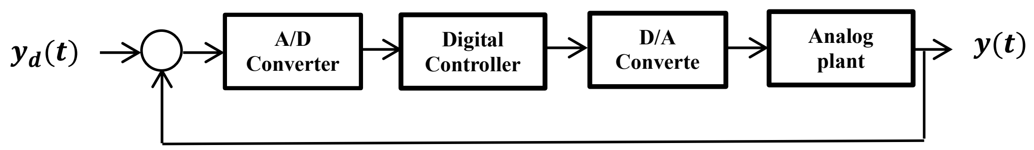


Fig I.1 A block diagram of a negative feedback control system with an analog plant and a digital controller.

I.2 Analog-to-Digital Conversion

Analog-to-digital conversion is a two-step process and is not instantaneous. There is a delay between the input analogue voltage and the output digital word. In an analogue-to-digital converter, the analogue signal is first converted to a sampled signal and then converted to a sequence of binary numbers, the digital signal. In order to avoid distortion, the sampling rate must to be at least twice the signal's bandwidth. The Nyquist sampling rate is the name given to this minimum sampling frequency [5].

In Fig I.2 (a), we start with the analog signal. In Fig I.2(b), we see the analog signal sampled at periodic intervals and held over the sampling interval by a zero-order sample-and-hold (ZOH) device that yields a staircase approximation to the analog signal. After sampling and holding, the analog-to-digital converter converts the sample to a digital number.

The dynamic range of the analog signal's voltage is divided into discrete levels, and each level is assigned a digital number. For example, in Fig I.2(b), the analog signal is divided into eight levels which are represented with a three-bit digital number. Thus, the dynamic range of the quantization levels is M=8 volts, where M is the maximum analog voltage. In general, for any system, this dynamic range is $M/2^n$ volts, where n is the number of binary bits used for the analog-to-digital conversion.

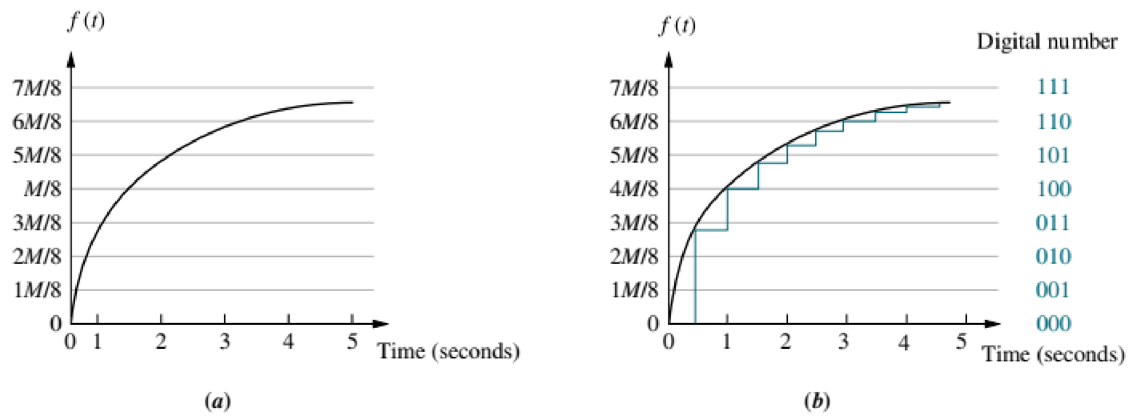


Fig I.2. Steps in analog to digital conversion: a. analog signal; b. analog signal after sample-and-hold [6].

I.2.1 Ideal periodic sampling of continuous-time signals

In the most common form of sampling, known as periodic or uniform sampling, a sequence of samples x(n) is obtained from a continuous-time signal $x_a(t)$ by taking values at equally spaced points in time. Periodic sampling is defined by the relation [6]:

$$x(n) = x_a(t)|_{t=nT_s} = x_a(nT_s) \quad \text{I.1}$$

where T_s, the fixed time interval between samples, is known as the sampling period. The reciprocal of the sampling period, $f_s = 1/T_s$, is called sampling frequency (when expressed in cycles per second or Hz) or sampling rate (when expressed in samples per second). The system that implements the operation is known as an ideal analog-to-digital converter (ADC) or ideal sampler and is depicted in Fig I.3.

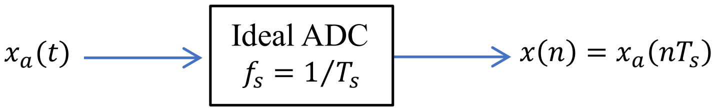


Fig I.3. Representation of the ideal analog-to-digital converter (ADC) or ideal sampler.

By definition, ideal sampling means instantaneous sampling, where each sample is measured with infinite accuracy. The main difference between an ideal ADC and a practical ADC is the finite number of bits (typically 12 or 16 bits) used to represent the value of each sample.
Sampling a signal $x_a(t)$ is equivalent to multiplying it with a train pulse or Dirac comb $\delta_{T_s}(t)$ of period T_s

$$x_s(t) = x_a(t).\delta_{T_s}(t) \quad \text{I.2}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_a(nT_s)\,\delta(t - nT_s) \quad \text{I.3}$$

with $\delta_{T_s}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$

The Nyquist frequency

The highest frequency f_h, in Hz, present in a bandlimited signal $x_a(t)$ is called the Nyquist frequency.

The Nyquist rate

The minimum sampling frequency required to avoid overlapping bands is $2f_h$, which is called the Nyquist rate.

The folding frequency

The actual highest frequency that the sampled signal $x(n)$ contains is $f_s/2$, in Hz, and is termed as the folding frequency.

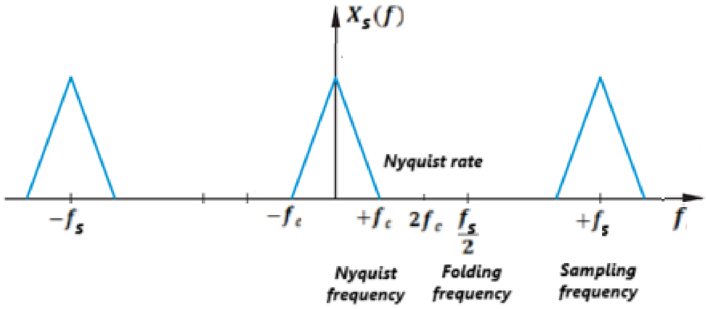


Fig I.4. Frequency terminology related to sampling operation.

***Sampling theorem* (*Shannon's theorem*)**

The theorem states that continuous-time signal with a frequency band limited to f_c can only be reconstructed from samples $x_s(t)$ if these have been taken with a period of

$$T_s \leq \frac{T_c}{2} \qquad \text{I.4}$$

Or

$$f_s \geq 2f_c \qquad \text{I.5}$$

I.2.2 Spectrum of the sampled signal

Using the Fourier transform applied to sampled signals, we show that the spectrum of the signal $x_s(t)$ consists of a sequence of replicas of the spectrum of $x_a(t)$ shifted with a periodicity of $f_s = 1/T_s$.

The Fourier transform of the Dirac comb is a frequency Dirac comb:

$$TF\{\delta_{T_s}(t)\} = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty}\delta(f - nf_s) \qquad \text{I.6}$$

$$X_s(t) = TF\{x_s(t)\}$$

$$X_a(t) = TF\{x_a(t)\}$$

$$x_s(t) \overset{TF}{\rightarrow} x_a(t) * TF\{\delta_{T_s}(t)\} \qquad \text{I.7}$$

Thus

$$X_s(f) = f_s\sum_{n=-\infty}^{+\infty}X_a(f - nf_s) \qquad \text{I.8}$$

The spectrum of the sampled signal is simply the periodic reproduction of the spectrum with period f_s.

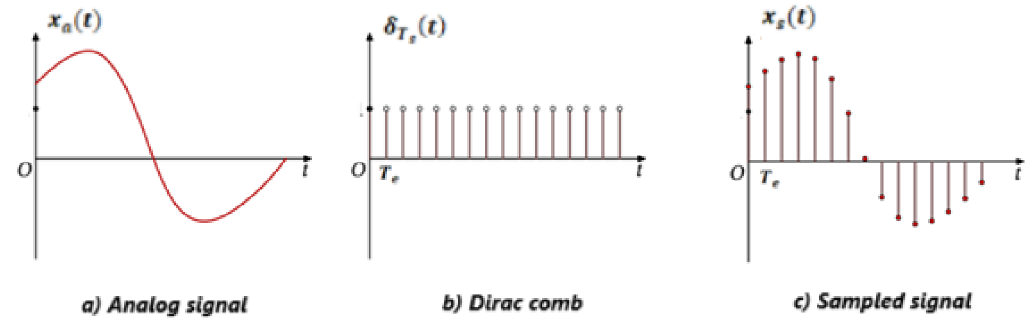


Fig I.5. Signal sampling process is seen as the multiplication of the analog signal with a Dirac comb.

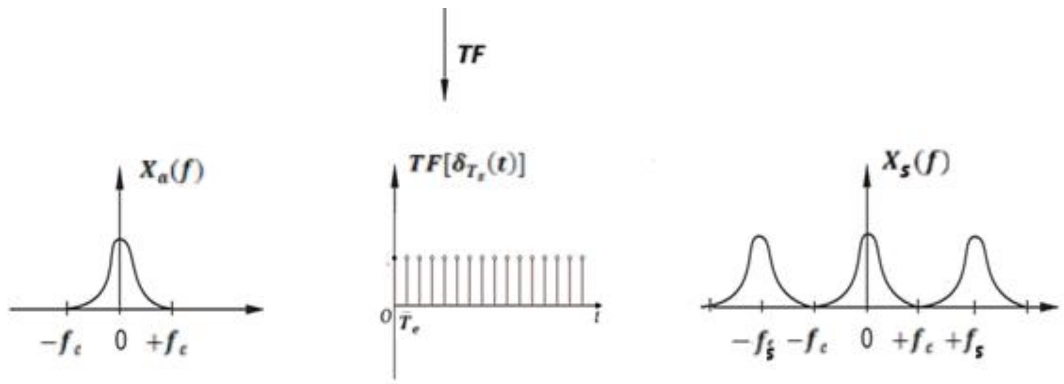


Fig I.6. Analog signal spectrum, b) Dirac comb spectrum, c) Sampled signal spectrum.

In the case where $f_s \geq 2f_c$, the $X_s(f)$ spectra do not overlap (or fold). If $f_s \leq 2f_c$, in this case, the spectra overlap, causing problems in the reconstruction of $x_a(t)$.

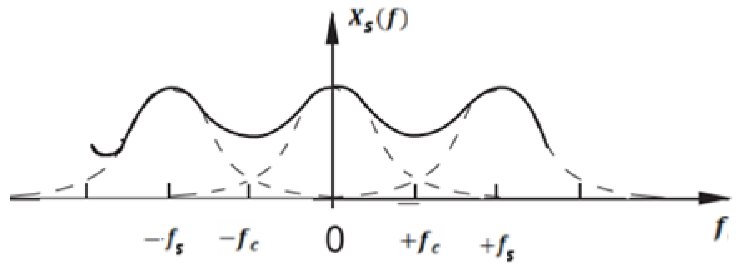


Fig I.7. Spectral overlapping for a very high sampling value.

I.3 Reconstruction of the analog signal $x_a(t)$ from its samples

Sampling has introduced a periodicity to the spectrum in frequency space; restoring the original signal means removing this periodicity, i.e. eliminating the image bands, an operation that can be performed using a low-pass filter.

To reconstruct $x_a(t)$, we can use an ideal low-pass filter with a cut-off frequency :

$$f_c = f_s/2 \quad \text{I.9}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_a(nT_s)\,\delta(t - nT_s) \quad \text{I.10}$$

Hence,

$$X_s(f) = X_a(f).\prod_{2f_c}(f) \xrightarrow{TFI} x_a(t) = x_s(t) * f(t) \quad \text{I.11}$$

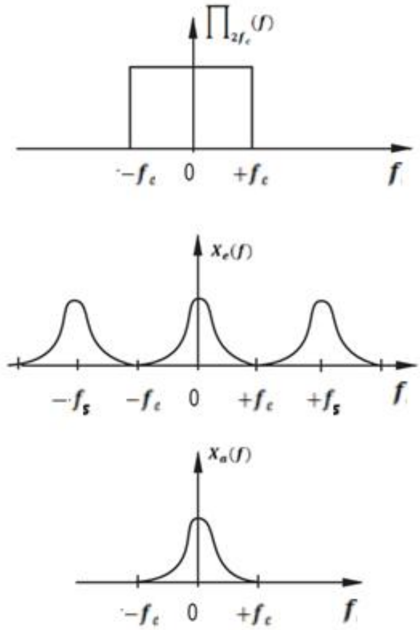


Fig I.8. Spectrum of the recovered analog signal.

where $f(t)$ is the impulse response of the ideal low-pass filter.

$$f(t) = \frac{1}{T_s}\, sinc(\pi \frac{t}{T_s}) \quad \text{I.12}$$

Hence, the filter output signal, $x_a(t)$ corresponds to the convolution product of the sequence $x_a(nT_s)$ by the function $f(t)$, i.e.

$$x_a(t) = \sum_{n=-\infty}^{+\infty} x_a(nT_s)\, sinc(\pi \frac{(t-nT_s)}{T_s}) \quad \text{I.13}$$

Note

The impulse response of the ideal low-pass filter $f(t)$ is not causal. Therefore, the ideal low-pass filter is not practically feasible. Hence, reconstruction using the ideal low-pass filter is only approximate.

I.4 Modeling the Sampler

At this point, our objective is to derive a mathematical model for the digital computer as represented by a sampler and zero-order hold. Our goal is to represent the computer as a transfer function similar to that of any subsystem [5].

Consider the models for sampling shown in Fig I.9 and Fig I.10. The model in Fig I.9 is a switch turning on and off at a uniform sampling rate.

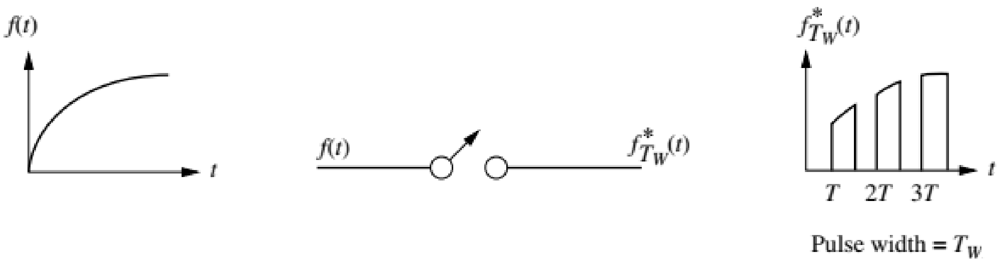


Fig I.9. Switch opening and closing uniform-rate sampling.

In Fig I.10, sampling can also be considered to be the product of the time waveform to be sampled, $f(t)$, and a sampling function, $s(t)$. If s(t) is a sequence of pulses of width T_w, constant amplitude, and uniform rate as shown, the sampled output, $f_w^*(t)$, will consist of a sequence of sections of $f(t)$, at regular intervals. This view is equivalent to the switch model of Fig I.9.

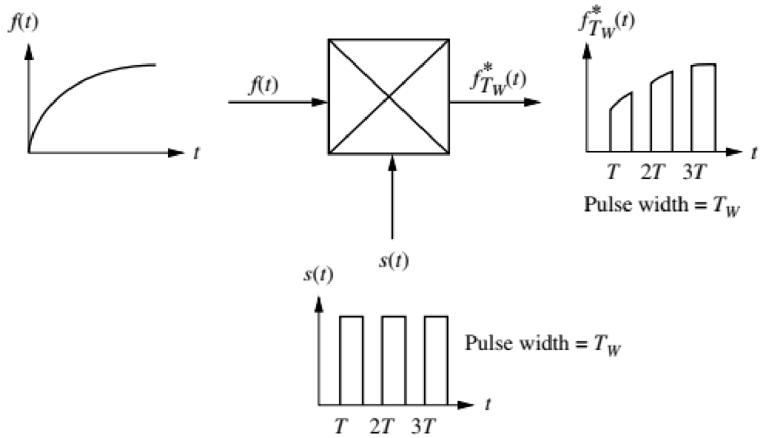


Fig I.10. Sampling as a product of time waveform and sampling waveform [3].

We can now write the time equation of the sampled waveform, $f_w^*(t)$. Using the model shown in Fig I.10, we have [5]:

$$f_w^*(t) = f(t)s(t) = f(t)\sum_{k=-\infty}^{+\infty} u(t-kT) - u(t-kT-T_w) \qquad \text{I.14}$$

where k is an integer between $-\infty$ and $+\infty$, T is the period of the pulse train, and T_w is the pulse width.

I.4.1 Modeling the Zero-Order Hold

The final step in modeling the digital computer is modeling the zero-order hold that follows the sampler. Fig I.11 shows a block diagram of the ideal sampling and holding processes with a zero-order hold and their input and output signals.

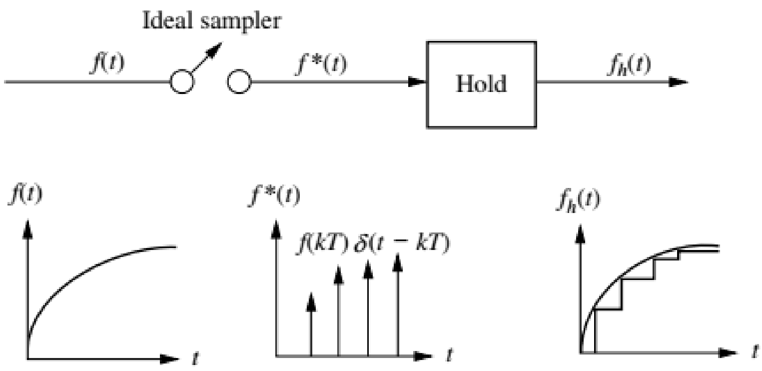


Fig I.11. Schematic diagram and input, sampled, and output signals of an analog-to-digital (A/D) converter with an ideal sampler and a zero-order hold.

Since the input to the ZOH is a series of impulses, $f^*(t)$ while the output, $f_h(t)$ is a series of steps with amplitude $f(kT)$, it follows that the impulse response, $h(t)$, of the ZOH must be a step that starts at t = 0 and ends at t = T, and is represented as follows [11]:

$$h(t) = u_s(t) - u_s(t - T) \qquad \text{I.15}$$

Then, taking the Laplace transform, the transfer function of the ZOH is given by:

$$G(p) = \frac{1 - e^{-pT}}{p} \qquad \text{I.16}$$

Note that we have used a special property of the Laplace transform called the time-shift property, which is denoted by

$$L\{y(t - T)\} = e^{-pT} L\{y(t)\} \qquad \text{I.17}$$

The ideal sampler is regarded as the sampler that produces a series of impulses, $f^*(t)$, weighted by the input value, $f(kT)$ as follows:

$$f^*(t) = \sum_{k=0}^{+\infty} f(kT)\delta(t - kT) \qquad \text{I.18}$$

The ideal sampler output, $f^*(t)$ does not depend upon T_w, which is regarded as a characteristic of the real sampler described by

$$f^*_{Tw}(t) = \sum_{k=0}^{+\infty} f(kT)T_w\, \delta(t - kT) = T_w \sum_{k=0}^{+\infty} f(kT)\, \delta(t - kT) \qquad \text{I.19}$$

Equation denotes the fact that the sampled signal, $f^*_{Tw}(t)$, is obtained by sampling $f(t)$ at the sampling rate, $1/T$, with pulses of duration t_w. The ideal sampler is thus a real sampler with $T_w = 1$ second.

Similarly, taking the Laplace transform of Eq. I.18, we can write the Laplace transform of the ideally sampled signal, $F^*(p)$, as follows:

$$F^*(p) = \sum_{k=0}^{+\infty} f(kT)e^{-kTp} \qquad \text{I.20}$$

I.5 The z-transform

The z-transform is an important tool in the analysis and design of discrete-time systems. It simplifies the solution of discrete-time problems by converting LTI difference equations to algebraic equations and convolution to multiplication. Thus, it plays a role similar to that served by Laplace transforms in continuous-time problems [1].

If we define a variable z such that $z = e^{Tp}$, the Equation below:

$$F^*(p) = \sum_{k=0}^{+\infty} f(kT)e^{-kTp} \qquad \text{I.21}$$

Can be written as follows:

$$F(z) = \sum_{k=0}^{+\infty} f(kT)z^{-k} \qquad \text{I.22}$$

$F(z)$ is called the z-transform of $f(kT)$, and is denoted by $TZ\{f(kT)\}$.

I.5.1 z-Transform Properties

The properties of the z-transform have a very similar meaning to the properties of the Laplace transform. The most important properties of the z transform, commonly used for problem-solving, are presented below [3].

a. Time Shift

If $X(z)$ denotes the z-transform of $x(n)$ function, then the corresponding transform of $x(n-N)$ is given by $z^{-N}X(z)$. The time shift operation adds or subtracts the axes' origin or infinity from the region of convergence of $X(z)$.

$$x(n) \rightarrow X(z)$$
$$x(n-N) \rightarrow z^{-N}X(z) \qquad \text{I.23}$$

b. Linearity

Let $x(n)$ be a function, which arises from the linear combination of two functions $x_1(n)$ and $x_2(n)$ with regions of convergence Π_1 and Π_2, respectively. The region of convergence of $x(n)$ includes the intersection of $\Pi_1 \cap \Pi_2$.

$$x(n) \rightarrow X(z)$$
$$\alpha x_1(n) + \beta x_2(n) \leftrightarrow \alpha X_1(z) + \beta X_2(z) \qquad \text{I.24}$$

c. Time Reverse

If the z-transform of the $x(n)$ function is $X(z)$, with region of convergence Π, then the transform of $x(-n)$ is with region of convergence $1/\Pi$.

$$x(n) \rightarrow X(z)$$
$$x(-n) \rightarrow X(z^{-1}) \qquad \text{I.25}$$

e. Convolution

Let two functions $x_1(n)$ and $x_2(n)$ with corresponding z-transforms and regions of convergence $x_1(n) \leftrightarrow X_1(z)$ where $z \in \Pi_1$ and $x_2(n) \leftrightarrow X_2(z)$ where $z \in \Pi_2$. The transform for the signals' convolution $x_1(n)$ and $x_2(n)$ is given by:

$$x(n) \rightarrow X(z)$$
$$x_1(n) * x_2(n) \leftrightarrow X_1(z)X_2(z) \qquad \text{I.26}$$

where the region of convergence of $X(z)$ is identical or includes the intersection of regions of convergence of $X_1(z)$ and $X_2(z)$.

f. Differentiation in z-Domain

Let $X(z)$ be the transform of $x(n)$ function with region of convergence Π. Then,

$$x(n) \rightarrow X(z)$$

$$nx(n) \rightarrow -z\,\frac{dX(z)}{dz} \qquad \text{I.27}$$

with the same region of convergence.

g. Initial Value Theorem

For causal stable systems, it holds that:

$$\lim_{n\to 0} x(n) = \lim_{Z\to +\infty} X(Z) \qquad \text{I.28}$$

h. Final Value Theorem

$$\lim_{n\to \infty} x(n) = \lim_{Z\to 1}(z-1)X(Z) \qquad \text{I.29}$$

I.6 Inverse z-Transform

The implementation of z-transform results in the transportation from the discrete-time domain to z-domain. The opposite procedure is implemented with the aid of the inverse z-transform. The inverse z-transform is defined by:

$$\mathrm{x(n)} = Z^{-1}[X(z)] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\,dz \qquad \text{I.30}$$

where c is a closed contour within the region of convergence of $X(z)$ which includes the intersection of real and imaginary axes of the z-complex plane.

Because calculating the involved integral is quite cumbersome, it is usually done using tables, which provide the timing functions of basic complex functions. These tables generally cover only some cases; thus, other methods can be used for calculating the inverse z-transform.

I.6.1 Long division

This approach relates a time sequence to its z-transform directly. We first use long division to obtain as many terms as desired of the z-transform expansion; then we use the coefficients of the expansion to write the time sequence. The following two steps give the inverse z-transform of a function $F(z)$ [1]:

1. Using long division, expand F(z) as a series to obtain

$$F(z) = f_0 + f_1 z^{-1} + \cdots + f_i z^{-i} = \sum_{k=0}^{i} f_k z^{-k} \qquad \text{I.31}$$

2. Write the inverse transform as the sequence: $\{f_0, f_1, \dots, f_i, \dots\}$

The number of terms i obtained by long division is selected to yield a sufficient number of points in the time sequence.

Example

Obtain the inverse z-transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.2z + 0.1}$$

Solution

Long Division

$$\begin{array}{r l}
 & z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \ldots\ldots \\ \hline
z^2 + 0.2z + 0.1 \,\big) & z + 1 \\
 & \underline{z + 0.2 + 0.1z^{-1}} \\
 & \quad 0.8 - 0.10z^{-1} \\
 & \quad \underline{0.8 + 0.16z^{-1} + 0.08z^{-2}} \\
 & \quad -0.26z^{-1} - \ldots
\end{array}$$

Thus,

$$F(z) = 0 + z^{-1} + 0.8z^{-2} - 0.26z^{-3}$$
$$\{f_k\} = \{0, 1, 0.8, -0.26, \ldots\}$$

I.6.2 Partial fraction expansion

This method is almost identical to that used in inverting Laplace transforms. However, because most z-functions have the term z in their numerator, it is often convenient to expand $F(z)/z$ rather than $F(z)$. As with Laplace transforms, partial fraction expansion allows us to write the function as the sum of simpler functions that are the z-transforms of known discrete-time functions.

The procedure for inverse z-transformation is

1. Find the partial fraction expansion of $F(z)/z$ or $F(z)$.
2. Obtain the inverse transform $f(k)$ using the z-transform tables.

We consider three types of z-domain functions $F(z)$: functions with simple (non-repeated) real poles, functions with complex conjugate and real poles, and functions with repeated poles.

Case1 Simple real roots

The most convenient method to obtain the partial fraction expansion of a function with simple real roots is the method of residues. The residue of a complex function $F(z)$ at a simple pole z_i is given by

$$A_i = (z - z_i)F(z)|_{z \to z_i} \qquad \text{I.32}$$

This is the partial fraction coefficient of the i^th term of the expansion

$$F(z) = \sum_{i=1}^{n} \frac{A_i}{(z - z_i)} \qquad \text{I.33}$$

Because most terms in the z-transform tables include a z in the numerator, it is often convenient to expand $F(z)/z$ and then to multiply both sides by z to obtain an expansion whose terms have a z in the numerator. Except for functions that already have a z in the numerator, this approach is slightly longer but has the advantage of simplifying inverse transformation.

Example

Obtain the inverse z-transform of the function

$$F(z) = \frac{z + 1}{z^2 + 0.3z + 0.02}$$

Solution

It is instructive to solve this problem using two different methods. First, we divide by z; then we obtain the partial fraction expansion.

1. Partial Fraction Expansion Dividing the function by z, we expand as

$$\frac{F(z)}{z} = \frac{z + 1}{z(z^2 + 0.3z + 0.02)}$$
$$= \frac{A}{z} + \frac{B}{z + 0.1} + \frac{C}{z + 0.2}$$

where the partial fraction coefficients are given by

$$A = (z - 0)\frac{F(z)}{z}\bigg|_{z=0} = F(0) = 50$$
$$B = (z + 0.1)\frac{F(z)}{z}\bigg|_{z=-0.1} = -90$$

$$C = (z+0.2)\frac{F(z)}{z}\bigg|_{z=-0.2} = 40$$

Thus, the partial fraction expansion is

$$F(z) = \frac{50z}{z} - \frac{90z}{z+0.1} + \frac{40z}{z+0.2}$$

Table Lookup

$$f(k) = 50\delta(k) - 90(-0.1)^k + 40(-0.2)^k \qquad k \geq 0$$

Now, we solve the same problem without dividing by z.

1. Partial Fraction Expansion

We obtain the partial fraction expansion directly

$$F(z) = \frac{z+1}{(z^2+0.3z+0.02)}$$
$$= \frac{A}{z+0.1} + \frac{B}{z+0.2}$$

where the partial fraction coefficients are given by

$$A = (z+0.1)F(z)|_{z=-0.1} = 9$$
$$B = (z+0.2)F(z)|_{z=-0.2} = -8$$

Thus, the partial fraction expansion is

$$F(z) = \frac{9}{z+0.1} - \frac{8}{z+0.2}$$

2. Table Lookup

Standard z-transform tables do not include the terms in the expansion of $F(z)$. However, $F(z)$ can be written as

$$F(z) = \frac{9z}{z+0.1}z^{-1} - \frac{8z}{z+0.2}z^{-1}$$

Then we use the delay theorem to obtain the inverse transform

$$f(k) = 9(-0.1)^{k-1} - 8(-0.2)^{k-1} \qquad k \geq 1$$

Case 2 Repeated roots

For a function F(z) with a repeated root of multiplicity r, r partial fraction coefficients are associated with the repeated root. The partial fraction expansion is of the form:

$$F(z) = \frac{N(z)}{(z-z_1)^r \prod_{j=r+1}^{n}(z-z_j)} = \sum_{i=1}^{r} \frac{A_i}{(z-z_1)^{r+1-i}} + \sum_{j=r+1}^{n} \frac{A_i}{(z-z_j)} \qquad \text{I.34}$$

The coefficients for repeated roots are governed by:

$$A_{1,i} = \frac{1}{(i-1)} \frac{d^{i-1}}{dz^{i-1}} (z - z_1)F(z)|_{z\to z_1}, \qquad i = 1, 2, \dots, r \qquad \text{I.35}$$

The coefficients of the simple or complex conjugate roots can be obtained as before.

Example

Obtain the inverse z-transform of the function:

$$F(z) = \frac{1}{z^2(z-0.5)}$$

Solution

1. Partial Fraction Expansion Dividing by z gives:

$$\frac{F(z)}{z} = \frac{z+1}{z^3(z-0.5)}$$
$$= \frac{A}{z^3} + \frac{B}{z^2} + \frac{C}{z} + \frac{D}{z-0.5}$$

where

$$A = z^3 \frac{F(z)}{z}\bigg|_{z=0} = -2$$

$$B = \frac{1}{1}\frac{d}{dz} z^3 \frac{F(z)}{z}\bigg|_{z=0} = -4$$

$$C = \frac{1}{2}\frac{d^2}{dz^2} z^3 \frac{F(z)}{z}\bigg|_{z=0} = -8$$

$$D == (z-0.5)\frac{F(z)}{z}\bigg|_{z=0.5} = 8$$

Thus, we have the partial fraction expansion:

$$F(z) = \frac{8z}{z-0.5} - 2z^{-2} - 4z^{-1} - 8$$

2. Table Lookup

The z-transform tables and Definition yield:

$$f(k) = 8(0.5)^k - 2\delta(k-2) - 4\delta(k-1) - 8\delta(k) \qquad k \geq 0$$

I.6.3 Method of Complex Integration

This method is quite general and is used when one or more partial fractions of the expanded $F(z)$ are not included into the lookup tables of z-transform. This method relies on the definition formula of the inverse z-transform [3].

The use of the inverse z-transform equation defined by:

$$x(n) = z^{-1}[X(z)] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz \qquad \text{I.36}$$

requires the use of the residue theorem, which is given by:

$$\oint F(z)z^{n-1}dz = 2\pi j \sum residues\,[F(z)z^{n-1}] \qquad \text{I.37}$$

In the latter expression, also known as Cauchy's formula, Σ stands for the um of residues for the poles of F(z), which includes the c curve.

Combining the above expressions, we have that

$$f(n) = \sum residues\,[F(z)z^{n-1}] \qquad \text{I.38}$$

• If there is a simple first-order pole of $F(z)z^{n-1}$, then its residual is given by

$$F(z)z^{n-1}(z-\alpha)|_{z=\alpha} \qquad \text{I.39}$$

• If there is an m-order pole of $F(z)z^{n-1}$, then its residual is given by

$$\frac{1}{(m-1)}\frac{d^{m-1}}{dz^{m-1}}F(z)z^{n-1}(z-\alpha)^m|_{z\to z_1} \qquad \text{I.40}$$

	x[n]	X(z)	Π.Σ.
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{z}{z-1}$	$\|z\| > 1$
3	$-u[-n-1]$	$\frac{z}{z-1}$	$\|z\| < 1$
4	$\delta[n-m]$	z^{-m}	All z except 0 ($m > 0$) or ∞ ($m < 0$)
5	$a^n u[n]$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
6	$-a^n u[-n-1]$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
7	$na^n u[n]$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
8	$-na^n u[-n-1]$	$\frac{az}{(z-a)^2}$	$\|z\| < \|a\|$
9	$(n+1)a^n u[n]$	$\left[\frac{z}{z-a}\right]^2$	$\|z\| > \|a\|$
10	$(\cos \Omega n)u[n]$	$\frac{z^2(\cos\Omega)z}{z^2-(2\cos\Omega)z+1}$	$\|z\| > 1$
11	$(\sin \Omega n)u[n]$	$\frac{(\sin\Omega)z}{z^2-(2\cos\Omega)z+1}$	$\|z\| > 1$
12	$(r^n \cos \Omega n)u[n]$	$\frac{z^2-(r\cos\Omega)z}{z^2-(2r\cos\Omega)z+r^2}$	$\|z\| > r$
13	$(r^n \sin \Omega n)u[n]$	$\frac{(r\sin\Omega)z}{z^2-(2r\cos\Omega)z+r^2}$	$\|z\| < r$

$$X(z) = \sum_{-\infty}^{+\infty} u[n] \cdot z^{-n} = \sum_{n=0}^{+\infty} 1 \cdot z^{-n} = \sum_{n=0}^{+\infty} (z^{-1})^n = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Table I.1. The z-Transform for Elementary Functions.

I.7 Discretization Methods

The most important comparison points of the discretization methods to be mentioned are

- Ease of use
- Stability maintenance
- Impulse response maintenance
- Harmonic response maintenance

I.7.1 Impulse-Invariance Method or z-Transform Method

The equivalent discrete-time filter of the impulse response is the filter whose impulse response is identified with the impulse response of the continuous time filter $G(p)$, at time instances kT, $k = 0, 1, 2, 3, \ldots, T$ where T denotes the sampling period [3].

The impulse response in the z-domain is the inverse z-transform of the transfer function $G(z)$. While in the s-domain the impulse response is the inverse Laplace transform of the transfer function $G(p)$.

Consider the systems of Fig I.12.

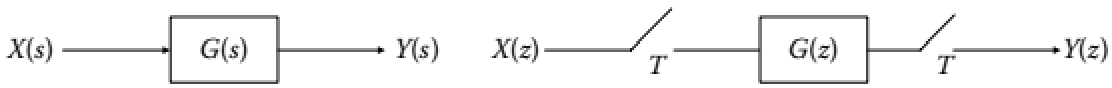


Fig I.12. Analog and discrete system.

The digital transfer function $G(z)$ is obtained from the related analog $G(p)$ by following these steps:

- From $G(p)$, we extract $g(t)$ in the time domain using the inverse z-transform.
- From $g(t)$, through discretization, we derive the function $g(kT)$, where T is the sampling interval.
- From $g(kT)$, using the z-transform, we obtain $G(z)$ in the z-domain.

Thus,

$$G(s) \xrightarrow{L^{-1}} g(t) \xrightarrow{t=kT} g(kT) \xrightarrow{ZT} G(z)$$

Or

$$G(z) = TZ[g(kT)]$$

Where

$$g(kT) = [TL^{-1}G(p)]_{t=kT}$$

Since the z-transform always projects a stable pole in the s-domain to a stable pole in the z-domain, we conclude that the discrete system will be stable if the original analog system is stable. Using this method, both the frequency and step responses are not preserved (frequency warping is observed due to overlap).

If the original analog system is stable, the discrete system will also be stable since the z-transform always projects a stable pole in the p-domain to a stable pole in the z-domain. Both the frequency and step responses are lost when using this method (overlap causes frequency warping).

I.7.2 Step-Invariance Method or z-Transform Method with Sample and Hold

This method aims to construct a discrete system $G(z)$ whose step response will consist of step response samples of the continuous system $G(p)$ at time instances kT, k=0,1,2,3…, where T represents the sampling period. Consider the system of Fig I.13.

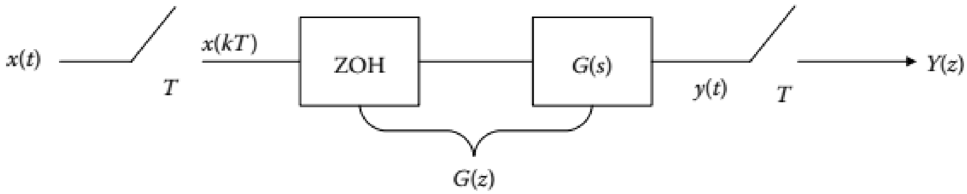


Fig I.13. Sampled data system using ZOH.

The zero-order hold (ZOH) circuit (filter) is used with the transfer function $G_h(p)$. Fig I.14 summarizes the ZOH operation, which is the preservation of the last sampled value of signal $f(t)$.

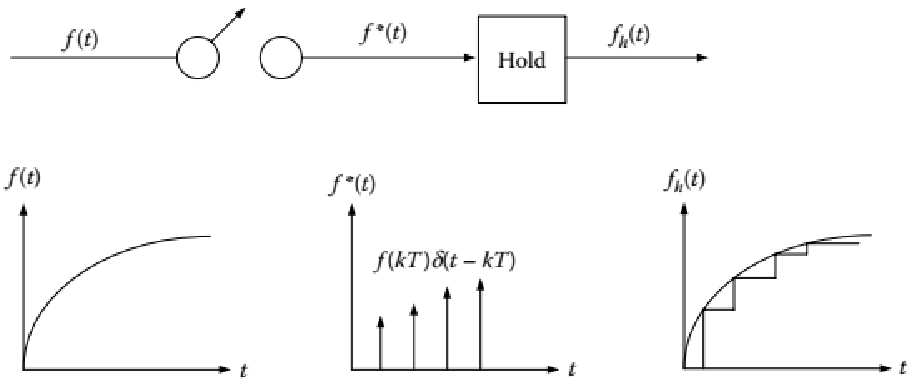


Fig I.14. Zero-order hold operation.

The impulse response, say $f_h(t)$, of the restraint network is a gate function, such that

$$f_h(t) = u(t) - u(t-T)$$

Or

$$F_h(p) = TL\{u(t) - u(t-T)\}$$
$$= TL\{u(t)\} - TL\{u(t-T)\}$$
$$= \frac{1}{p} - \frac{1}{p}e^{-Tp}$$

$$\Rightarrow F_h(p) = \frac{1-e^{-Tp}}{p}$$

Since the input is the impulse function, the restraint system transfer function is given by

$$G_h(p) = \frac{1-e^{-Tp}}{p} \qquad \text{I.41}$$

Furthermore, there are higher-order restraint filters (first, second, etc.), which are used in practice.

The digital transfer function $G(z)$ arises from the analog $G(p)$, using the expression (I.41)

$$G(z) = TZ[G_h(p).G(p)] = TZ[\frac{1-e^{Tp}}{p}G(p)]$$

$$\Rightarrow G(z) = (1-z^{-1})TZ\left(\frac{G(p)}{p}\right) \qquad \text{I.42}$$

If the original analog system $G(p)$ is stable then the equivalent discrete $G(z)$ obtained by the method of the invariance of the step response is stable. Using this method, neither the frequency (harmonic) nor the impulse responses are preserved.

I.7.2.1Transfer Function of First-Order Hold

Although we do not use first-order holds in control systems, it is worthwhile to see what the transfer function of first-order holds may look like. We show that the transfer function of the first-order hold may be given by:

$$G_{h1}(p) = \left(\frac{1-e^{-Tp}}{p}\right)\frac{Tp+1}{T} \qquad \text{I.43}$$

For example, the output of a First-Hrder-hold (FOH) filter is not partially stable, as in ZOH, but partially linear with a slope:

$$\frac{\{u(kT)-[u(k-1)T]\}}{T} \qquad \text{I.44}$$

and it has a transfer function given by:

$$y(kT+m) = u(kT) + \frac{\{u(kT)-[u(k-1)T]\}}{T} \qquad \text{I.45}$$

I.7.3 Backward Difference Method

One way of calculating the derivative $\dot{g}(t)$ at time instance $t = kT$ is by using the difference between the current and the previous sample divided by the sampling period, such that [3]:

$$\dot{g}(t) = \frac{dg(t)}{dt} = \frac{g(k)-g(k-1)}{T} \qquad \text{I.46}$$

The derivative of $\dot{x}(t)$ in s-domain is $pX(p)$, while in z-domain is

$$TZ[\dot{g}(t)] = TZ\left[\frac{g(k)-g(k-1)}{T}\right] = \frac{1-z^{-1}}{T}G(z) \qquad \text{I.47}$$

Comparing the above derivatives, the conversion of $G(p)$ in z-domain is presented as:

$$G(z) = G(p)|_{p=\frac{1-z^{-1}}{T}} \qquad \text{I.48}$$

$p = \frac{1-z^{-1}}{T}$ represents the projection from the p-domain to the z-domain, which is depicted in Fig I.15. This occurs by using the substitution $p = j\omega_s$ in the expression $z = (1/1 - pT)$. If we set $z = (1/1 - pT)$ where $p = \sigma + j\omega_s$, we get the same equation, and the σ values define points within the circle (1/2, 1/2). Using the backward difference method preserves the stability, but the harmonic response does not.

$$z = \frac{1}{1-pT}\bigg|_{p=j\omega_s} = \frac{1}{1-j\omega_s T} = \frac{1+j\omega_s T}{1+(\omega_s T)^2} = x + jy \qquad \text{I.49}$$

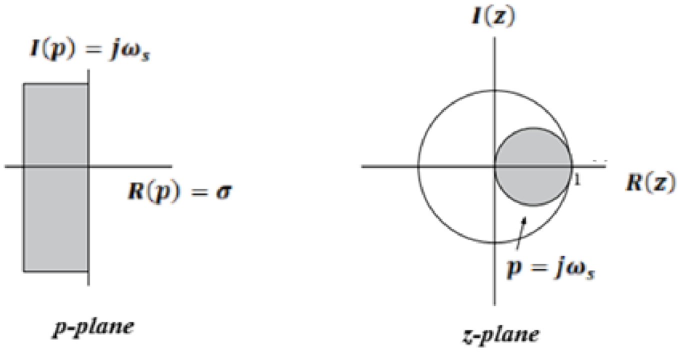


Fig I.15. Projection from s-domain to z-domain using the backward difference method [3].

I.7.4 Forward Difference Method

Another way of calculating the derivative $\dot{g}(t)$ at time instance $t = kT$ is by using the difference of the next sample and the current one, divided by the sampling period:

$$\dot{g}(t) = \frac{dg(t)}{dt} = \frac{g(k+1)-g(k)}{T} \qquad \text{I.50}$$

The derivative of $\dot{x}(t)$ in s-domain is $pX(p)$ while in z-domain is:

$$TZ[\dot{g}(t)] = TZ\left[\frac{g(k+1)-g(k)}{T}\right] = \frac{z-1}{T}G(z) \qquad \text{I.51}$$

Comparing the above derivatives, the conversion of $G(p)$ in z-domain is presented as:

$$G(z) = G(p)|_{p=\frac{z-1}{T}} \tag{I.52}$$

Using the forward difference method, the stability is not always preserved neither its harmonic response; thereby it is usually not preferred.

I.7.5 Bilinear or Tustin Method

Using this method, the conversion of $G(p)$ in z-domain is presented as [3]:

$$G(z) = G(p)|_{p=\frac{2\,(z-1)}{T(z+1)}} \tag{I.53}$$

The Tustin transform is defined by:

$$p = \frac{2}{T}\frac{z-1}{z+1} \text{ and } z = \frac{1+p(T/2)}{1-p(T/2)} \tag{I.54}$$

It represents one of the most popular methods because the stability of the analog system is preserved, while the impulse response can be also maintained when the nonlinear relation between the analog and digital frequency is considered.

By substituting $p = j\omega_s$ in the expression (I.54), we have [3]:

$$z = \frac{1+p(T/2)}{1-p(T/2)}\bigg|_{p=j\omega_s} = \frac{1+j(\omega_s T/2)}{1-j(\omega_s T/2)}$$

$$= \frac{1-(\omega_s T/2)^2+j\omega_s T}{1+(\omega_s T/2)^2}$$

$$\Rightarrow z = x + jy = \frac{1-(\omega_s T/2)^2}{1+(\omega_s T/2)^2} + j\frac{\omega_s T}{1+(\omega_s T/2)^2}$$

$$\Rightarrow x^2 + y^2 = \left[\frac{1-(\omega_s T/2)^2}{1+(\omega_s T/2)^2}\right]^2 + \left[\frac{\omega_s T}{1+(\omega_s T/2)^2}\right]^2 = 1 \tag{I.55}$$

The expression $x^2 + y^2 = 1$ defines a circle with center 0 and radius 1, as Fig I.16 shows. Based on Fig I.16, the left p half plane is presented within the unit circle in z-domain, using the Tustin method, therefore the discrete system will always be stable. The drawback of the Tustin method is the warping level in the ω frequency axis, which is caused by the nonlinear relation between the frequencies ω_s (in p-domain) and ω (in z-domain).

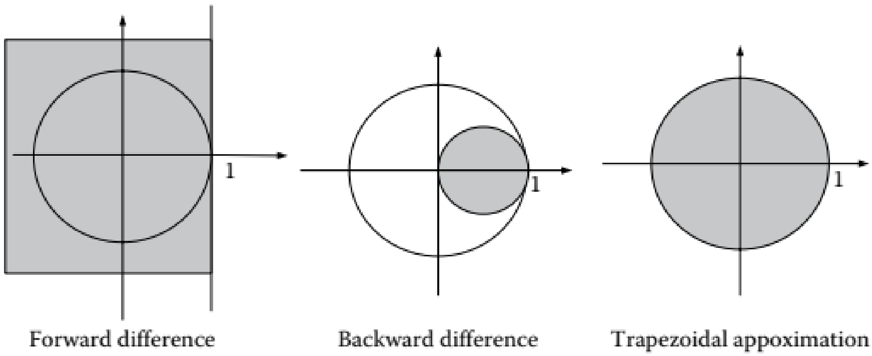


Fig I.16. Stability regions using different methods [3].

Chapter II: Performance analysis of sampled-data systems

II.1 Introduction

In most engineering applications, a physical system or plant must be controlled to behave in accordance with specified design specifications. Usually, the control action is updated regularly, the plant is analog, and the control is piecewise constant. The total system produced by this configuration is conveniently described using a discrete-time model.

A digital computer often hosts the controller algorithm in a feedback control system. Since the computer receives data only at specific intervals, it is necessary to develop a method for describing and analyzing the performance of computer control systems. Discrete-time models provide mathematical relations between the system variables at discrete time instants and the z-transform of a transfer function is used to analyze the stability and transient response of a system.

II.2 Difference Equations

Difference equations correspond to discrete-time systems, while differential equations correspond to continuous-time systems. The general form of an N-degree difference equation is [2]:

$$\sum_{k=0}^{N} b_k\, y(n-k) = \sum_{k=0}^{M} a_m\, x(n-m) \qquad \text{II.1}$$

and to solve it, $(N+M)$ initial conditions of $y(-1), y(-2), \dots, y(-N)$ and $x(-1), x(-2), \dots, x(-M)$ should be known.

The sample signal extracted from the sampling of its analog counterpart is mathematically expressed as:

$$y^*(t) = x(t).\,\delta_T(t) = x(t).\sum_{k=0}^{\infty} \delta(t-kT) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) \qquad \text{II.2}$$

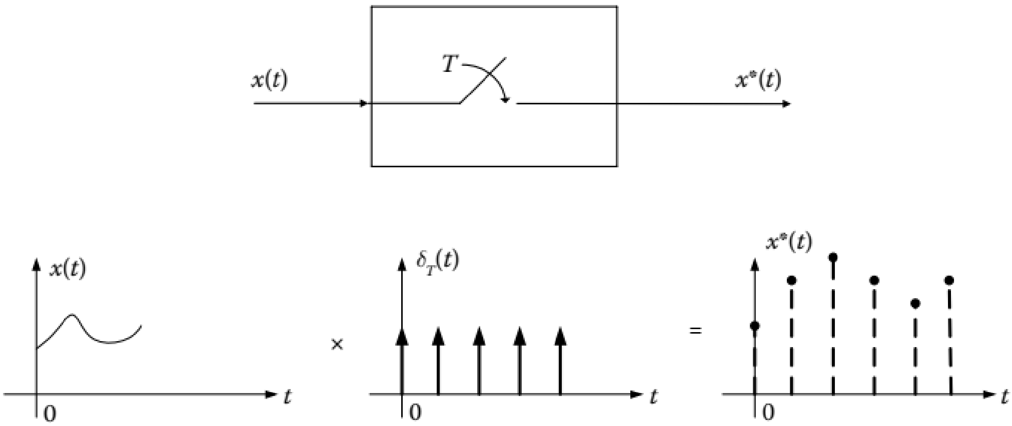


Fig II.1. Sampled signal.

II.3 Transfer function

The transfer function is defined as the ratio of the z-transform of the output for a linear invariant system to the z-transform of its input, when the initial conditions are zero and corresponds to a relation which describes the dynamics of the system under consideration.

Consider the system of Fig II.2, its transfer function is given by:

$$G(z) = \frac{Y(z)}{X(z)} \qquad \text{II.3}$$

The transfer function represents the z-transform of the impulse response [7].

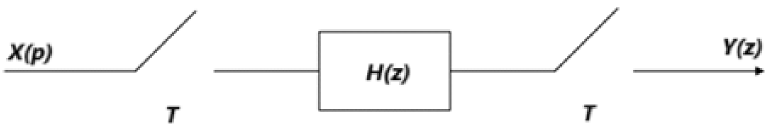


Fig II.2. Sampled data system.

II.4 Open-Loop Sampled-Data Control System

Consider the system of Fig II.3, which corresponds to an open-loop sampled-data system.

In this case, the output (Y(z)) of the given system is

$$Y(z) = TZ\left[G_0(p)G_p(p)\right]G_c(z)X(z) = G(z)G_c(z)X(z) \qquad \text{II.4}$$

where

$G_p(p)$ is the transfer function of the system under control,

$G_c(z)$ is the transfer function of the digital controller,

$G(z)$ is the transfer function of the analog system in discrete time, and

$G_0(p)$ is the transfer function of the system with which $G_p(p)$ is discretized.

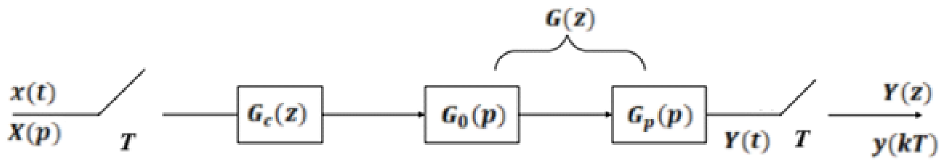


Fig II.3. Open-loop sampled-data system.

II.5 Closed-loop Sampled-Data Control System

Any system in which the output quantity is monitored and compared with the input, any difference being used to actuate the system until the output equals the input is called a *closed-loop* or *feedback* control system. The elements of a closed-loop control system are represented in block diagram form using the transfer function approach. The general form of such a system is shown in Fig II.4 [7].

The transfer function is defined as the ratio of the z-transform of the output $R(z)$ for a linear invariant system to the z-transform of its input $C(z)$, when the initial conditions are zero and corresponds to a relation which describes the dynamics of the system under consideration [3].

Consider the system of Fig II.4, which corresponds to a closed-loop sampled-data control system. In this case, the output (Y(z)) of the given system is:

$$Y(z) = \frac{G_c(z)\, G(z)}{1+G_c(z)\, TZ[H(p)G_0(p)G_p(p)]} \quad \text{II.5}$$

where

X: Laplace transform of reference input $x(t)$;

Y: Laplace transform of controlled output $y(t)$;

B: Primary feedback signal;

ε: Actuating or error signal, of value $(X(z) - B(z))$;

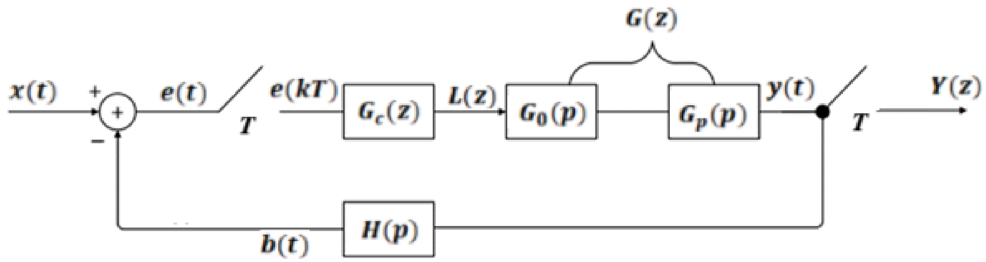


Fig II.4. Closed-loop sampled-data system.

The expression (II.5) is deduced from the above block diagram. According to Fig II.4, we have that:

$$Y(z) = G(z)G_c(z)\,\varepsilon(z) \qquad \text{II.6}$$

$$\varepsilon(z) = X(z) - B(z) \qquad \text{II.7}$$

$$B(z) = TZ[H(p)G_0(p)G_p(p)]L(z) \qquad \text{II.8}$$

$$L(z) = G_c(z)\varepsilon(z) \qquad \text{II.9}$$

From (II.8) and (II.9), we have

$$\Rightarrow B(z) = TZ[H(p)G_0(p)G_p(p)]G_c(z)\varepsilon(z) \qquad \text{II.10}$$

Substituting (II.10) in (II.7), we obtain

$$\Rightarrow \varepsilon(z) = X(z) - TZ[H(p)G_0(p)G_p(p)]G_c(z)\varepsilon(z) \qquad \text{II.11}$$

Thus

$$\varepsilon(z) = \frac{X(z)}{1+G_c(z)\,TZ[H(p)G_0(p)G_p(p)]} \qquad \text{II.12}$$

$$\Rightarrow\; Y(z) = \frac{G_c(z)\,G(z)\,X(z)}{1+G_c(z)\,TZ[H(p)G_0(p)G_p(p)]} \qquad \text{II.13}$$

The transfer function of the closed-loop system is

$$\Rightarrow\; G_{cl}(z) = \frac{Y(z)}{X(z)} = \frac{G_c(z)\,G(z)}{1+G_c(z)\,TZ[H(p)G_0(p)G_p(p)]} \qquad \text{II.14}$$

For $H(p) = 1$, it holds that

$$G_{cl}(z) = \frac{Y(z)}{X(z)} = \frac{G_c(z)G(z)}{1+G_c(z)G(z)} \qquad \text{II.15}$$

It should be noted that the expression (II.15), which provides the system's output in z-domain, can be dynamically changed according to the form of the sampled data system, correspondingly. Several block diagrams for digital control systems are presented in Table I.1, including their corresponding transfer function in terms of z [3].

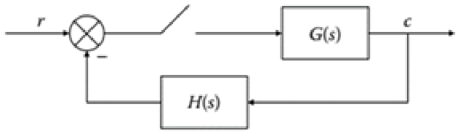
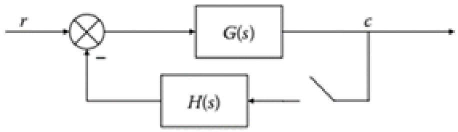
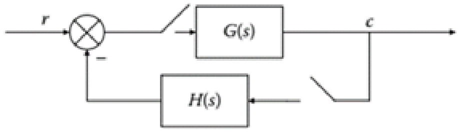
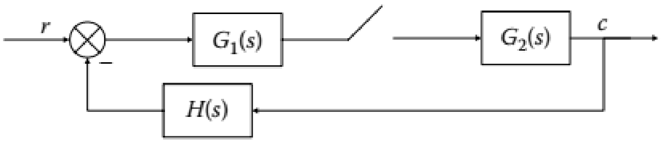
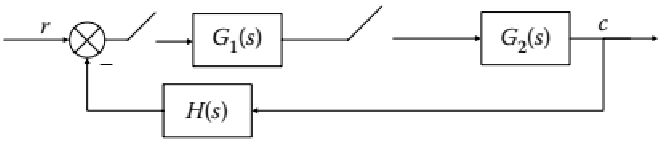
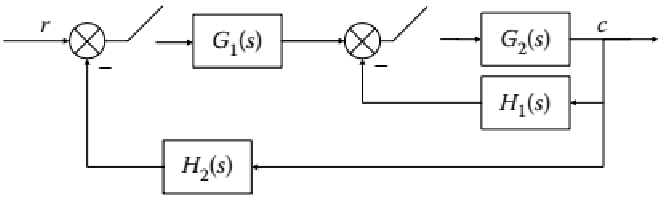
System	Transfer Function $C(z)$
	$C(z)=\frac{R(z)G(z)}{1+GH(z)}$ $GH(z)=Z[G(s)H(s)]$
	$C(z)=\frac{RG(z)}{1+GH(z)}$ $RG(z)=Z[R(s)G(s)]$
	$C(z)=\frac{R(z)G(z)}{1+G(z)H(z)}$
	$\frac{R(z)G_1(z)G_2(z)}{1+G_1(z)G_2H(z)}$
	$\frac{R(z)G_1(z)G_2(z)}{1+G_1(z)G_2H(z)}$
	$\frac{R(z)G_1(z)G_2(z)}{1+G_2H_1(z)+G_1(z)G_2H_2(z)}$

Table II.1. Transfer Function of Closed-Loop Sampled-Data Systems.

II.6 Stability

Stability is a basic requirement for digital and analog control systems. Digital control is based on samples and is updated every sampling period, and there is a possibility that the system will become unstable between updates. The most commonly used definitions of stability are based on the magnitude of the system response in the steady state. If the steady-state response is unbounded, the system is said to be unstable. This obviously makes stability analysis different in the digital case [1].

A system is stable if, for finite input, the output is also finite. This fundamental principle is known as bounded input–bounded output (BIBO) stability criterion.

The output of a stable system is within acceptable limits while the corresponding output of an unstable system theoretically tends to infinity.

The stability of a discrete control system is directly connected with the positions of roots of the characteristic equation (poles) of the transfer function:

$$P(Z) = 1 + GH(Z) = 0 \quad \text{II.16}$$

We examine different definitions and tests of the stability of linear time-invariant (LTI) digital systems based on transfer function models. In particular, we consider input-output stability and internal stability. We provide several tests for stability: the Routh-Hurwitz criterion, the Jury criterion, and the Nyquist criterion.

II.6.1 Digital System Stability via the z-Plane

In the p-plane, the region of stability is the left half-plane. If the transfer function, $G(p)$, is transformed into a sampled-data transfer function, $G(z)$, the region of stability on the z-plane can be evaluated from the definition, $z = e^{pT}$ [1,5] :

$$z = e^{pT} \quad \text{II.17}$$

Letting $p = \sigma + j\omega$, we obtain:

$$z = e^{pT} = e^{(\sigma + j\,w)\,T} = e^{\sigma\,T}.\arg(\omega T) \quad \text{II.18}$$

Each region of the p -plane can be mapped into a corresponding region on the z-plane as shown in Fig II.5. Points that have positive values of α are in the right half of the p-plane, region C. The magnitudes of the mapped points are $e^{\alpha T}$. Thus points in the right half of the p-plane map into points outside the unit circle on the z-plane.

Points on the $j\omega$-axis, region B, have zero values of a and yield points on the z-plane with magnitude equal 1, the unit circle. Hence, points on the $j\omega$-axis in the s-plane map into points on the unit circle on the z-plane.

Finally, points on the p -plane that yield negative values of α map into the inside of the unit circle on the z-plane.

Thus, a linear time invariant discrete system is stable if the poles of the closed loop system are inside the unit circle (i.e., they have real parts between -1 and 1), while it is unstable if at least one pole is located outside the unit circle.

If the system characteristic equation has roots in the circumference of the unit cycle with all other roots being located inside, then the steady-state output will operate unabated oscillations of finite amplitude when its input is a finite function. Such behavior makes the system marginally stable. All the above are illustrated in Fig II.5.

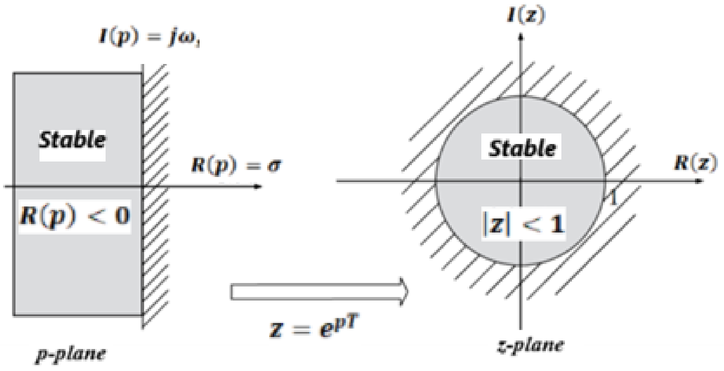


Fig II.5. Stability of analog and discrete system [1].

The most popular techniques for determining the stability of a discrete time system are:

- Unit-circle criterion
- Routh criterion using the bilinear mobius transformation
- Jury criterion
- Root locus method
- Nyquist stability criterion
- Bode stability criterion

II.6.2 Unit-Circle Criterion

The key relation between analog and digital domain is $z = e^{pT}$

or

$$z = e^{j(\sigma+\omega)T} = e^{\sigma T}.e^{j\omega T} = e^{\sigma T}.e^{j\omega T}.e^{\pm j2k\pi} \quad \text{II.19}$$

$$\Rightarrow \quad = e^{\sigma T}.e^{j(\omega T \pm j2k\pi)} \quad \text{II.20}$$

Hence, the variable z is a vector of length $e^{\sigma T}$ and phase $(\omega T \pm j2k\pi)$. As we know from the analog control system theory, a system is stable when its poles are located on the left half complex plane s, that is, when $Re(p) = \sigma < 0$ with marginal stability $\sigma = 0$. By moving into the z-domain, observe that for

$$\sigma = 0 \;\Rightarrow\; |z| = |e^{0T}| = 1$$

and

$$\sigma \to -\infty, \Rightarrow\; |z| = |e^{-\infty T}| = 0$$

Thereby, the key relation between stability and poles location is transferred from the left half-plane s into the unit circle with center being the intersection of complex z-plane axes.

II.6.3 Routh Criterion Using the Bilinear Mobius Transformation

The Routh-Hurwitz criterion determines conditions for left half plane (LHP) polynomial roots and cannot be directly used to investigate the stability of discrete-time systems. The bilinear transformation transforms the inside of the unit circle to the LHP [1,3]:

$$\omega = \frac{z+1}{z-1} \quad \Rightarrow \quad z = \frac{\omega+1}{\omega-1} \qquad \text{II.21}$$

Fig II.6 illustrates the unit circle of the z-plane in the left-half w-plane. For a point inside the unit circle, the angle of ω is of a magnitude greater than 90°, which corresponds to points in the LHP. For a point on the unit circle, the angle is $\pm 90°$, which corresponds to points on the imaginary axis, and for points outside the unit circle, the magnitude of the angle is less than 90°, which corresponds to points in the RHP.

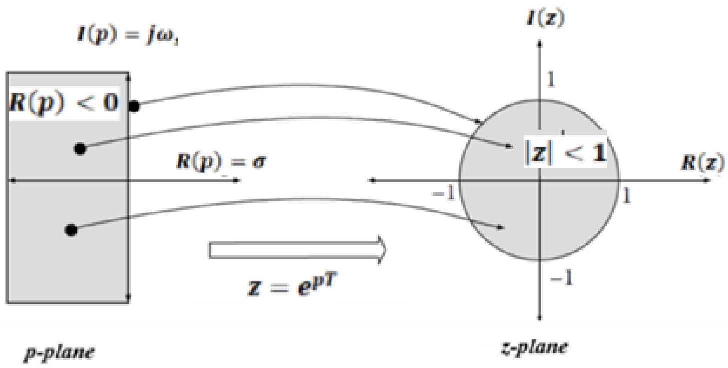


Fig II.6. Relation between the left *p* half-plane with the interior of unit circle in *z*-plane [1].

The left w half-plane is illustrated within the unit circle of z-plane.
Consider the characteristic equation:

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \qquad \text{II.22}$$

Using the bilinear transform, the characteristic equation becomes:

$$P(z) = a_0 \left(\frac{1+\omega}{1-\omega}\right)^n + a_1 \left(\frac{1+\omega}{1-\omega}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1+\omega}{1-\omega}\right) + a_n = 0$$

$$Q(\omega) = b_0 \omega^n + b_1 \omega^{n-1} + \cdots + b_{n-1}\omega + b_0 = 0 \qquad \text{II.23}$$

Hence, we transform $P(z) = 0$ into $Q(\omega) = 0$ and study the stability of the discrete control system using the Routh criterion similar to the continuoustime control systems. The Möbious transform is illustrated in Fig II.7.

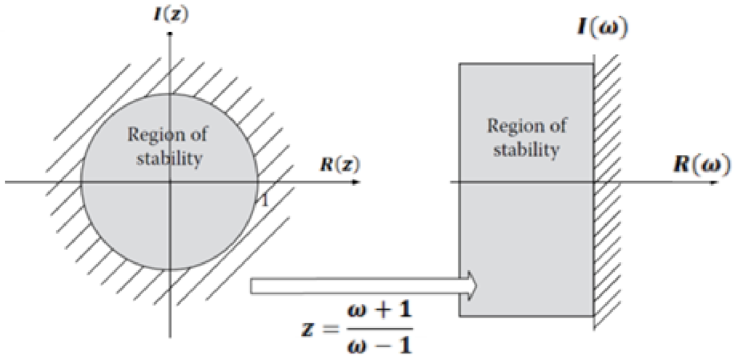


Fig II.7. z and ω -domains via the Möbious bilinear transform [1].

The Routh-Hurwitz approach becomes progressively more difficult as the order of the z-polynomial increases. But for low-order polynomials, it easily gives stability conditions. For high-order polynomials, a symbolic manipulation package can be used to perform the necessary algebraic manipulations.

II.6.4 Jury Criterion

It is possible to investigate the stability of z-domain polynomials directly using the Jury test for real coefficients or the Schur-Cohn test for complex coefficients. These tests involve determinant evaluations as in the Routh-Hurwitz test for p-domain polynomials but are more time consuming [1].

Let the characteristic equation of a sampled data system be:

$$\alpha(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_0 = 0 \qquad \text{II.24}$$

The Jury table is given by [4]:

z^0	z^1	z^2	z^3	$\cdots$	z^{n-1}	z^{n-1}	z^n
a_0	a_1	a_2	a_3	$\cdots$	a_{n-2}	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	a_2	$\cdots$	a_1	a_0	
b_0	b_1	b_2	b_3	$\cdots$	b_{n-2}	b_{n-1}	b_n
b_n	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_2	b_1	b_0
p_0	p_1	p_2	p_3	$\cdots$			
p_3	p_2	p_1	p_0	$\cdots$			
q_0	q_1	q_2					

Table II.2. Jury table.

The roots of the polynomial are inside the unit circle if and only if [1]:

1. $|a_n| < |a_0|$
2. $P(z)|_{z=1} > 0$
3. $P(z)|_{z=-1} \begin{cases} > 0 & for\ n\ pair \\ < 0 & for\ n\ impair \end{cases}$
4. $|b_{n-1}| > |b_0|$
5. $|C_{n-1}| > |C_0|$
6. $\vdots$
7. $|q_2| > |q_0|$

The entries of the table are calculated as follows:

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad k = 0,1,2,\ldots,n-1$$

$$c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, \quad k = 0,1,2,\ldots,n-2$$

$$d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \quad k = 0,1,2,\ldots,n-3$$

$$q_0 = \begin{vmatrix} p_0 & p_3 \\ p_3 & p_0 \end{vmatrix}, \quad q_2 = \begin{vmatrix} p_0 & p_1 \\ p_3 & p_2 \end{vmatrix}$$

Based on the Jury table and the Jury stability conditions, we make the following observations:

1. The first row of the Jury table is a listing of the coefficients of the polynomial F(z) in order of increasing power of z.
2. The number of rows of the table 2n-3 is always odd, and the coefficients of each even row are the same as the odd row directly above it with the order of the coefficients reversed.

II.6.5 Nyquist Stability Criterion

The Nyquist stability criterion is based on the graphical representation of the open-loop transfer function for a particular closed path in the complex frequency domain and provides information not only on the stability of the closed systems but for their relative stability as well. The special closed road is called *Nyquist path* or *Nyquist plot* and includes the right complex half plane. In Fig II.8 Nyquist path Γ_C is presented.

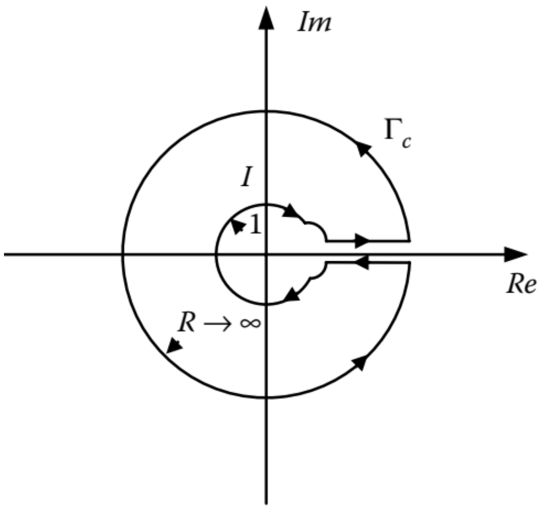


Fig II.8. Nyquist path.

The Nyquist stability criterion studies the stability of the closed-loop system, when the open-loop transfer function is considered as known. To apply the Nyquist stability criterion in discrete-time systems, it suffices to set $z = e^{j\omega T}$ in the open-loop transfer function and to design the polar diagram with the circular frequency ω as a parameter.

Consider the discrete system of Fig II.9.

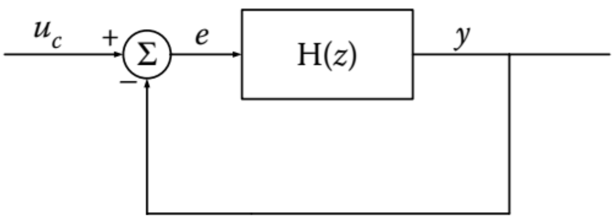


Fig II.9. Closed discrete system.

Its transfer function is presented as:

$$G_{cl}(z) = \frac{Y(z)}{U_c(z)} = \frac{H(z)}{1+H(z)} \qquad \text{II.25}$$

The characteristic polynomial is given by $(1 + H(z))$. It holds that:

$$N = Z - P \qquad \text{II.26}$$

where

Z is the roots of the characteristic equation $(1 + H(z) = 0)$, except the unit circle.

N is the number of encirclements of the point $(-1 + j0)$ clockwise to $H(z)$.
P is the number of poles of $H(z)$outside the unit circle.
Based on Nyquist criterion, to preserve the stability of the closed system, then P = 0 should hold, thus:

$$N = Z \tag{II.27}$$

Nyquist stability criterion

If the open-loop system is stable, then the stability of the closed-loop system is determined by the case when the point $(-1 + j0)$ is surrounded by the Nyquist diagram of $H(e^{j\omega T})$ for $\omega_c T$, from 0 to π.

The gain margin k_g is defined as the quantity arising from the expression (II.25) and it is the inverse value of gain $|H(e^{j\omega T})|$ into the frequency for which the phase angle tends to -180° [3]:

$$k_g = \frac{1}{H(e^{j\omega_c T})} \tag{II.28}$$

where ω_c is the critical frequency where the Nyquist diagram of $H(e^{j\omega T})$ intersects the axis Re[GH], that is

$$arg\left(H(e^{j\omega_c T})\right) = -\pi \tag{II.29}$$

A closed system is stable if $k_g > 0$.

The gain margin is the amount of gain increase or decrease required to make the loop gain unity at the frequency where the phase angle is –180°.

The phase margin φ_{marg} is the quantity arising from the expression (II.29) and it is the angle at which the diagram of $H(e^{j\omega T})$ should be rotated so as the point of $|H(e^{j\omega T})| = 1$ pass through the point $(-1 + j0)$ of the coordinates plane of $H(e^{j\omega T})$:

$$\varphi_{marg} = \pi + arg\left(H(e^{j\omega_c T})\right) \tag{II.30}$$

where ω_c is the frequency where the amplitude $|H(e^{j\omega_c T})|$ equals to unity. This stability measure is practically equal to the added phase delay, which is required before the system is turned to unstable. A closed system is stable if $\varphi_{marg} > 0$.

II.6.6 Bode Stability Criterion

Using the bilinear transform:

$$z = \frac{(1+(T/2)\omega)}{(1-(T/2)\omega)} \tag{II.31}$$

The internal of the unit circle of the complex z-plane is depicted to the left w half-plane. In this way, we study the stability of a digital control system in w-domain, by utilizing the same methods applied to analog systems. Consider a control system with the loop transfer function:

$$G(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + \alpha_n} \quad m \leq n \qquad \text{II.32}$$

$$\Rightarrow G(\omega) = \frac{b_0 \omega^m + b_1 \omega^{m-1} + \cdots + b_{m-1} \omega + b_m}{\omega^n + a_1 \omega^{n-1} + \cdots + a_{n-1} \omega + \alpha_n} \qquad \text{II.33}$$

Based on $G(\omega)$ and setting $\omega = jv$ (where v is the system angular frequency), the Bode diagram can be designed for $G(jv)$, while some useful outcomes regarding the stability of the closed-loop system can be extracted [3]. Fig II.10 illustrates the relation between ω and $(T/2)v$.

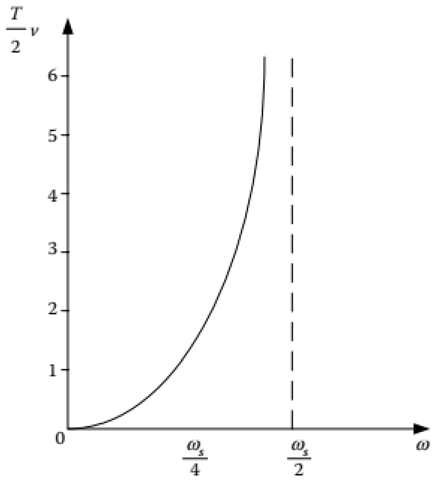


Fig II.10. Relationship between the analog and digital frequency.

II.7 Influence of sampling frequency on stability

II.7.1 Choice of sampling frequency

In control systems, the sampling frequency is not dictated only by Shannon's Theorem but also by the characteristics of the system because it's not always possible to know a priori the spectra of the signals in the system.

The rule traditionally adopted by automation engineers for choosing the sampling frequency is to evaluate the bandwidth f_c fpas of the controlled system and choose a sampling frequency such that [7]:

$$6f_c \leq f_e \leq 25f_c \qquad \text{II.34}$$

II.8 Steady-State Errors

An important factor related to the operation of control systems corresponds to the error in the steady state (steady-state error $e_{ss}(kT)$), which appears to the system output after the transient response period. The steady state is quite important since the design of an automatic control system is designed, among others, to maintain a predetermined steady state for the output $y_{ss}(kT)$, which is usually the input function $u(kT)$. In other words, the system is designed so as to hold $y_{ss}(kT) = u_{ss}(kT)$), when the system is stimulated by $u(kT)$. Otherwise, we have a static error or steady-state error [3].

Note that the steady-state error of closed-loop stable system is usually much smaller than its open-loop counterpart. From the theory of analog control systems, it is known that the steady-state error depends on the input function and the system features. The system features are studied with the aid of three error factors, namely, position, velocity, and acceleration (k_p, k_v, and k_a).

Consider the digital system in Fig II.11, where the digital computer is represented by the sampler and zero-order hold. The transfer function of the plant is represented by $G_p(p)$ and the transfer function of the ZOH by $((1 - e^{-Tp})/p)$.

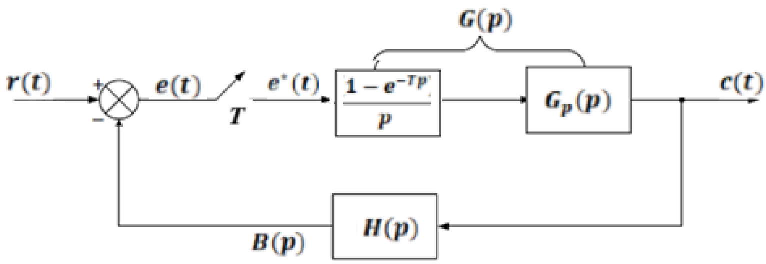


Fig II.11. Sampled data system [11].

Consider the open-loop transfer function FTBO in the general form:

$$FTBO = \frac{1.B(z)}{(z-1)^N A(z)} = \frac{K.\prod(z-z_i)^N}{(z-1)^N A(z)}, \quad z_i, z_j \neq 1 \qquad \text{II.35}$$

N plays a crucial role since it influences the system features and defines the system as a zero-order, first-order or second-order, when N = 0, 1, and 2.

From Fig II.11, it holds that

$$e(t) = r(t) - b(t) \qquad \text{II.36}$$

To make the analysis feasible, the signal e*(t) is used, thus the steady-state error, during sampling, is:

$$e_{ss}^{*} = \lim_{k\mapsto\infty} e^{*}(t) = \lim_{k\mapsto\infty} e(kT) \qquad \text{II.37}$$

Applying the *z*-transform and the final value theorem, we get:

$$e_{ss}^{*} = \lim_{t\mapsto\infty} e^{*}(t) = \lim_{z\mapsto 1}(z-1)\,E(z) \qquad \text{II.38}$$

For the system of the scheme, we have that:

$$E(z) = \frac{R(z)}{1+TZ[G(p).H(p)]} \qquad \text{II.39}$$

where

$$GH(z) = TZ[GH(p)] = (1-\ z^{-1})TZ\left[\frac{G_S(p).H(p)}{p}\right] \qquad \text{II.40}$$

Thus, the steady-state error is given by:

$$e_{ss}^{*} = \lim_{t\mapsto\infty} e^{*}(t) = \lim_{z\mapsto 1}(z-1)\frac{R(z)}{1+TZ[G(p).H(p)]} \qquad \text{II.41}$$

From the expression (II.41), it is explicitly indicated that the steady-state error depends on the input signal r(t) and the system features.

Next, we study the impact of input function for three different signal types. The most common ones are the step, ramp and parabolic input functions.

1. Unit Step input

The z-transform for a unit step input of range A is:

$$R(z) = \frac{A\,z}{z-1} \qquad \text{II.42}$$

Substituting equation (II.42) in (II.41), we get

$$e_{ss}^{*} = \lim_{z\mapsto 1}\frac{A}{1+\lim\limits_{z\mapsto 1} GH(z)} = \frac{A}{1+K_p} \qquad \text{II.43}$$

where K_p is the position error constant:

$$K_p = \lim_{z\mapsto 1} GH(z) \qquad \text{II.44}$$

Consequently, for N = 0, the position error is

$$e_{ss}^{*} = \frac{A}{1+K_p} \qquad \text{II.45}$$

while for $N \geq 1$ the position error is zero ($e_{ss}^{*} = 0$).

$e_{ss}^{*} = 0$ for $k_p \rightarrow \infty$, which requires GH(Z) to have at least one pole at Z=1.

2. Unit Ramp input

Following the procedure for the step input. The z-transform for a ramp input of range A is:

$$R(z) = \frac{A\,Tz}{(z-1)^2} \qquad \text{II.46}$$

Substituting equation (II.46) in (II.41), we have:

$$e_{ss}^{*} = \lim_{z \mapsto 1} \frac{A}{1+\lim\limits_{z \mapsto 1} GH(z)} = \frac{AT}{(z-1)(1+GH(z))} \qquad \text{II.47}$$

$$e_{ss}^{*} = \lim_{z \mapsto 1} \frac{A}{1+\lim\limits_{z \mapsto 1} GH(z)} = \frac{AT}{(z-1/T)GH(z)} = \frac{A}{K_v} \qquad \text{II.48}$$

where k_v is the velocity error constant [5,3] :

$$k_v = \lim_{z \mapsto 1} (z - 1/T)\, GH(z) \qquad \text{II.49}$$

$e_{ss}^{*} = 0$ for $k_v \rightarrow \infty$, which requires GH(Z) to have a double pole at Z=1.

For $N = 0 \Rightarrow k_v = 0$ and $e_{ss} = \infty$

For $N \geq 1 \Rightarrow k_v = \infty$ and $e_{ss} = 0$

For $N = 1 \Rightarrow k_v = (k_p/T)$

So, for $N = 1$, the velocity error is:

$$e_{ss}^{*} = \frac{A}{K_v} = \frac{AT}{K_p} \qquad \text{II.50}$$

3. Parabolic input

Similarly, the z-transform for a parabolic input of range A is:

$$R(z) = \frac{A\,T^2 z(z+1)}{2(z-1)^3} \qquad \text{II.51}$$

$$e_{ss}^{*} = \frac{AT^2}{2} \lim_{z \mapsto 1} \frac{z+1}{(z-1)^2\,(1+GH(z))} \qquad \text{II.52}$$

$$e_{ss}^{*} = \frac{A}{\lim_{z \mapsto 1}(z-1/T)^2 GH(z)} = \frac{A}{K_a} \qquad \text{II.53}$$

where k_a is the acceleration error constant [3,5]:

$$k_a = \lim_{z \mapsto 1} (z-1/T)^2 GH(z) \qquad \text{III.54}$$

For $N \leq 1 \Rightarrow k_a = 0$ and $e_{ss} = \infty$

For $N = 2 \Rightarrow k_a = \left(k_p/T^2\right) \Rightarrow \; e_{ss}^{*} = \frac{A}{K_a} = \frac{AT^2}{K_a}$ II.55

$e_{ss}^{*} = 0$ for $k_a \rightarrow \infty$, which requires GH(z) to have a triple pole at z=1.

Chapter III: State Space Analysis

III.1 Introduction

A modern control system often includes multiple inputs and outputs that can interact in complex ways. State-space methods are particularly effective for analyzing and synthesizing optimal control systems with these multiple inputs and outputs. The system equations are represented using state-space techniques, which articulate them in terms of 'n' first-order differential or difference equations. Combining these equations can create one first-order vector-matrix differential or difference equation. The use of vector-matrix notation greatly simplifies the mathematical representation of these equation systems. Engineers can develop control systems that satisfy particular performance standards using the state-space concept in system design. Additionally, instead of being restricted to particular functions like impulses or sinusoidal functions, this approach enables the development of systems that can handle a variety of input sources. The ability to include initial conditions in the design, which is generally impractical with conventional design approaches, is another significant benefit of state-space methods [10].

III.2 Discrete-Time State-Space Equations

The state equations of a discrete-time system is a first-order difference equation system of the form:

$$x(k+1) = Ax(k) + Bu(k) \qquad \text{III.1}$$

The system output vector $y(k)$ is:

$$y(k) = Cx(k) + Du(k) \qquad \text{III.2}$$

where

A is a square matrix of dimension $(n \times n)$, it is called the *state matrix* and represents the physical (actual) system;

The matrix *B* of dimension $(n \times r)$ is called the *input matrix*;

The matrix *C* of dimension $(m \times n)$ is called the *output matrix*; and the matrix *D* of dimension $(m \times r)$ is called the *feedforward matrix*. The matrices *A*, *B*, *C*, *D* are called *state-space matrices*

In the case of a single input–single output (SISO) system ($m = r = 1$), the system is described by the difference equations [3]:

$$\begin{cases} x(k+1) = Ax(k) + bu(k) \\ y(k) = cx(k) + du(k) \\ x(0) = x_0 \end{cases} \qquad \text{III.3}$$

where

c is a column vector with n elements

b is a column vector with n elements

d is a scalar and

$x(0) = x_0$ is a column vector of the initial conditions of state variables.

In practice, it is impossible to measure all the system states. But if the system mathematical model is available, then we can calculate (estimate) the state vector, by using the already measured inputs and outputs.

III.2.1 Solution of discrete-time state-space equations

In this section, we develop the general solution of linear time-invariant state equations. It will be seen that the key to the solution of the state equations is the calculation of the state transition matrix. Two related techniques, based on the z-transform, for calculating the state transition matrix are presented. Then the solution of state equations via the digital computer is mentioned [8,9].

III.2.2 Recursive solution

We will first assume that the system is time-invariant and that $x(0)$ and $u(k)$, $k = 0, 1, 2, \dots, k-1$ are known. Now the system equations are:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \qquad \text{III.4}$$

Recursively, At $k = 0$, and $k = 1$, we have:

$$x(1) = Ax(0) + Bu(0) \qquad \text{III.5}$$
$$x(2) = Ax(1) + Bu(1)$$

Substituting from the first equation III.4 into the second equation in III.5 gives:

$$x(2) = A[Ax(0) + Bu(0)] + Bu(1)$$
$$= A^2x(0) + ABu(0) + Bu(1) \quad \text{III.6}$$

We then rewrite equation III.6 as:

$$x(2) = A^2x(0) + \sum_{i=0}^{2-1} A^{2-i-1}Bu(i) \quad \text{III.7}$$

Similarly, we can show that:

$$x(3) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2) = A^3x(0) + \sum_{i=0}^{2-1} A^{2-i-1}Bu(i) \quad \text{III.8}$$

By induction, the general solution of the state vector is given by:

$$x(k) = A^kx(0) + \sum_{i=0}^{k-1} A^{k-i-1}Bu(i) \quad \text{III.9}$$

Let us define $\phi(\mathrm{k}) = A^k$ which is known as the state-transition matrix, and represents the response of the system only by the influence of the initial conditions (i.e., the free response of the system).
Then, the general solution can be written in the form:

$$x(k) = \phi(\mathrm{k})x(0) + \sum_{i=0}^{k-1} \phi(\mathrm{k})(k - 1 - i)Bu(i) \quad \text{III.10}$$

This equation is called the state transition matrix equation; it describes the change of state relative to the initial conditions $x(0)$ and the input $u(k)$.
From III.4 and III.10, we can write the output as [3,9]:

$$y(k) = C\phi(\mathrm{k})x(0) + \sum_{i=0}^{k-1} \mathrm{C}\phi(\mathrm{k})(k - 1 - i)Bu(i) + Du(k) \quad \text{III.11}$$

The solution III.11 includes two terms as in the continuous-time case. The first is the zero-input response due to nonzero initial conditions and zero input. The second is the zero-state response due to nonzero input and zero initial conditions. Because the system is linear, each term can be computed separately and then added to obtain the total response for a forced system with nonzero initial conditions [3].

III.3 Eigenvalues and Eigenvectors

The elements of the system matrix depend on the components comprising the system.

Consider an n-order system with column vectors $x = X_i$, $(n = 1, 2, \ldots, n)$ and real or complex values for the parameter λ, which satisfy the equation:

$$Ax = \lambda x \;\Rightarrow\; (\lambda I - A)x = 0 \qquad \text{III.12}$$

The matrix $(\lambda I - A)$ is called the *characteristic matrix* of the system. The values of parameter λ satisfying $(\lambda I - A)x = 0$ represent a column vector called *eigenvalues* or *characteristic values* of the system, which arise from the solution of the linear system.
Eliminating the determinant of the characteristic matrix, the *characteristic polynomial* of the system is revealed, such as:

$$P(\lambda) = det(\lambda I - A) = \lambda^n + a_{n-1}\lambda_{n-1} + \cdots + a_1\lambda + a_0 = 0 \qquad \text{III.13}$$

The roots of $P(\lambda)$, namely, its eigenvalues, denote the *poles* of the closed system. The characteristic equation of the system is given by

$$(z I - A) = 0 \qquad \text{III.14}$$

III.4 Transfer Functions

Given the state equations of a discrete-time system, the transfer function can be obtained by taking the z-transform of the state equations and eliminating $X(z)$. Since,

$$x(k+1) = Ax(k) + Bu(k) \qquad \text{III.15}$$

Taking the *z*-transform yields

$$zX(z) - zx(0) = AX(z) + BU(z) \qquad \text{III.16}$$

Since, in deriving transfer functions, we ignore initial conditions, the equation III.16 can be expressed as [8]:

$$[zI - A]X(z) = BU(z) \qquad \text{III.17}$$

Solving for *X*(z), we obtain:

$$X(z) = [zI - A]^{-1}BU(z) \qquad \text{III.18}$$

Also, the output state vector is defined as

$$y(k) = Cx(k) + Du(k) \qquad \text{III.19}$$

Then

$$Y(z) = CX(z) + DU(z) \qquad \text{III.20}$$

Substituting III.18 into III.20 yields:

$$Y(z) = [C[zI - A]^{-1}B + D]U(z) \qquad \text{III.21}$$

The system transfer function is then seen to be [9]:

$$G(z) = C[zI - A]^{-1}B + D \qquad \text{III.22}$$

III.5 Solution in z**-domain**

1. Solution in the time domain [3]

The general solution to the state equations was developed above:

$$x(k + 1) = Ax(k) + Bu(k) \qquad \text{III.23}$$

and is given by:

$$x(k) = \phi(\mathrm{k})x(0) + \sum_{i=0}^{k-1} \phi(\mathrm{k})(k - 1 - i)Bu(i) \qquad \text{III.24}$$

where ϕ(k), the state transition matrix, is given by ϕ(k)=A^k.

One technique for evaluating ϕ(k) as a function of k is through the use of the z-transform.

Let $u(k) = 0$, then the z-transform of this equation yields:

$$zX(z) - zx(0) = AX(z) \qquad \text{III.25}$$

Solving for $X(z)$, we see that:

$$X(z) = z[zI - A]^{-1}x(0) \qquad \text{III.26}$$

Then

$$x(k) = TZI[X(z)] = TZI[z[zI - A]^{-1}x(0)] \qquad \text{III.27}$$

Comparing III.27 with III.24, we see that:

$$\phi(\mathrm{k}) = TZI[z[zI - A]^{-1}] \qquad \text{III.28}$$

III.6 Properties of the state transition matrix

Three properties of the state transition matrix will now be derived [3].

$$x(k) = \phi(\mathrm{k})x(0) \qquad \text{III.29}$$

then evaluating this expression for k = 0 yields the first property:

$$\phi(0) = 1 \qquad \text{III.30}$$

where I is the identity matrix. Next, since $\phi(k) = A^k$, then the second property is given by:

$$\phi(\mathrm{k}) = A^{k_1+k_2} = A^{k_1}A^{k_2} = \phi(k_1)\phi(k_2) \qquad \text{III.31}$$

The third property is seen from the relationships:

$$\phi(-\mathrm{k}) = A^{-k} = (A^k)^{-1} = \phi^{-1}(\mathrm{k}) \qquad \text{III.32}$$

or, taking the inverse of this expression, we obtain an equivalent expression:

$$\phi(\mathrm{k}) = \phi^{-1}(-\mathrm{k}) \qquad \text{III.33}$$

III.7 State-Space Representation

III.7.1 Canonical Form (First companion form or phase variable canonical form)

A direct realization structure for the system described by equation III.4 is shown in Fig III.1. Notice that n delay elements have been used in this realization. The coefficients $(a_0, a_2, \dots, a_{n-1})$, appear as feedback elements, and the coefficients $(b_0, b_1, \dots, b_{n-1})$, appear as feedforward elements [2].

Given the transfer function [12]:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_{n-1}z^{n-1}+b_{n-2}z^{n-2}+\dots+ b_1\mathrm{z}+b_0}{z^n+a_{n-1}z^{n-1}+\dots+ a_1\mathrm{z}+a_0} \qquad \text{III.34}$$

The digital diagram is produced following a suitable decomposition of the transfer function's quotient. The method involves dividing the diagram-building process into two stages and connecting them with a new variable, $X(z)$.

$$\frac{Y(z)}{U(z)} = \frac{Num}{Den} \Rightarrow G(z) = \frac{Y(z)}{X(z)}.\frac{X(z)}{U(z)} \qquad \text{III.35}$$

$$\Rightarrow \frac{X(z)}{U(z)} = \frac{1}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0} \qquad \text{III.36}$$

$$\frac{Y(z)}{X(z)} = b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0 \qquad \text{III.37}$$

where the auxiliary variable $X(z)$ has been introduced. We now let:

$$U(z) = (z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)X(z) \qquad \text{III.38}$$

$$Y(z) = (b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0)\, X(z) \qquad \text{III.39}$$

Recall from the real translation property the correspondence:

$$\begin{cases} X(z) \to x(k) \\ zX(z) \to x(k+1) \\ z^2X(z) \to x(k+2) \\ \vdots \\ z^nX(z) \to x(k+n) \end{cases} \qquad \text{III.40}$$

Under this correspondence, we define the state variables:

$$\begin{cases} x_1(k) = x(k) \\ x_2(k) = x(k+1) = x_1(k+1) \\ x_3(k) = x(k+2) = x_2(k+1) \\ \vdots \\ x_n(k) = x(k+n-1) = x_{n-1}(k+1) \end{cases} \qquad \text{III.41}$$

Thus, the state equations are given by:

$$\begin{cases} x_1(k) = x(k) \\ x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ \vdots \\ x_n(k+1) = u(k) - a_0x_1(k) - a_1x_2(k) - \cdots - a_{n-1}x_n(k) \end{cases} \qquad \text{III.42}$$

and the output equation:

$$y(k) = b_0x_1(k) + b_1x_2(k) + \cdots + b_{n-1}x_n(k) \qquad \text{III.43}$$

The state equations and the system output in canonical form can be expressed in vectors/ matrices as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k) \qquad \text{III.44}$$

$$y(n) = [b_0 \quad b_1 \quad \dots \quad b_{n-1}]x(k) \qquad \text{III.45}$$

Or simply

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \qquad \text{III.46}$$

The block diagram that reflects the discrete system in canonical form, which corresponds to the expressions III.46, is given in Fig III.1.

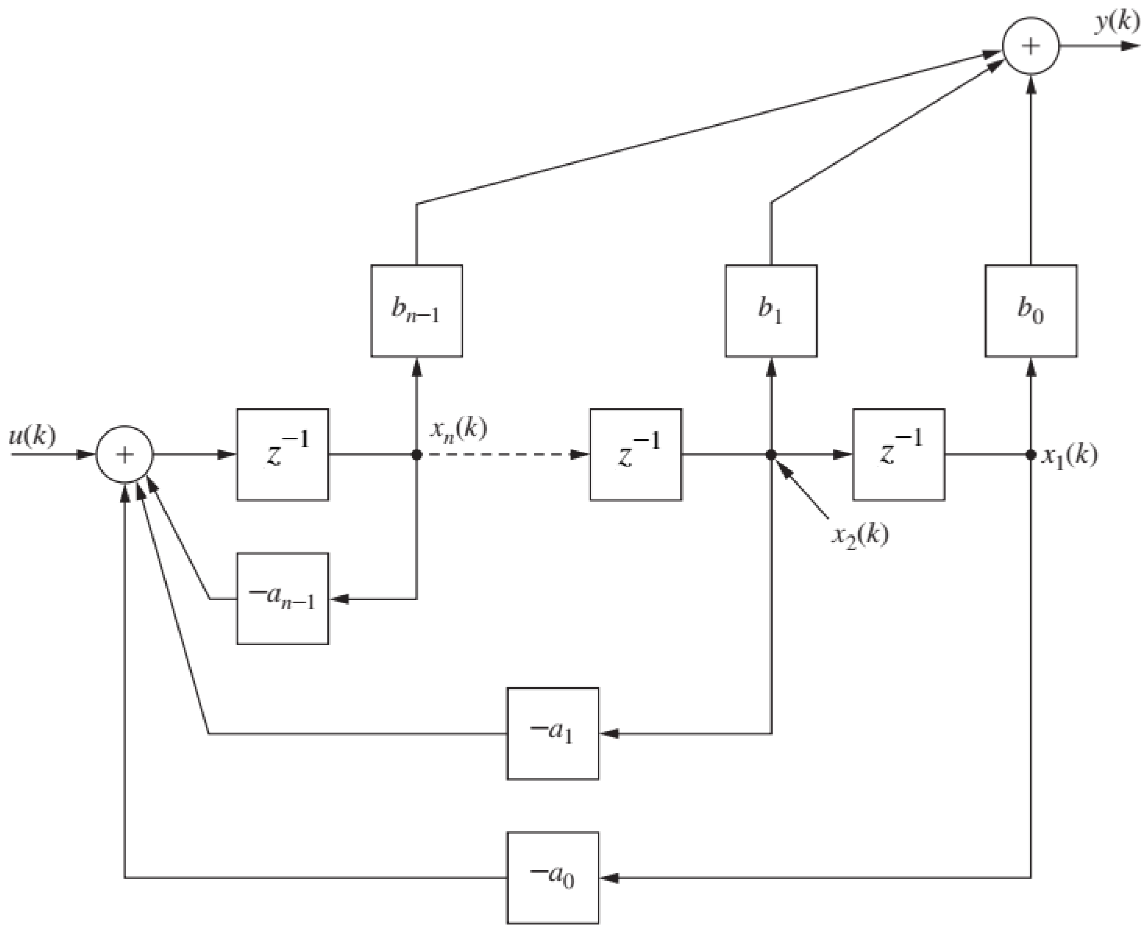


Fig III.1. Block diagram in canonical form or variable phase [12].

The matrix A has a very special structure the coefficients of the denominator of the transfer function preceded by minus signs form a string along the bottom row of the matrix. The rest of the matrix is zero except for the 'superdiagonal' terms which are all unity. A matrix with this structure is said to be in companion form. We refer the state variable model the first companion form state model [2].

III.7.2 Observable Canonical Form (Second companion form)

In the first companion form, the coefficients of the transfer function denominator appear in one of the rows of the A matrix. There is another companion form in which the coefficients appear in a column of the A matrix. The observable companion representation can be derived from the previously transformed form of G(z) in the previous section [2].

Given the transfer function:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_{n-1}z^{-1} + b_{n-2}z^{-2} + \cdots + b_1 z^{-n+1} + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_1 z^{-n+1} + a_0 z^{-n}} \qquad \text{III.47}$$

Solving for $Y(z)$, we have:

$$Y(z) = -[a_0 z^{-n} + a_1 z^{-n+1} + \cdots + a_{n-1}z^{-1}]Y(z) + [b_0 z^{-n} + b_1 z^{-n+1} + \cdots + b_{n-1}z^{-1}] \qquad \text{III.48}$$

Or

$$Y(z) = [b_0 U(z) - a_0 Y(z)]z^{-n} + [b_1 U(z) - a_1 Y(z)]z^{-n+1} + \cdots + [b_{n-1}U(z) - a_{n-1}Y(z)]z^{-1} \qquad \text{III.49}$$

The state equations are derived directly from this representation:

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, B_0 = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}, C_0 = [0 \ 0 \ 0 \ \dots 1],$$

The corresponding state equations are given by

$$\begin{cases} x_n(k+1) = x_{n-1}(k) - a_{n-1}x_n(k) + b_{n-1}u(k) \\ x_{n-1}(k+1) = x_{n-2}(k) - a_{n-2}x_{n-1}(k) + b_{n-2}u(k) \\ \vdots \\ x_1(k+1) = -a_0 x_n(k) + b_0 u(k) \\ y(k) = x_n(k) \end{cases} \qquad \text{III.50}$$

Comparing the A_0, B_0, and C_0 matrices of the second companion form with that of the first, we observe that A, B and C matrices of one companion form correspond to the transpose of A, C, and B matrices respectively of the other ($A_0 = A^T$, $B_0 = C^T$, and $C_0 = B^T$). Both the companion forms of state variable models play an important role in pole-placement design through state feedback.

The block diagram that reflects the discrete system in the observable canonical form is presented in Fig III.2.

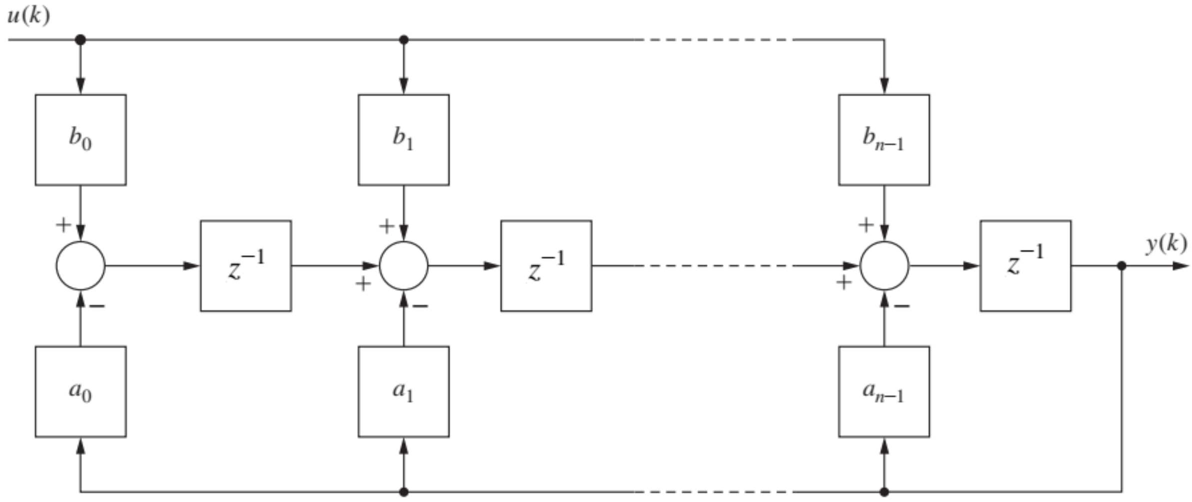


Fig III.2. Block diagram in observable canonical form.

III.7.3 Jordan Canonical Form (or parallel realization)

The transfer function of the expression III.47 can be rewritten in a partial fraction expansion as [1,7,13]:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{a_1}{z-p_1} + \frac{a_2}{z-p_2} + \cdots + \frac{a_n}{z-p_n} \qquad \text{III.51}$$

The summation gives the parallel configuration, which justifies the name parallel realization.

$$\Rightarrow Y(z) = \mathrm{U(z)}[\frac{a_1}{z-p_1} + \frac{a_2}{z-p_2} + \cdots + \frac{a_n}{z-p_n}]$$

$$\Rightarrow Y(z) = U(z)\frac{a_1}{z-p_1} + U(z)\frac{a_2}{z-p_2} + \cdots + U(z)\frac{a_n}{z-p_n}$$

$$\Rightarrow Y(z) = a_1X_1(z) + a_2X_2(z) + \cdots + a_nX_n(z) \qquad \text{III.52}$$

where

$$\begin{cases} X_1(z) = \frac{U(z)}{z-p_1} \\ X_2(z) = \frac{U(z)}{z-p_2} \\ \quad\vdots \\ X_n(z) = \frac{U(z)}{z-p_n} \end{cases} \qquad \text{III.53}$$

$$X_1(z) = \frac{U(z)}{z-p_1} \Rightarrow (z-p_1)X_1(z) = U(z)$$
$$\Rightarrow\ zX_1(z) - p_1X_1(z) = U(z)$$
$$\Rightarrow\ zX_1(z) = p_1X_1(z) + U(z) \qquad \text{III.54}$$

Using the inverse z transform, we have:

$$\Rightarrow x_1(k+1) = p_1x_1(k) + u(k) \qquad \text{III.55}$$

which can be represented by a positive feedback loop with forward transfer function z^{-1} and feedback gain p_i.

Similarly, we obtain:

$$\begin{cases} x_1(k+1) = p_1x_1(k) + u(k) \\ x_2(k+1) = p_2x_2(k) + u(k) \\ \vdots \\ x_n(k+1) = p_nx_n(k) + u(k) \end{cases} \qquad \text{III.56}$$

The output equation is obtained by the inverse transform of equation (III.52):

$$ZT[Y(z)] = ZT[a_1X_1(z) + a_2X_2(z) + \cdots + a_nX_n(z)]$$
$$\Rightarrow y(k) = a_1x_1(k) + a_2x_2(k) + \cdots + a_nx_n(k) \qquad \text{III.57}$$

The state equations and the system output in canonical form can be expressed in vectors/ matrices as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k) \qquad \text{III.58}$$

$$y(n) = [a_1 \quad a_2 \quad \cdots \quad a_n]x(k) \qquad \text{III.59}$$

Thus

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \qquad \text{III.60}$$

where A is a diagonal matrix with eigenvalues corresponding to the roots of the transfer function.

$$A = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & p_n \end{bmatrix} \qquad \text{III.61}$$

The parallel implementation of the transfer function of the latter expression is presented in Fig III.3.

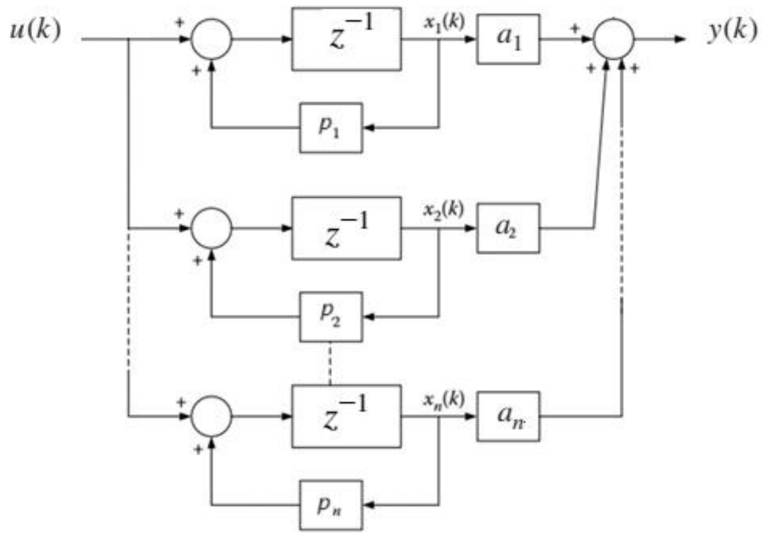


Fig III.3. Block diagram in Jordan canonical form.

Assuming the outputs of the delay elements as state variables, it can easily be shown that the following matrices represent the model at state space:

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, C = [a_1 a_2\ a_3, \dots, a_n]$$

III.7.4 Series realization

The transfer function below can be realized using n cascades elementary blocks as shown in Fig III.4 [7,13].

$$G(z) = \frac{Y(z)}{U(z)} = \frac{a}{(z-p_1)(z-p_2)\dots(z-p_n)} \qquad \text{III.62}$$

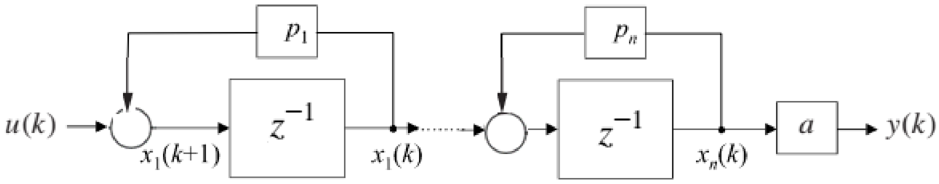


Fig III.4. Series realization.

$$X_1(z) = \frac{U(z)}{z-p_1} \Rightarrow (z-p_1)X_1(z) = U(z)$$

$$\Rightarrow zX_1(z) - p_1X_1(z) = U(z)$$

$$\Rightarrow zX_1(z) = p_1X_1(z) + U(z) \quad \text{III.63}$$

$$\Rightarrow x_1(k+1) = p_1x_1(k) + u(k) \quad \text{III.64}$$

Similarly

$$X_2(z) = \frac{X_1(z)}{z-p_2} \Rightarrow (z-p_2)X_2(z) = X_1(z)$$

$$\Rightarrow zX_2(z) - p_2X_2(z) = X_1(z)$$

$$\Rightarrow zX_2(z) = p_2X_2(z) + X_1(z) \quad \text{III.65}$$

Finally, the state equations are obtained as:

$$\begin{cases} x_1(k+1) = p_1x_1(k) + u(k) \\ x_2(k+1) = p_2x_2(k) + x_1(k) \\ \vdots \\ x_n(k+1) = p_nx_n(k) + x_{n-1}(k) \end{cases} \quad \text{III.66}$$

and the output state vector is given by:

$$y(k) = ax_n(k) \quad \text{III.67}$$

The state equations and the system output in canonical form can be expressed in vectors/ matrices as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 1 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k) \quad \text{III.68}$$

$$y(n) = [0 \quad 0 \quad \dots \quad a]x(k) \quad \text{III.69}$$

III.8 Controllability and Observability

The *controllability* of a system refers to whether it is possible to move a system from a given initial state to any final state in finite time.
Consider the system described in the state space by Equations:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \qquad \text{III.70}$$

III.8.1 Controllability

For the linear system given by equations III.70, if there exists an input $u(k)$; $k \in [0, N-1]$ with N a finite positive integer, which transfers the initial state $x(0) = x^0$ to the state x^1 at $k = N$, the state x^0 is said to be controllable. If all A initial states are controllable, the system is said to be completely controllable or simply controllable. Otherwise, the system is said to be uncontrollable [2].

The following theorem gives a simple controllability test.

Theorem 1

The necessary and sufficient condition for the system to be completely controllable is that the *matrix controllability U*:

$$C = [\,B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \qquad \text{III.71}$$

has rank equal to n, i.e., $rank(C) = n$.

III.8.2 Observability

For the linear system given by equations III.70, if the knowledge of the input $u(k)$; $k \in [0, N-1]$ and the output $y(k)$; $k \in [0, N-1]]$ with N a finite positive integer, suffices to determine the state $x(0) = x^0$, the state x is said to be observable. If all initial conditions are observable, the system is said to be completely observable, or simply observable. Otherwise, the system is said to be unobservable [2].

The following theorem gives a simple observability test.

Theorem 2

The necessary and sufficient condition for the system to be completely observable is that the $n \times n$ *observability matrix*:

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \text{III.72}$$

has rank equal to n, i.e., $rang(O) = n$.

The *observability* of a system refers to whether each position $x(k)$; can be determined by observing the output $y(k)$ at a finite time.

Overall, a system is controllable when we can control the system operation process, given an initial state, while a system is observable when all the provided information about the system state must be recovered from knowledge of the obtained measurements.

Chapter IV: Control system correction

IV.1. Introduction

Similarly, to continuous-time systems, sampled systems must generally meet requirements that impose a set number of closed-loop performances: accuracy, speed, stability margin, and overshoot limits [7].

Consider a system consisting of a direct a feedback loop. In most cases, the transfer functions $A(z)$ and $B(z)$ are imposed by the design of the control system being developed.

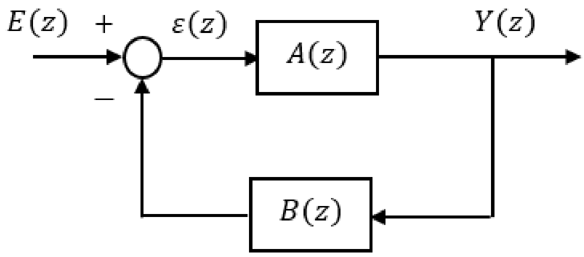


Fig IV.1. Block diagram of a sampled system.

IV.2 Role of the controller

The main idea is to introduce into the direct loop, upstream of the $A(z)$ system, an additional device with transfer function $C(z)$, called a digital controller whose principal role is to modify the performance of the initial system as shown in Fig IV.2. In other words, we transform the open-loop and closed-loop transfer functions in such a way as to force the system to operate according to the desired specifications [7].

Specifications are obtained by appropriately calculating the action to be sent to the actuator, based on the available information (setpoint, output, intermediate signals available on the process).

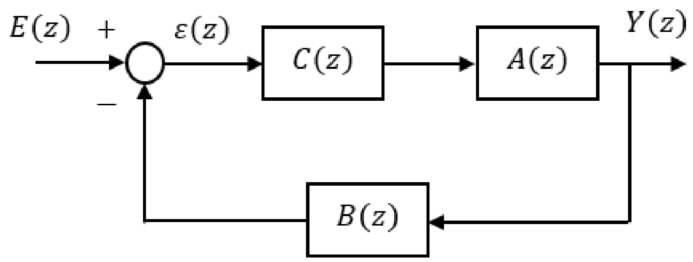


Fig IV. 2. Block diagram of a sampled controlled system.

IV.3 Equivalence of an association of several systems

The equivalent $G(z)$ of a system with a continuous-time transfer function $G(z)$ can only be determined if if its input and output signals are sampled.

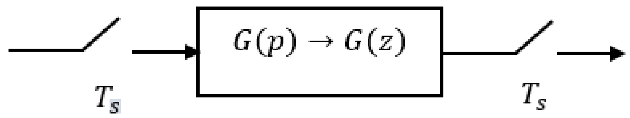


Fig IV.3. Laplace - Z equivalence principle.

As a result, when two systems are cascaded together as shown in Fig IV.4, it is impossible to calculate the equivalent of the global transfer function $G_0(p)$ by purely and simply multiplying of $G_1(z)G_2(z)$ as:

$$G_0(p) = G_1(p)G_2(p) \neq G_1(z)G_2(z) \qquad \text{IV.1}$$

In fact, when looking for the $G_0(z)$ equivalent of $G_0(p)$, we implicitly assume that only the input and output signals of G_0 are sampled. And when we write $G_1(z)G_2(z)$, we assume that the signal leaving $G_1(p)$ and entering $G_1(p)$ is also sampled, otherwise we wouldn't be able to find these two equivalents [7].

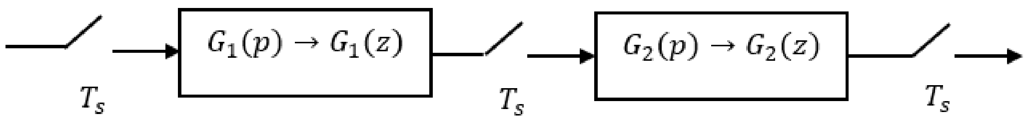


Fig IV.4. Laplace - Z equivalence principle for a cascade association.

IV.4 Control of continuous systems in discrete time

Control systems often include both discrete-time and continuous-time elements. Among these systems, we find in particular analog control systems for which digital controller with computer is required. In this case, the setpoint and output signals are continuous; only the signals entering and leaving the controller are sampled as shown in Fig IV.5 [7].

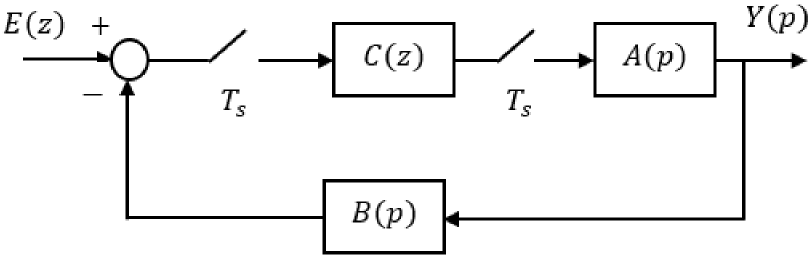


Fig IV.5. Continuous time control system with digital controller.

In other cases, the complete control of a continuous system is driven by a sampled signal as shown in Fig IV.6 [7].

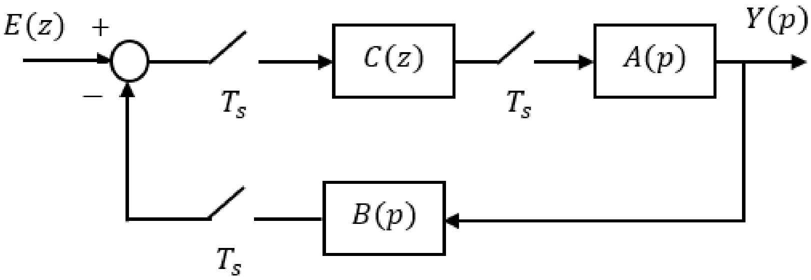


Fig IV.6. A continuous system piloted by a sampled signal.

In reality, the signal delivered by the $C(z)$ system is generally a sequence of numbers delivered in binary form, which is not compatible with the input of a continuous-time system. continuous-time.

In order to recover an "admissible" signal, it is necessary to convert the digital sequence back into discrete pulses using a digital-to-analog converter, and then to hold the signal using a system called a Zero-Order Hold (ZOH).

The function transfer of the Zero-Order Hold (ZOH) is given by [7]:

$$B_0(p) = \frac{1-e^{-pT_e}}{p} \tag{IV.2}$$

IV.5 Continuous-time discrete-time relationship in a closed loop

it is not possible to determine the closed-loop z-transfer function from the equivalent equivalence of the continuous-time closed-loop transfer function $H(p)$. It is necessary to determine independently the z-transfer functions of each subsystem as shown in Fig IV.7 and calculate the closed-loop transfer function $H(z)$ from the expression:

$$H(z) = \frac{A(z)}{1+A(z)+B(z)} \quad \text{IV.3}$$

If $G_i(z)$ and $H_i(z)$ are the open-loop and closed-loop transfer functions of the original system and $G_c(z)$ and $H_c(z)$ are the open-loop and closed-loop transfer functions of the corrected system, we have :

$$G_i(z) = A(z)B(z)H_i(z) = \frac{A(z)}{1+A(z)B(z)} \quad \text{IV.4}$$

$$G_c(z) = A(z)B(z)C(z)H_c(z) = \frac{A(z)C(z)}{1+A(z)B(z)C(z)} \quad \text{IV.5}$$

The control of sampled systems consists in choosing the right transfer function $C(z)$ for this digital controller so as to set each performance to its required value, without, disturbing system operation.

IV.6 Digital controller of a continuous-time system

For the sake of flexibility and accuracy, we often choose to control a continuous-time system numerically. The corresponding control loop diagram is shown in Fig IV.7. A Zero Order Hold (ZOH) must be inserted between the digital controller and the system [7].

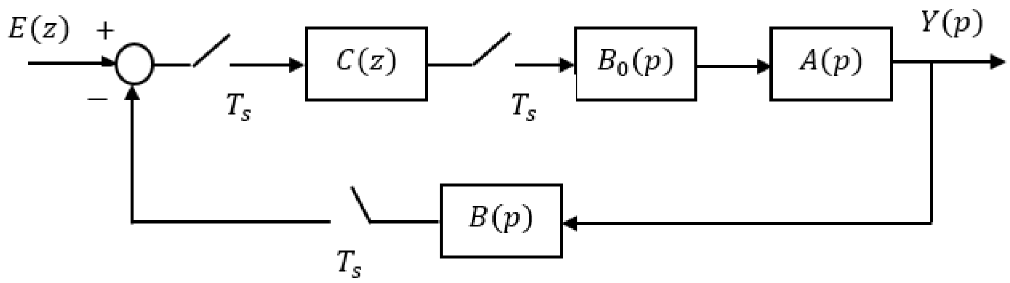


Fig IV.7. Digital controller of a continuous time system.

IV.7 State feedback control

The control system's stability is an important factor in the design of feedback systems. The stability of the control system must be guaranteed, regardless of our objectives. In some cases, a feedback design's primary objective is to increase a system's stability if transitory phenomena do not go away quickly enough or to stabilize a system if it is initially unstable.

In many applications, all the state variables cannot be measured because of cost considerations or because of the lack of suitable transducers. In these cases, those state variables that cannot be measured must be estimated from the ones that are measured. Fortunately, we can separate the design into two phases. During the first phase, we design the system as though all states of the system will be measured. The second phase is concerned with the design of the state estimator.

State feedback uses the state vector to determine the control action for given system dynamics Fig IV.8. shows a linear system (A, B, C) with constant state feedback gain matrix K. Using the principles of matrix multiplication, we deduce that the matrix K is m × n so that for a single-input system K is a row vector [1].

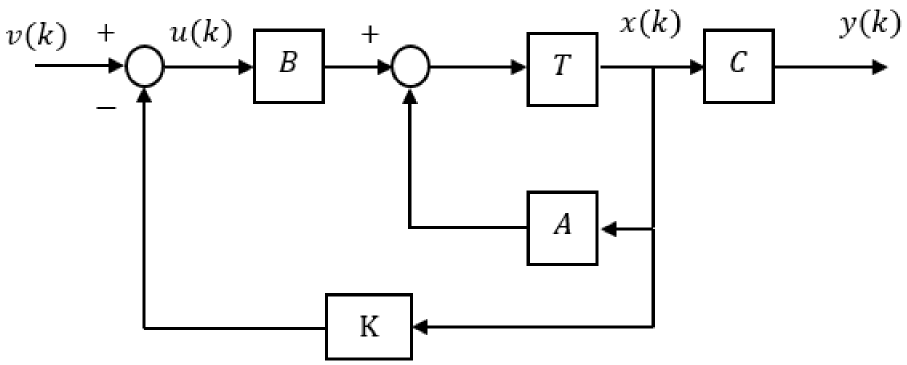


Fig IV.8. Block diagram of constant state feedback control.

The equations that describe the linear system and the feedback control law, respectively, are given by the following equations:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \\ u(k) = -Kx(k) + v(k) \end{cases} \qquad \text{IV.6}$$

The two equations can be combined to yield the closed-loop state equation:

$$\begin{aligned} x(k+1) &= Ax(k) + B[-Kx(k) + v(k)] \\ &= [A - BK]x(k) + Bv(k) \end{aligned} \qquad \text{IV.7}$$

We define the closed-loop state matrix as:

$$A_{cl} = A - BK \qquad \text{IV.8}$$

and rewrite the closed-loop system state–space equations in the form

$$x(k+1) = A_{cl}x(k) + Bv(k) \qquad \text{IV.9}$$

$$y(k) = Cx(k) \quad \text{IV.10}$$

The dynamics of the closed-loop system is influenced by the eigenstructure, which includes the eigenvalues and eigenvectors of the matrix A_{cl}. By appropriately selecting the gain matrix K, we can determine the desired system dynamics.
In many physical systems, measuring all state variables can be very costly or impractical. In such cases, output measurements y must be used to compute the control input u, as illustrated in Fig IV.9 [1].

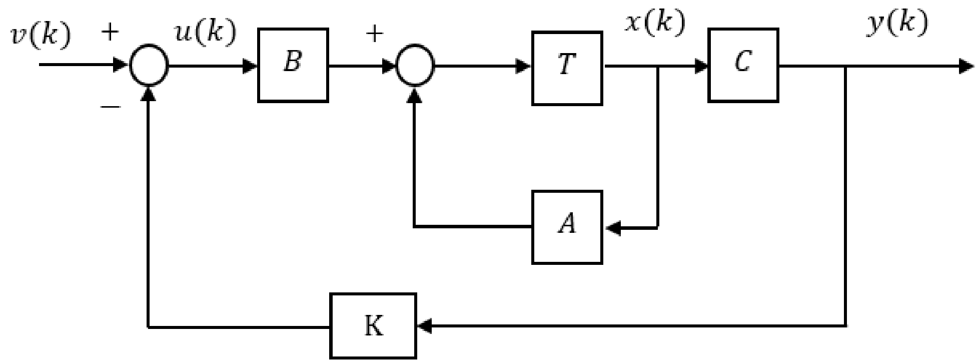


Fig IV.9. Block diagram of constant output feedback control.

The feedback control for output feedback is:

$$\begin{aligned} u(k) &= -K_y x(k) + v(k) \\ &= -K_y C x(k) + v(k) \end{aligned} \quad \text{IV.11}$$

Substituting in the state equation gives the closed-loop system

$$\begin{aligned} x(k+1) &= [Ax(k) + B[K_y C x(k) + v(k)] \\ &= [A - BK_y C]x(k) + Bv(k) \end{aligned} \quad \text{IV.12}$$

The corresponding state matrix is:

$$A_y = A - BK_y C \quad \text{IV.13}$$

We can conclude that output feedback typically provides less information than state feedback when forming a control law. This is due to the limited information used in output feedback. Furthermore, the post-multiplication by the C matrix in equation (IV.13) restricts the selection of closed-loop dynamics. However, output feedback presents a broader design challenge because state feedback is a specific case where the C matrix is the identity matrix.

IV.7.1 Pole placement via state feedback

This part describes a design method often known as pole assignment or pole placement. This approach allows for the placement of the poles of the closed-loop transfer function (which correspond to the zeros of the characteristic equation) to specified locations [9]. We notice that under a mildly restrictive condition (the system must be completely controllable), all the eigenvalues of $(A - BK)$ can be arbitrarily located in the complex plane by choosing k suitably (with the restriction that complex eigenvalues occur in complex-conjugate pairs). If all the eigenvalues of $(A - BK)$ are placed in the left-half plane, the closed-loop system is of course asymptotically stable; $x(1)$ will decay to zero irrespective of the value of $x(0)$, the initial perturbation in the state. The system state is thus maintained at zero value inspite of disturbances that act upon the system. Systems with this property are called regulator systems [1,10].

Theorem of State Feedback.

If the pair (A, B) is controllable, then there exists a feedback gain matrix K that arbitrarily assigns the system poles to any set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Furthermore, if the pair (A, B) is stabilizable, then the controllable modes can all be arbitrarily assigned [1].

Consider the nth-order plant modeled by:

$$x(k+1) = Ax(k) + Bu(k) \qquad \text{IV.14}$$

We generate the control input $u(k)$ by the relationship:

$$u(k) = -Kx(k) \qquad \text{IV.15}$$

where

$$K = [K_1 K_2 \dots\ K_n] \qquad \text{IV.16}$$

Then, expression IV.14 can be written as:

$$x(k+1) = (A - BK)x(k) \qquad \text{IV.17}$$

and, in (IV.17), $A_f = (A - BK)$ We choose the desired pole locations:

$$z = \lambda_1, \lambda_2, \dots, \lambda_n \qquad \text{IV.18}$$

Then the closed-loop system characteristic polynomial is

$$\alpha_c(z) = |zI - A + BK| = (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n) \qquad \text{IV.19}$$

In this equation there are n unknowns, $K_1, K_2, \dots, K_n$, and n known coefficients in the right hand-side polynomial. We can solve for the unknown gains by equating coefficients in (IV.19). An The gain matrix K was determined by equating coefficients in the characteristic equation of (IV.19). This calculation becomes significantly simpler if the state model of the plant is in control canonical form, which is the following form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k) \qquad \text{IV.20}$$

The characteristic equation of this plant is:

$$\alpha(z) = |zI - A + BK| = z^n + a_{n-1}z^{n-1} + \dots + a_1 \mathrm{z} + a_0 = 0 \qquad \text{IV.21}$$

For this state model, the term BK is calculated as:

$$BK = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} [K_1\ K_2 \dots\ K_n] = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ K_1 & K_2 & K_3 & \dots & K_n \end{bmatrix} \qquad \text{IV.22}$$

Hence the closed-loop system matrix is:

$$A - BK = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -(a_0 + K_1) & -(a_1 + K_2) & -(a_2 + K_3) & \dots & -(a_{n-1} + K_n) \end{bmatrix} \qquad \text{IV.23}$$

and the characteristic equation becomes:

$$|zI - A + BK| = z^n + (a_{n-1} + K_n)z^{n-1} + \dots + (a_1 + K_2)\mathrm{z} + (a_0 + K_1) = 0 \qquad \text{IV.24}$$

If we write the desired characteristic equation as:

$$\alpha_c(z) = z^n + \alpha_{n-1}z^{n-1} + \dots + \alpha_1 \mathrm{z} + \alpha_0 = 0 \qquad \text{IV.25}$$

we calculate the gains by equating coefficients in (IV.24) and (IV.25) to yield:

$$K_{i+1} = \alpha_i - a_i, \quad i = 0, 1, \dots, n-1 \qquad \text{IV.26}$$

Procedure

pole placement by equating coefficients

1. Evaluate the desired characteristic polynomial from the specified eigenvalues using the expression:

$$\alpha_c(z) = \prod_{i=1}^{n} (z - \lambda_i)$$

2. Evaluate the closed-loop characteristic polynomial using the expression:

$$\det(zI - A + BK) = |zI - A + BK|$$

3. Equate the coefficients of the two polynomials to obtain n equations to be solved for the entries of the matrix K.

Example

Assign the eigenvalues (0.3 ± j0.2) to the pair $A = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution

For the given eigenvalues the desired characteristic polynomial is:

$$\alpha_c(z) = \prod_{i=1}^{2} (z - \lambda_i) = (z - 0.3 - j0.2)(z - 0.3 + j0.2) = z^2 - 0.6z + 0.13$$

The closed-loop state matrix is:

$$A - BK = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \; K_2] = \begin{bmatrix} 0 & 1 \\ 3 - K_1 & 4 - K_2 \end{bmatrix}$$

The closed-loop characteristic polynomial is:

$$\det(zI - A + BK) = |zI - A + BK|$$
$$= \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 3 - K_1 & 4 - K_2 \end{bmatrix}$$

$$= \begin{bmatrix} z & -1 \\ -3+K_1 & z-4+K_2 \end{bmatrix}$$

$$= z^2 - (4-K_2)z - (3-K_1)$$

Equating coefficients gives the two equations:

$$\begin{cases} 4-K_2 = 0.6 \\ -3+K_1 = 0.13 \end{cases} \Rightarrow \begin{cases} K_2 = 3.4 \\ K_1 = 3.13 \end{cases}$$

that is $K = [3.13\ 3.4]$

Because the system is in controllable form, the same result can be obtained as the coefficients of the open-loop characteristic polynomial minus those of the desired characteristic polynomial.

IV.7.2 Pole placement using a matrix polynomial (Ackermann's Formula)

The gain vector for pole placement can be expressed in terms of the desired closed-loop characteristic polynomial. In general, the techniques used to develop state models for plants do not yield the control canonical form. A more practical method for calculating the gain matrix K is via Ackermann's formula which is based on transformations that convert a general system matrix A into the control canonical form [1].

Consider the state-space model of a SISO system given by:

$$x(k+1) = Ax(k) + Bu(k) \qquad \text{IV.27}$$

The control input is:

$$u(k) = -Kx(k) \qquad \text{IV.28}$$

Thus the closed loop system will be:

$$x(k+1) = (A-BK)x(k) \qquad \text{IV.29}$$

The matrix polynomial $\alpha_c(z)$ is formed using the coefficients of the desired characteristic equation :

$$\alpha_c(z) = A^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 \mathrm{z} + \alpha_0 I = 0 \qquad \text{IV.30}$$

Desired characteristic Equation:

$$|zI - A + BK| = 0 \qquad \text{IV.31}$$

$$\text{Or,} \quad (z - z_1)(z - z_2) \dots (z - z_n) = 0 \qquad \text{IV.32}$$

$$\text{Or,} \quad z^n + \alpha_1 z^{n-1} + \dots + \alpha_{n-1} z + \alpha_n = 0 \qquad \text{IV.33}$$

Then Ackermann's formula for the gain matrix K is given by:

$$K = [0 \;\; 0 \; \dots \; 1] \, C_A^{-1} \alpha_c(A) \qquad \text{IV.34}$$

$$= [0 \;\; 0 \; \dots \; 1] \, [\, B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \, \alpha_c(A) \qquad \text{IV.35}$$

where $\alpha_c(A)$ is the closed loop characteristic polynomial and C_A is the controllability matrix.

$$C_A = [\, B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \qquad \text{IV.36}$$

Example

Find out the state feedback gain matrix K for the following system using two different methods such that the closed loop poles are located at 0.5, 0.6 and 0.7.

$$x(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Solution

$$C_A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

The above matrix has rank 3, so the system is controllable.

Open loop characteristic equation: $z^3 + 3z^2 + 2z + 1 = 0$

Desired characteristic equation:

$$(z - 0.5)(z - 0.6) \dots (z - 0.7) = 0$$

$$\text{Or,} \quad z^3 - 1.8z^2 + 1.07z - 0.21 = 0$$

Since the open loop system is already in controllable canonical form, T = I.

$$K = [\, \alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1]$$

where

$$\alpha_3 = -0.21, \alpha_2 = 1.07, \alpha_1 = -1.8, \text{and } a_3 = 1, a_2 = 2, a_1 = 3.$$

Using Ackermann's formula:

$$K = [\,-1.21 \quad -0.93 \quad -4.8]$$

$$C_A^{-1} = -\begin{bmatrix} -2 & -3 & -1 \\ -3 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\alpha_c(A) = A^3 - 1.8z^2 + 1.07A - 0.21I$$

$$= \begin{bmatrix} -1.21 & -0.93 & -4.8 \\ 4.8 & 8.39 & 13.47 \\ -13.47 & -22.14 & -30.02 \end{bmatrix}$$

Thus

$$K = [0 \ \ 0 \ \ 1]\, C_A^{-1}\alpha_c(A)$$

$$\Rightarrow K = [0 \ \ 0 \ \ 1] \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \alpha_c(A) = [-1.21 - 0.93 - 4.8]$$

IV.7.3 Pole placement by transformation to controllable form

The problem is first solved for the controllable canonical form. Let us denote the controllability matrix by C_A and consider a transformation matrix T as [1,3]:

$$T = C_A W \qquad \text{IV.37}$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

ai's are the coefficients of the characteristic polynomial:

$$|zI - A| = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \qquad \text{IV.38}$$

Define a new state vector $x = Tx_t$. This will transform the system given by:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases}$$

into controllable canonical form, as:

$$x_t(k+1) = A_t x_t(k) + B_t u(k) \qquad \text{IV.39}$$

You should verify that

$$A_t = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \qquad \text{IV.40}$$

and

$$B_t = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad \text{IV.41}$$

We first find K_t such that $u(k) = -K_t\, x_t(k)$ places poles in desired locations. Since eigenvalues remain unaffected under similarity transformation, $u(k) = -K_t T^{-1} x(k)$ will also place the poles of the original system in desired locations.

If poles are placed at $z_1, z_2, \dots, z_n$, the desired characteristic equation can be expressed as:

$$(z - 0.5)(z - 0.6) \dots (z - 0.7) = 0$$

$$\text{Or,} \quad z^3 - 1.8z^2 + 1.07z - 0.21 = 0 \qquad \text{IV.42}$$

$$\text{Or,} \quad (z - z_1)(z - z_2) \dots (z - z_n) = 0$$

$$\text{Or,} \quad z^n + \alpha_1 z^{n-1} + \dots + \alpha_{n-1} z + \alpha_n = 0 \qquad \text{IV.43}$$

Since the pair (A_t, B_t) are in controllable-companion form then, we have:

$$A_t - B_t K_t = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_n - K_{t,1}) & -(a_{n-1} - K_{t,2}) & \dots & \dots & -(a_1 - K_{t,n}) \end{bmatrix} \qquad \text{IV.44}$$

Note that the characteristic equation of both original and canonical form is expressed as:

$$|zI - A| = |zI - A_t| = z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0 \qquad \text{IV.45}$$

The characteristic equation of the closed loop system with $u(k) = -K_t\, x_t(k)$ is given by:

$$z^n + (a_1 + K_{t,n}) z^{n-1} + (a_2 + K_{t,n-1}) z^{n-2} + \dots + (a_n + K_{t,1}) = 0 \qquad \text{IV.46}$$

Comparing Eqs IV.45 and IV.46, we get:

$$K_{t,n} = \alpha_1 - a_1, K_{t,n-1} = \alpha_2 - a_2, \ldots, K_{t,1} = \alpha_n - a_n \qquad \text{IV.47}$$

We need to compute the transformation matrix T to find the actual gain matrix:

$$K = K_t T^{-1} \qquad \text{IV.48}$$

where $K_t = [K_{t,1}, K_{t,2}, \ldots, K_{t,n}]$.

Example

Find out the state feedback gain matrix K for the following system by converting the system into controllable canonical form such that the closed loop poles are located at 0.5 and 0.6.

$$x(k+1) = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Solution

$$C_A = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$$

The above matrix has rank 2, so the system is controllable.

Open loop characteristic equation:

$$z^2 + 3z + 2 = 0$$

Desired characteristic equation:

$$(z - 0.5)(z - 0.6) = 0$$

$$\text{Or,} \quad z^2 - 1.1z + 0.3 = 0$$

To convert into controllable canonical form:

$$W = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

The transformation matrix:

$$T = C_A W = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

Verify

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now

$$\alpha_1 = -1.1, \alpha_2 = 0.3, a_1 = 3, a_2 = 2$$

Thus

$$K = [\alpha_2 - a_2 \; \alpha_1 - a_1] = [-1.7 \quad -4.1]$$

Then

$$K = K_t T^{-1} = [-1.7 \quad -4.1]\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = [-2.4 \quad -4.1]$$

IV.7.4 Improvement of stability by state feedback

The placement of poles is vital in control systems for several reasons, impacting both stability and performance attributes of a system. Key factors include:

- ***Stability Assurance***: By ensuring all poles lie in the left half of the complex plane, the system can achieve stable behavior.
- ***Dynamic Response Shaping***: Control over parameters such as natural frequency and damping ratio
- ***Time Domain Specifications***: Enhancing features like rise time, overshoot, and settling time.

IV.8 Diophantine equation

Consider the system defined by the pulse transfer function [10]:

$$\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)} \quad \text{IV.49}$$

where

$$A(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \quad \text{IV.50}$$

$$B(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1}z + b_n \quad \text{IV.51}$$

Assume that this pulse transfer function system is completely state controllable and completely observable. That is, there is no pole-zero cancellation in the pulse transfer function, or $A(z)$ and $B(z)$ have no common factors. When polynomials $A(z)$ and $B(z)$ have no cancellation, these polynomials are called coprime polynomials. A polynomial in z is called monic if the coefficient of the highest-degree term is unity. Thus, polynomial $A(z)$ is monic.

Next, let us define a stable (2n-1)th-degree polynomial $D(z)$ as follows [10]:

$$D(z) = d_0 z^{2n-1} + d_1 z^{2n-2} + \cdots + d_{2n-2}z + d_{2n-1} \quad \text{IV.52}$$

Then there exist unique (n - 1)th-degree polynomials $\alpha(z)$ and $\beta(z)$ such that

$$\alpha(z)A(z) + \beta(z)B(z) = D(z) \qquad \text{IV.53}$$

where

$$\alpha(z) = \alpha_0 z^{n-1} + \alpha_1 z^{n-2} + \cdots + \alpha_{n-2}\text{z} + \alpha_{n-1} \qquad \text{IV.54}$$

$$\beta(z) = \beta_0 z^{n-1} + \beta_1 z^{n-2} + \cdots + \beta_{n-2}\text{z} + \beta_{n-1} \qquad \text{IV.55}$$

Equation (IV.53) is called a Diophantine equation. The Diophantine equation can be solved for $\alpha(z)$ and $\beta(z)$ by use of the following $(2n \times 2n)$ Sylvester matrix E, which is defined in terms of the coefficients of coprime polynomials $A(z)$ and $B(z)$ as follows:

$$E = \begin{bmatrix} a_n & 0 & \dots & 0 & \dots & b_n & & \dots & 0 \\ a_{n-1} & a_n & \dots & 0 & \dots & b_{n-1} & & \dots & 0 \\ \vdots & a_{n-1} & \dots & 0 & \dots & \vdots & & \vdots & \vdots \\ a_1 & \vdots & \vdots & \vdots & \vdots & b_1 & & \dots & 0 \\ 1 & a_1 & \dots & a_n & \dots & 0 & \cdots & & b_n \\ 0 & 1 & \cdots & a_{n-1} & \cdots & \cdots & \cdots & & b_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \dots & 0 & \dots & & b_1 \end{bmatrix} \qquad \text{IV.56}$$

Polynomial A(z) must be monic, otherwise, we must modify Equation (IV.53).] If n = 4, then this matrix becomes as follows:

$$E = \begin{bmatrix} a_4 & 0 & 0 & 0 & b_4 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & b_3 & b_4 & 0 & 0 \\ a_2 & a_3 & a_4 & 0 & b_2 & b_3 & b_4 & 0 \\ a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & b_4 \\ 1 & a_1 & & & b_0 & b_1 & b_2 & b_3 \\ 0 & 1 & & & 0 & b_0 & b_1 & b_2 \\ 0 & 0 & 1 & a_1 & 0 & 0 & b_0 & b_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_0 \end{bmatrix}$$

The Sylvester matrix E is nonsingular if and only if $A(z)$ and $B(z)$ are coprime, or have no common factors. This fact may be seen from the following: Referring to the preceding 8 x 8 matrix E, the determinant E becomes as follows:

$$|E| = \begin{vmatrix} a_4 & 0 & 0 & 0 & b_4 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & b_3 & b_4 & 0 & 0 \\ a_2 & a_3 & a_4 & 0 & b_2 & b_3 & b_4 & 0 \\ a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & b_4 \\ 1 & a_1 & a_2 & a_3 & b_0 & b_1 & b_2 & b_3 \\ 0 & 1 & a_1 & a_2 & 0 & b_0 & b_1 & b_2 \\ 0 & 0 & 1 & a_1 & 0 & 0 & b_0 & b_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_0 \end{vmatrix}$$

$$\begin{aligned} &= b_0(\lambda_1-\lambda_5)(\lambda_1-\lambda_6)(\lambda_1-\lambda_7)(\lambda_1-\lambda_8) \\ &\quad .(\lambda_2-\lambda_5)(\lambda_2-\lambda_6)(\lambda_2-\lambda_7)(\lambda_2-\lambda_8) \\ &\quad .(\lambda_3-\lambda_5)(\lambda_3-\lambda_6)(\lambda_3-\lambda_7)(\lambda_3-\lambda_8) \\ &\quad .(\lambda_4-\lambda_5)(\lambda_4-\lambda_6)(\lambda_4-\lambda_7)(\lambda_4-\lambda_8) \end{aligned} \qquad \text{IV.57}$$

where

$a_1, \dots, a_4$ and $b_1, \dots, b_4$ are coefficients of $A(z)$ and $B(z)$, respectively, and $\lambda_1, \dots, \lambda_4$ and $\lambda_5, \dots, \lambda_8$ are characteristic roots of $A(z)$ and $B(z)$, respectively:

$$A(z) = z^4 + a_1 z^3 + a_2 z^2 + a_3 \mathrm{z} + a_4 = (z-\lambda_1)(z-\lambda_2)(z-\lambda_3)(z-\lambda_4)$$
$$B(z) = b_0 z^4 + b_1 z^3 + b_2 z^2 + b_3 \mathrm{z} + b_4 = b_0(z-\lambda_5)(z-\lambda_6)(z-\lambda_7)(z-\lambda_8)$$

It is clear that the determinant |E| is nonzero if and only if all multiplicative factors on the right-hand side of the equation are nonzero, that is, if and only if no cancellation occurs between $A(z)$ and $B(z)$.

Now define vectors D and M such that:

$$D = \begin{bmatrix} d_{2n-1} \\ d_{2n-2} \\ \vdots \\ d_1 \\ d_0 \end{bmatrix}, M = [\alpha_{n-1} \quad \alpha_{n-2} \quad \dots \quad \alpha_0 \quad \beta_{n-1} \quad \beta_{n-2} \quad \dots \quad \beta_0]^T$$

Then the coefficients $\alpha_0, \alpha_1 \dots, \alpha_{n-1}$ and $\beta_0, \beta_1, \dots, \beta_{n-1}$ can be determined from:

$$M = E^{-1} D \qquad \text{IV.58}$$

Equation (IV.58) gives the solution to the Diophantine equation.

Example

Consider the following A(z) (a monic polynomial of degree 2), B(z) (a polynomial of degree 1), and D(z) (a polynomial of degree 3):

$$A(z) = z^2 + \text{ z} + 0.5$$
$$B(z) = \text{z} + 2$$
$$D(z) = z^3$$

Clearly, there is no common factor between A(z) and B(z). The problem here is to find unique polynomials a(z) and ß(z) such that

$$\alpha(z)A(z) + \beta(z)B(z) = D(z)$$

Where

$$\alpha(z) = \alpha_0 z^{n-1} + \alpha_1$$
$$\beta(z) = \beta_0 \text{z} + \beta_1$$

Or

$$(\alpha_0 z^{n-1} + \alpha_1)(z^2 + \text{ z} + 0.5) + (\beta_0 \text{z} + \beta_1)(\text{z} + 2) = z^3$$

To solve this Diophantine equation for $\alpha(z)$ and $\beta(z)$, first determine a1, a2, b0, b1 and b2, and then write the Sylvester matrix E as follows:

$$E = \begin{bmatrix} a_2 = 0.5 & 0 & b_2 = 2 & 0 \\ a_1 = 1 & a_2 = 0.5 & b_1 = 1 & b_2 = 2 \\ 1 & a_1 = 1 & b_0 = 0 & b_1 = 1 \\ 0 & 1 & 0 & b_0 = 0 \end{bmatrix}$$

Since $D(z) = z^3$
we have $d_0 = 1$, $d_1 = 0$, $d_2 = 0$, and, $d_3 = 0$,

Thus, matrix D and M matrix becomes:

$$D = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

The solution to the Diophantine equation is obtained from $M = E^{-1}D$ as follows:

$$\alpha(z) = \alpha_0 z + \alpha_1 = z - 1.2$$
$$\beta(z) = \beta_0 \text{z} + \beta_1 = 0.2z + 0.3$$

The polynomials $\alpha(z)$ and $\beta(z)$ thus determined will satisfy the Diophantine equation given by equation. To verify, notice that

$$(z-1.2)(z^2+\ z+0.5)+(0.2z+0.3)(z+2)=z^3$$

IV.9 Polynomial Equations Approach to Design Regulator System.

Consider the block diagram shown in Fig IV.10. The feedback pulse transfer function B(z)/a(z) serves as a regulator [10].

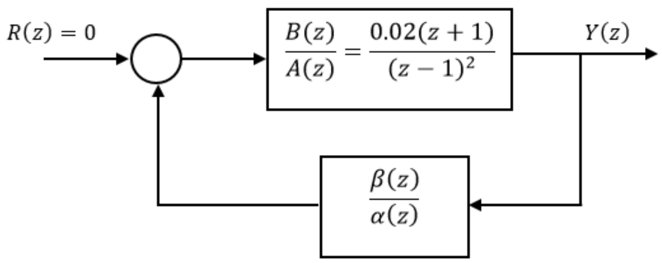


Fig IV.10. Block diagram of regulator system.

Let us determine $\alpha(z)$ and $\beta(z)$ by use of the polynomial equations approach. First note that the pulse transfer function of the plant is:

$$\frac{Y(z)}{U(z)}=\frac{B(z)}{A(z)}=\frac{0.02(z+1)}{(z-1)^2} \qquad \text{IV.59}$$

$A(z)$ is a monic polynomial of degree 2 and there is no cancellation between $A(z)$ and $B(z)$. Then, although $R(z)=0$, the closed-loop pulse transfer function for the system can be given by:

$$\frac{Y(z)}{U(z)}=\frac{\alpha(z)\beta(z)}{\alpha(z)A(z)+\beta(z)B(z)}=\frac{0.02(z+1)\alpha(z)}{\alpha(z)(z-1)^2+\beta(z)0.02(z+1)} \qquad \text{IV.60}$$

The desired closed-loop poles for state feedback are:

$$z_1=0.6+j0.4,\ z_2=0.6-j0.4$$

Or the desired characteristic polynomial is given by:

$$H(z)=(z-0.6-j0.4)(z-0.6+j0.4)=z^2-1.2z+0.52$$

The desired minimum-order observer error polynomial is: $F(z)=z$

To determine $\alpha(z)$ and $\beta(z)$ we solve the following Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = F(z)H(z) = D(z) \qquad \text{IV.61}$$

$$D(z) = d_0 z^3 + d_1 z^2 + d_2 z + d_3 = z^3 - 1.2z^2 + 0.52z$$

where $H(z)$ is the desired characteristic polynomial for pole placement part, and $F(z)$ is the characteristic polynomial for the minimum-order observer. Both polynomials H(z) and $F(z)$ are stable polynomials. The degree of polynomial $H(z)$ is n and the degree of polynomial $F(z)$is n-1. We assume that the system output is the only measurable state variable. Therefore, the order of the minimum-order observer is (n – 1).

Note that $D(z)$ is a stable (2n-1)th-degree polynomial in z (where n=2 in the present case). Since,

$$A(z) = z^2 - 2\,\mathrm{z} + 1$$

$$B(z) = 0.02\mathrm{z} + 0.02$$

We have

$$a_1 = -2,\, a_2 = 1,\, b_0 = 0,\, b_1 = 0.02,\, b_2 = 0.02$$

By substituting the polynomial expressions for $A(z)$, $B(z)$ and $D(z)$ into equation (IV.61), we obtain:

$$\alpha(z)(z^2 - 2\,\mathrm{z} + 1) + \beta(z)(0.02\mathrm{z} + 0.02) = z^3 - 1.2\,z^2 + 0.52z$$

To solve this Diophantine equation for $\alpha(z)$ and $\beta(z)$, we first define 2nx2n (where n=2) Sylvester matrix E:

$$E = \begin{bmatrix} 1 & 0 & 0.02 & 0 \\ -2 & 1 & 0.02 & 0.02 \\ 1 & -2 & 0 & 0.02 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E^{-1} = \begin{bmatrix} 0.25 & -0.25 & 0.25 & 0.75 \\ 0 & 0 & 0 & 1 \\ 37.5 & 12.5 & -12.5 & -37.5 \\ -12.5 & 12.5 & 37.5 & 62.5 \end{bmatrix}$$

are polynomials of degree n-1=2-1=1.

$$\alpha(z) = \alpha_0 z + \alpha_1$$

$$\beta(z) = \beta_0 \mathrm{z} + \beta_1$$

$$D=\begin{bmatrix}d_3\\d_2\\d_1\\d_0\end{bmatrix}=\begin{bmatrix}0\\0.52\\-1.2\\1\end{bmatrix},\quad M=\begin{bmatrix}\alpha_1\\\alpha_0\\\beta_1\\\beta_0\end{bmatrix}$$

$$M=E^{-1}D=\begin{bmatrix}0.25 & -0.25 & 0.25 & 0.75\\0 & 0 & 0 & 1\\37.5 & 12.5 & -12.5 & -37.5\\-12.5 & 12.5 & 37.5 & 62.5\end{bmatrix}\begin{bmatrix}0\\0.52\\-1.2\\1\end{bmatrix}=\begin{bmatrix}0.32\\1\\-16\\24\end{bmatrix},$$

Hence

$$\alpha_1=0.32, \alpha_0=1,\ \beta_1=-16, \beta_0=24$$

Therefore $\alpha(z)$ and $\beta(z)$ are determined as

$$\alpha(z)=\alpha_0 z+\alpha_1=z+0.32$$
$$\beta(z)=\beta_0\mathrm{z}+\beta_1=24z-16$$

And the feedback regulator is obtained as

$$\frac{\beta(z)}{\alpha(z)}=24\left(\frac{z-0.6667}{z+0.32}\right)$$

IV.9.1 Polynomial Equations Approach to control Systems design

In the precedent section, we designed a regulator system by use of the polynomial equations approach. The block diagram of the regulator system designed is shown in Fig IV.12.

$\alpha(z)$ and $\beta(z)$ were obtained from the following Diophantine equation [10]:

$$\alpha(z)A(z)+\beta(z)B(z)=F(z)H(z) \qquad \text{IV.62}$$

where $A(z)$ is a monic polynomial of degree n, $B(z)$ is a polynomial of degree m $(m\le n)$ and we assume that there are no common factors between $A(z)$ and $B(z)$.

$H(z)$ is the desired characteristic polynomial for pole placement part, and $F(z)$ is the characteristic polynomial for the minimum-order observer. Both polynomials $H(z)$ and $F(z)$ are stable polynomials. The degree of polynomial $H(z)$ is n and the degree of polynomial $F(z)$ is n-1. We assume that the system output is the only measurable state variable. Therefore, the order of the minimum-order observer is (n – 1).

In the following we discuss the design of control systems based on the polynomial equations approach. We consider two different control system configurations.

IV.9.2 Control System Configuration 1

The regulator system shown in Fig IV.11 can be modified to a control system such that the output follows the reference input. A possible block diagram for the control system is shown in Fig IV.12. As a control system, it is necessary to have an adjustable gain K_0. This gain K_0 should be set such that the steady-state output $y(k)$ is equal to unity when the input $r(k)$ is a unit-step sequence [10].

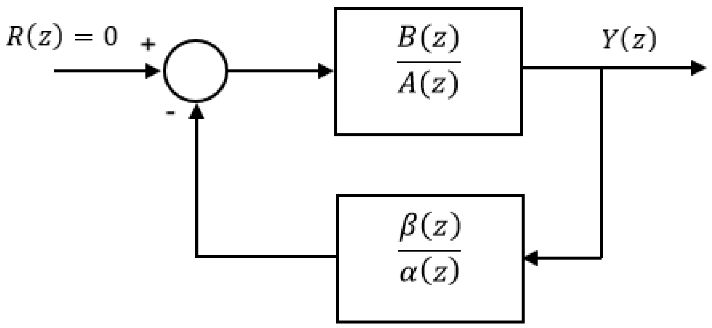


Fig IV.11. Block diagram of regulator system.

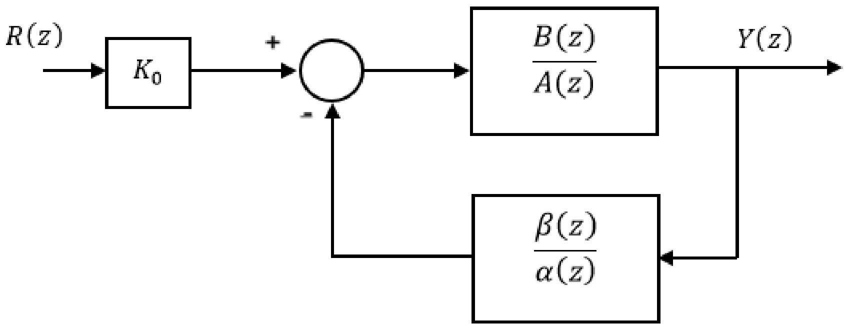


Fig IV.12. Block diagram of a control system.

The closed loop function transfer $Y(z)/R(z)$ is given by [10]:

$$\frac{Y(z)}{R(z)} = K_0 \frac{B(z)/A(z)}{1 + \left(\frac{B(z)\beta(z)}{A(z)\alpha(z)}\right)}$$

$$= K_0 \frac{\alpha(z)B(z)}{\alpha(z)A(z)+\beta(z)B(z)}$$

$$= K_0 \frac{\alpha(z)B(z)}{F(z)H(z)} \qquad \text{IV.63}$$

Notice that the closed-loop system is of (2n-1)th order, unless cancellation occurs between $\alpha(z)B(z)$ and $F(z)H(z)$ Notice also that the numerator dynamics has been changed from $B(z)$ to $K_0\alpha(z)B(z)$.

To determine gain K_0, we set:

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1 - z^{-1})Y(z)$$
$$= \lim_{z\to 1}\frac{z-1}{z}K_0\frac{\alpha(z)B(z)}{F(z)H(z)}\frac{z}{z-1}$$
$$= K_0\frac{\alpha(1)B(1)}{F(1)H(1)} = 1$$

From which we get:

$$K_0 = \frac{\alpha(1)B(1)}{F(1)H(1)} \qquad \text{IV.64}$$

Example

Consider the precedent regulator system example

$A(z) = (z-1)^2$, $B(z) = 0.02(z+1)$, $H(z) = z^2 - 1.2\,\mathrm{z} + 0.52$, $F(z) = \mathrm{z}$, $\alpha(z) = z + 0.32$, $\beta(z) = 24z - 16$.

The closed loop function transfer $Y(z)/R(z)$ is determined as:

$$\frac{Y(z)}{R(z)} = K_0\frac{(z+0.32)0.02(z+1)}{z^3 - 1.2\,z^2 + 0.52z}$$

And the gain K_0 is:

$$K_0 = \frac{\alpha(1)B(1)}{F(1)H(1)} = \frac{0.32\times 1}{1.32\times 0.04} = 1$$

Notice that the system is of third order.

IV.9.3 Control System Configuration 2

A control system with a different block diagram configuration may be designed by use of the polynomial equations approach. Consider the block diagram shown in Fig IV.13.

From Fig IV.13, we obtain the following equation [10]:

$$U(z) = -\left[\frac{\alpha(z)}{F(z)}U(z) - U(z) + \frac{\beta(z)}{F(z)}Y(z)\right] + K_0R(z) \qquad \text{IV.65}$$

Which can be simplified as:

$$\frac{\alpha(z)}{F(z)}U(z) = -\frac{\beta(z)}{F(z)}Y(z) + K_0R(z) \qquad \text{IV.66}$$

The pulse transfer function of the plant is given by:

$$\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)} \qquad \text{IV.67}$$

where $A(z)$ is a monic polynomial of degree n and $B(z)$ is a stable polynomial of degree m ($m \leq n$). Since

$$U(z) = \frac{A(z)}{B(z)}Y(z) \qquad \text{IV.68}$$

By substituting equation (IV.68) into equation (IV.66), we obtain:

$$\left[\frac{\alpha(z)}{F(z)}\frac{A(z)}{B(z)} + \frac{\beta(z)}{F(z)}\right]Y(z) = K_0R(z) \qquad \text{IV.69}$$

Then

$$\frac{Y(z)}{R(z)} = \frac{K_0}{\frac{\alpha(z)}{F(z)}\frac{A(z)}{B(z)} + \frac{\beta(z)}{F(z)}}$$

$$= \frac{K_0F(z)B(z)}{\alpha(z)A(z) + \beta(z)B(z)}$$

$$= K_0\frac{F(z)B(z)}{F(z)H(z)} \qquad \text{IV.70}$$

Thus,

$$\Rightarrow \quad \frac{Y(z)}{R(z)} = K_0\frac{B(z)}{H(z)} \qquad \text{IV.71}$$

Notice that the observer polynomial F(z) has been canceled, since F(z) is a stable polynomial, cancellation of F(z) is permissible, and the characteristic polynomial for the closed-loop system is given by H(z). H(z) is a desired, but in a sense "arbitrarily chosen," stable polynomial of degree n. Thus, the control system designed is of the nth order. In the case of control system configuration 1, the order of the system is 2n-1, unless cancellations occur in the designed system, resulting in the reduction of the system order. Notice also that the numerator dynamics of Y(z)/R(z) has not been changed in the present approach.

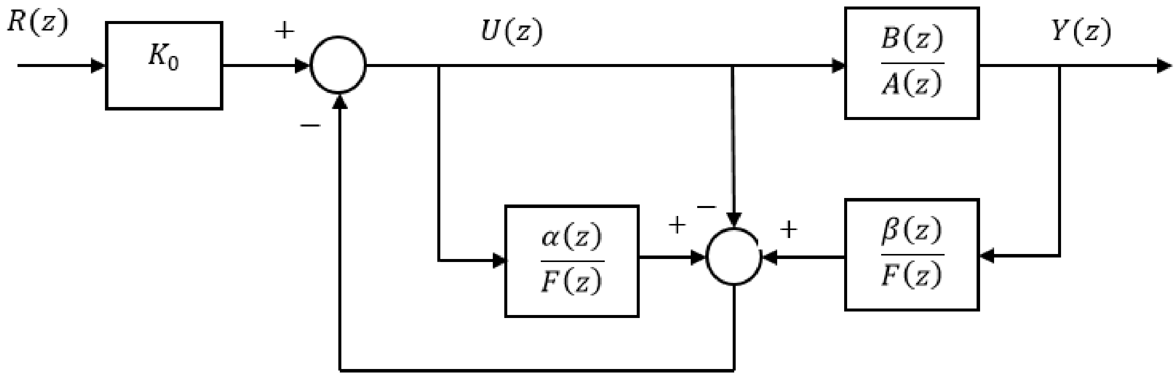


Fig IV.13. Block diagram of the control system.

Example

Let us design a control system based on the block diagram shown in Fig IV.13. The plant we consider is given by:

$$\frac{B(z)}{A(z)} = \frac{0.02(z+1)}{(z-1)^2}$$

The desired closed-loop poles for state feedback are:

$$z_1 = 0.6 + j0.4,\ z_2 = 0.6 - j0.4$$

Or the desired characteristic polynomial is given by:

$$H(z) = (z - 0.6 - j0.4)(z - 0.6 + j0.4) = z^2 - 1.2z + 0.52$$

The desired minimum-order observer error polynomial is $\phi(z) = z$ and the desired characteristic observer polynomial is $F(z) = z$.

Then, we solve the Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = H(z)F(z)$$
$$\alpha(z)(z^2 - 2\,\mathrm{z} + 1) + \beta(z)(0.02\mathrm{z} + 0.02) = z^3 - 1.2\,z^2 + 0.52z$$

Therefore $\alpha(z)$ and $\beta(z)$ are determined as:

$$\alpha(z) = \alpha_0 z + \alpha_1 = z + 0.32$$

$$\beta(z) = \beta_0 \mathrm{z} + \beta_1 = 24z - 16$$

The closed loop pulse transfer function $Y(z)/R(z)$ can be written as follows:

$$\frac{Y(z)}{R(z)} = K_0 \frac{B(z)}{H(z)} = \frac{K_0(0.02z + 0.02)}{z^2 - 1.2\,\mathrm{z} + 0.52}$$

and the constant gain K_0 is determined for a unit step response of unity.

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1 - z^{-1})Y(z)$$
$$= \lim_{z\to 1} \frac{z-1}{z} \frac{K_0(0.02z + 0.02)}{z^2 - 1.2\,\mathrm{z} + 0.52} \frac{z}{z-1}$$
$$= \frac{K_0}{8} = 1$$

$$\Rightarrow \; K_0 = 8$$

Finally, the closed loop pulse transfer function becomes:

$$\frac{Y(z)}{R(z)} = \frac{0.16z + 0.16}{z^2 - 1.2\,\mathrm{z} + 0.52}$$

Clearly, the system designed is of second order. A block diagram for the designed system is shown in Fig IV.14 a). Fig IV.14 b) Shows a simplified block diagram [10].

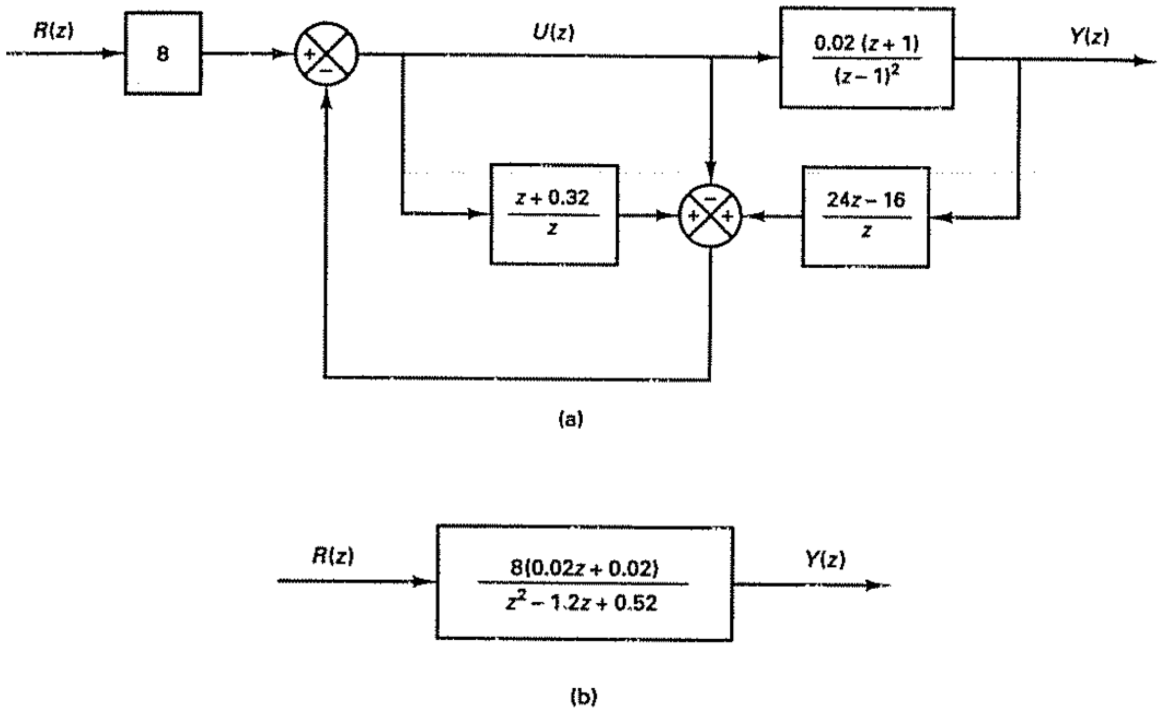


Fig IV.14. a) Block diagram of the control system designed by use of polynomial equations approach, b) Simplified block diagram.

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