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Étude de quelques systèmes différentiels couplés d'ordre fractionnaire

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Publications

1. S. Aibout, S. Abbas, M. Benchohra and M. Bohner, A coupled Caputo-Hadamard fractional differential system with multipoint boundary conditions, *Dynamics Con. Discrete Impul. Sys. Series A : Math. Anal.* **29** (2022) 191-209.

2. S. Aibout, A. Salim, S. Abbas and M. Benchohra, Coupled systems of Conformable fractional differential equations, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **..** (.),2023 1-1.

Introduction

The concept of fractional differential calculus has a long history. One may wonder what meaning may be ascribed to the derivative of a fractional order, that is $\frac{d^n y}{dx^n}$, where n is a fraction. In fact L'Hopital himself considered this very possibility in a correspondence with Leibniz, In 1695, L'Hopital wrote to Leibniz to ask, " *What if* n *be* $\frac{1}{2}$" From this question, the study of fractional calculus was born. Leibniz responded to the question, "$d^{\frac{1}{2}}x$ *will be equal to* $x\sqrt{dx : x}$. *This is an apparent paradox from which, one day, useful consequences will be drawn.*"

Many known mathematicians contributed to this theory over the years. Thus, 30 September 1695 is the exact date of birth of the "***fractional calculus***"! Therefore, the fractional calculus it its origin in the works by *Leibnitz*, *L'Hopital* (1695), *Bernoulli* (1697), *Euler* (1730), and *Lagrange* (1772). Some years later, *Laplace* (1812), *Fourier* (1822), *Abel* (1823), *Liouville* (1832), *Riemann* (1847), *Grünwald* (1867), *Letnikov* (1868), *Nekrasov* (1888), *Hadamard* (1892), *Heaviside* (1892), *Hardy* (1915), *Weyl* (1917), *Riesz* (1922), *P. Levy*(1923), *Davis* (1924), *Kober* (1940), *Zygmund* (1945), *Kuttner* (1953), *J. L. Lions* (1959), and *Liverman* (1964)... have developed the basic concept of fractional calculus.

In June 1974, Ross has organized the "*First Conference on Fractional Calculus and its Applications*" at the University of New Haven, and edited its proceedings [133]; Thereafter, Spanier published the first monograph devoted to "*Fractional Calculus*" in 1974 [123]. The integrals and derivatives of non-integer order, and the fractional integrodifferential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics. There exists the remarkably comprehensive encyclopedic-type monograph by *Samko*, *Kilbas* and *Marichev* which was published in Russian in 1987 . (for more details see [118]) The works devoted substantially to fractional differential equations are : the book of *Miller* and *Ross* (1993) [120], of *Podlubny* (1999) [126], by *Diethelm* (2010) [68], by *Ortigueira* (2011) [125], by *Abbas et al.* (2012) [8], and by *Baleanu et al.* (2012) [39].

In recent years, there has been a significant development in the theory of fractional differential equations. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, chaos and fractals, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, proteins, biosciences, bioengineering, etc. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. See for example [40, 41, 84, 86, 118, 126, 138, 145].

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator D^{α}_{a+} where $\alpha, a, \in \mathbb{R}$. Several approaches to fractional derivatives exist : Riemann-Liouville (RL),Caputo and Hadamard etc.
Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year ; see for instance [20] and the references therein.

There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\alpha(B)$ of a bounded set B in a metric space is defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of sets of diameter smaller than r. The Hausdorf measure of noncompactness $\chi(B)$ defined as infimum of numbers $r > 0$ such that B can be covered with a finite number of balls of radii smaller than r. Several authors have studied the measures of noncompactness in Banach spaces. See, for example, the books such as [28, 42, 146] and the articles [31, 43, 44, 49, 55, 57, 87, 121],and references therein.

Considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivative. See for example [19, 22, 25, 26, 38, 50, 51, 52, 55, 88, 98, 111, 112, 113, 115, 144, 159], and references therein.

In the theory of ordinary differential equations in a Banach space there are several types of data dependence . On the other hand, in the theory of functional equations there are some special kind of data dependence : Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers- Bourgin, Aoki-Rassias [134].

The stability problem of functional equations originated from a question of Ulam [148, 149] concerning the stability of group homomorphisms : " *Under what conditions does there exist an additive mapping near an approximately additive mapping* " Hyers [89] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers Theorem was generalized by Aoki [34] for additive mappings and by T.M. Rassias [129] for linear mappings by considering an unbounded Cauchy difference. A generalization of the T.M. Rassias theorem was obtained by Gavruta [74].

After, many interesting results of the generalized Hyers-Ulam stability to a number of functional equations have been investigated by a number of mathematicians ; see [4, 30, 47, 92, 93, 95, 96, 107, 127, 151, 153, 154] and the books [64, 130, 131] and references therein.
We have organized this thesis as follows :

Chapter 1.
This chapter consists of three Sections.
In Section one, we present "A brief visit to the history of the Fractional Calculus", and in Section two, we present some "Applications of Fractional calculus".
Finally, in the last Section, we recall some preliminarys : some basic concepts, and useful famous theorems and results (notations, definitions, lemmas and fixed point theorems) which are used throughout this thesis.

Chapter 2.
This chapter consists of two Sections.
In the first section ; we discuss and establish the existence,the uniqueness of solu-

tions for a coupled Caputo–Hadamard fractional differential system in Banach spaces.

We will give existence and uniqueness of solutions for a coupled system of fractional differential equations of the form

$$\begin{cases} ({}^{\mathrm{HC}}D_1^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ ({}^{\mathrm{HC}}D_1^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in I := [1, T],$$

with the multipoint boundary conditions

$$\begin{cases} a_1u(1) - b_1u'(1) = d_1u(\xi_1) \\ a_2u(T) + b_2u'(T) = d_2u(\xi_2) \\ a_3v(1) - b_3v'(1) = d_3v(\xi_3) \\ a_4v(T) + b_4v'(T) = d_4v(\xi_4), \end{cases}$$

where $T > 1$, $a_i, b_i, d_i \in \mathbb{R}$, $\xi_i \in (1, T)$, $i = 1, 2, 3, 4$, $\alpha_j \in (1, 2]$, $f_j : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $j = 1, 2$, are given continuous functions, $\mathbb{R}^m$ for $m \in \mathbb{N}$ is the Banach space with a suitable norm $\| \cdot \|$, ${}^{\mathrm{HC}}D_1^{\alpha_j}$ is the Caputo–Hadamard fractional derivative of order α_j, $j = 1, 2$.
Finally, an example will be included to illustrate our main results.

In the second section; two results for the following coupled system of implicit fractional differential equations in Banach spaces with Caputo-Hadamard fractional derivative are discussed. The argument are based on *Banach's fixed point theorem* and *Nonlinear alternative of Leray-Schauder type.*

We establish existence and uniqueness results for the following coupled system of implicit fractional differential equations :

$$\begin{cases} ({}^{Hc}D_1^{\alpha_1}u_1)(t) = f_1(t, u_1(t), u_2(t), ({}^{Hc}D_1^{\alpha_1}u_1)(t)) \\ ({}^{Hc}D_1^{\alpha_2}u_2)(t) = f_2(t, u_1(t), u_2(t), ({}^{Hc}D_1^{\alpha_2}u_2)(t)) \end{cases} ; \ t \in I := [1, T],$$

with the multipoint boundary conditions

$$\begin{cases} a_1u_1(1) - b_1u_1'(1) = d_1u_1(\xi_1) \\ a_2u_1(T) + b_2u_1'(T) = d_2u_1(\xi_2) \\ a_3u_2(1) - b_3u_2'(1) = d_3u_2(\xi_3) \\ a_4u_2(T) + b_4u_2'(T) = d_4u_2(\xi_4) \end{cases} ; \ w \in \Omega,$$

where $T > 1,\ a_i, b_i, d_i \in \mathbb{R},\ \xi_i \in (1,T);\ i = 1,2,3,4,\ \alpha_j \in (1,2],\ f_j : I \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m;\ j = 1,2$ are given continuous functions, $\mathbb{R}^m;\ m \in \mathbb{N}^*$ is the Euclidian Banach space with a suitable norm $\|\cdot\|$, ${}^{Hc}D_1^{\alpha_j}$ is the Caputo–Hadamard fractional derivative of order $\alpha_j;\ j = 1,2$.
At last and as application, an example is included.

Chapter 3.
This chapter consists of two Sections.
In the first section; we investigate the existence of solutions for the following coupled conformable fractional differential system :

$$\begin{cases} (\mathcal{T}_{0^+}^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{0^+}^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in I,$$

with the following coupled boundary conditions :

$$(u(0), v(0)) = (\delta_1 v(T), \delta_2 u(T)),$$

where $T > 0,\ I := [0,T],\ \alpha_i \in (0,1];\ i = 1,2\quad f_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ i = 1,2$ are given continuous functions, $\mathcal{T}_0^{\alpha_i}$ is the conformable fractional derivative of order $\alpha_i; i = 1,2$, and δ_1, δ_2 are real numbers with $\delta_1\delta_2 \neq 1$.

Next, we investigate the following coupled conformable fractional differential system :

$$\begin{cases} (\mathcal{T}_{a^+}^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{a^+}^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in [a, \infty),$$

with the coupled initial conditions :

$$(u(a), v(a)) = (u_a, v_a),$$

where $a > 0$, $\alpha_i \in (0, 1]$; $i = 1, 2$, $(E, \|\cdot\|)$ is a (real or complex) Banach space, $u_a, v_a \in E$ and $f_i : \mathbb{R}_+ \times E \times E \to E$; $i = 1, 2$ are given continuous functions.

In the second section ; we investigate the existence and stability of solutions for the following coupled Conformable fractional differential system :

$$\begin{cases} (\mathcal{T}_{0^+}^{\alpha_1} u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{0^+}^{\alpha_2} v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in I,$$

with the following coupled boundary conditions :

$$(u(0), v(0)) = (\delta_1 v(T), \delta_2 u(T)),$$

where $T > 0$, $I := [0, T]$, $\alpha_i \in (0, 1]$; $i = 1, 2$, $f_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $i = 1, 2$ are given continuous functions, $\mathcal{T}_0^{\alpha_i, \rho}$ is the conformable fractional derivative of order α_i ; $i = 1, 2$, and δ_1, δ_2 are real numbers with $\delta_1 \delta_2 \neq 1$.

Next, we investigate the attractivity of solutions for the following coupled conformable fractional differential system :

$$\begin{cases} (\mathcal{T}_{a^+}^{\alpha_1} u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{a^+}^{\alpha_2} v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in [a, \infty),$$

with the following coupled initial conditions :

$$(u(a), v(a)) = (u_a, v_a),$$

where $a > 0$, $\alpha_i \in (0, 1]$; $i = 1, 2$, $(\mathbb{R}, \|.\|)$ is a Banach space, $u_a, v_a \in \mathbb{R}$ and $f_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $i = 1, 2$ are given continuous functions.

At last and as application, an example is included.

Chapter 4.

This chapter consists of two Sections.

In the first section ; we investigate the existence of solutions for the following coupled Katugampola fractional differential system

$$\begin{cases} ({}^{\rho}D_0^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ ({}^{\rho}D_0^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ;\ t \in I := [0, T],$$

with the multipoint boundary conditions

$$\begin{cases} \mathcal{I}_0^{2-\alpha_1,\rho}u(0) = a_1; \mathcal{I}_{0^+}^{2-\alpha_1,\rho}u(T) = b_1 \\ \\ \mathcal{I}_0^{2-\alpha_2,\rho}v(0) = a_2; \mathcal{I}_{0^+}^{2-\alpha_2,\rho}v(T) = b_2, \end{cases}$$

where $T > 0,\ ,\ t \in (0, T);\ ,\ \alpha_i \in (1, 2],\ f_i : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m;\ i = 1, 2$ are given continuous functions, $\mathbb{R}^m;\ m \in \mathbb{N}^*$ is the Banach space with a suitable norm $\| \cdot \|$, $\mathcal{I}_0^{2-\alpha_i,\rho}$ is Katugampola fractional integral of order $2 - \alpha_i$.

In the second section ; we investigate the existence of solutions for the following coupled Caputo– Katugampola fractional differential system

$$\begin{cases} ({}^{c}D_{a^+}^{\alpha_1,\rho}u)(t) = f_1(t, u(t), v(t)) \\ ({}^{c}D_{a^+}^{\alpha_2,\rho}v)(t) = f_2(t, u(t), v(t)) \end{cases} ;\ t \in I := [a, b],$$

with the multipoint boundary conditions

$$\begin{cases} u(a) = \lambda_1 v(b); {}^{c}D_{a^+}^{\gamma_1,\rho}u(b) = \lambda_2 \sum_{i=1}^{N} ({}^{c}D_{a^+}^{\delta_1,\rho}v)(\eta_i) \\ \\ v(a) = \mu_1 u(b); {}^{c}D_{a^+}^{\gamma_2,\rho}v(b) = \mu_2 \sum_{i=1}^{M} ({}^{c}D_{a^+}^{\delta_2,\rho}u)(\xi_i) \end{cases} ;$$

where $a, b > 0,\ ,\ t \in (a, b);\ ,\ \alpha_i \in (1, 2], \gamma_1, \delta_1 \in (0, 1], \eta_i \in R$ for $i = 1, 2.....N(N \in \mathbb{N})\xi_i \in \mathbb{R}$ for $i = 1, 2.....M(M \in \mathbb{N})a < \xi_1 < \xi_2.... < b, \lambda_i, \mu_i, i = 1, 2$ are real positive constants $f_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ i = 1, 2$ are given continuous functions and ${}^{c}D_a^{\alpha_i,\rho}$ is caputo- Katugampola fractional derivative of order α_i ; $i = 1, 2$.

At last and as application, an example is included.

Chapitre 1

Preliminaries

1.1 A brief visit to the history of the Fractional Calculus

In 1695, in a letter to the French mathematician *L'Hospital*, *Leibniz* raised the following question : "*Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders*" *L'Hospital* was somewhat curious about that question and replied by another question to *Leibniz* : "*What if the order will be* $\frac{1}{2}$" *Leibnitz* in a letter dated September 30, replied : "*It will lead to a paradox, from which one day useful consequences will be drawn.*"

In 1783, *Leonhard Euler* made his first comments on fractional order derivative. He worked on progressions of numbers and introduced first time the generalization of factorials to *Gamma* function. A little more than fifty year after the death of *Leibniz*, *Lagrange*, in 1772, indirectly contributed to the development of exponents law for differential operators of integer order, which can be transferred to arbitrary order under certain conditions. In 1812, *Laplace* has provided the first detailed definition for fractional derivative. *Laplace* states that fractional derivative can be defined for functions with representation by an integral, in modern notation it can be written as $\int y(t)t^{-x}dt$. Few years after, *Lacroix* worked on generalizing the integer order derivative of function $y(t) = t^m$ to fractional order, where m is some natural number.

In modern notations, integer order n^{th} derivative derived by *Lacroix* can be given as

$$\frac{d^n y}{dt^n} = \frac{m!}{(m-n)!}t^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}t^{m-n}, \ m > n$$

where, Γ is the *Euler's Gamma function*. Thus, replacing n with $\frac{1}{2}$ and letting $m = 1$, one obtains the derivative or order $\frac{1}{2}$ of the function t

$$\frac{d^{\frac{1}{2}} y}{dt^{\frac{1}{2}}} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\sqrt{t}$$

Euler's Gamma function (or ***Euler's integral*** of the second kind) has the same importance in the fractional-order calculus and it is basically given by integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The exponential provides the convergence of this integral in ∞, the convergence at zero obviously occurs for all complex z from the right half of the complex plane ($Re(z) > 0$). This function is generalization of a factorial in the following form :

$$\Gamma(n) = (n-1)!.$$

Other generalizations for values in the left half of the complex plane can be obtained in following way. If we substitute e^{-t} by the well-known limit

$$e^{-t} = \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n$$

and then use n-times integration by parts, we obtain the following limit definition of the Gamma function

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\ldots(z+n)}.$$

Therefore, historically the first discussion of a derivative of fractional order appeared in a calculus written by Lacroix in 1819.

It was Liouville who engaged in the first major study of fractional calculus. Liouville's first definition of a derivative of arbitrary order ν involved an infinite series. Here, the series must be convergent for some ν. Liouville's second definition succeeded in giving a fractional derivative of x^{-a} whenever both x and are positive. Based on the definite integral related to Euler's gamma integral, the integral formula can be calculated for x^{-a}. Note that in the integral

$$\int_0^\infty u^{a-1}e^{-xu}du,$$

if we change the variables $t = xu$, then

$$\int_0^\infty u^{a-1}e^{-xu}du = \int \left(\frac{t}{x}\right)^{a-1} e^{-t}\frac{1}{x}dt = \frac{1}{x^a}\int_0^\infty t^{a-1}e^{-t}dt.$$

Thus,

$$\int_0^\infty u^{a-1}e^{-xu}du = \frac{1}{x^a}\int_0^\infty t^{a-1}e^{-t}dt.$$

From the ***Gamma*** function, we obtain the integral formula

$$x^{-a} = \frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}e^{-xu}du.$$

Consequently, by assuming that $\frac{d^\nu}{dx^\nu}e^{ax} = a^\nu e^{ax}$, for any $\nu > 0$, then

$$\frac{d^\nu}{dx^\nu}x^{-a} = \frac{\Gamma(a+\nu)}{\Gamma(a)}x^{-a-\nu} = (-1)^\nu\frac{\Gamma(a+\nu)}{\Gamma(a)}x^{-a-\nu}$$

In 1884 Laurent published what is now recognized as the definitive paper on the foundations of fractional calculus. Using Cauchy's integral formula for complex valued analytical functions and a simple change of notation to employ a positive ν rather than a negative ν will now yield Laurent's definition of integration of arbitrary order

$${}_{x_0}D_x^\alpha h(x) = \frac{1}{\Gamma(\nu)}\int_{x_0}^x (x-t)^{\nu-1}h(t)dts.$$

The **Riemann-Liouville** differential operator of fractional calculus of order α defined as

$$(D_{a+}^\alpha f)(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)ds & \text{if } n-1<\alpha<n, \\ \left(\frac{d}{dt}\right)^n f(t), & \text{if } \alpha = n, \end{cases} \tag{1.1}$$

where $\alpha, a, t \in R$, $t > a$, $n = [\alpha]+1$; $[\alpha]$ denotes the integer part of the real number α, and Γ is the *Gamma* function.

The **Grünwald-Letnikov** differential operator of fractional calculus of order α defined as

$$(D^{\alpha}_{a+}f)(t) := \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh).$$

Binomial coefficients with alternating signs for positive value of n are defined as

$$\binom{n}{j} = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!} = \frac{n!}{j!(n-j)!}.$$

For binomial coefficients calculation we can use the relation between Euler's *Gamma* function and factorial, defined as

$$\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\Gamma(\alpha)}{\Gamma(j+1)\Gamma(\alpha-j+1)}.$$

The Grünwald-Letnikov definition of differ-integral starts from classical definitions of derivatives and integrals based on infinitesimal division and limit. The disadvantages of this approach are its technical difficulty of the computations and the proofs and large restrictions on functions.[160]

The **Caputo** (**1967**) differential operator of fractional calculus of order α defined as

$$({}^{c}D^{\alpha}_{a+}f)(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds & \text{if } n-1<\alpha<n, \\ \left(\frac{d}{dt}\right)^n f(t), & \text{if } \alpha = n, \end{cases} \tag{1.2}$$

where $\alpha, a, t \in R$, $t > a$, $n = [\alpha] + 1$. This operator is introduced in 1967 by the Italian Caputo.
This consideration is based on the fact that for a wide class of functions, the three best known definitions ((GL), (RL), and Caputo) are equivalent under some conditions. [85]

Unfortunately, fractional calculus still lacks a geometric interpretation of integration or differentiation of arbitrary order.

We refer readers, for example, to the books such as [8, 39, 86, 114, 120, 123, 126, 140] and the articles [14, 18, 19, 36, 37, 46, 48, 50, 52, 53, 54, 55, 105, 144], and references therein.

1.1.1 Applications of Fractional calculus

The concept of fractional calculus has great potential to change the way we see, model and analyze the systems. It provides good opportunity to scientists and engineers for revisiting the origins. The theoretical and practical interests of using fractional order operators are increasing. The application domain of fractional calculus is ranging from accurate modeling of the microbiological processes to the analysis of astronomical images.
Next, we will present a brief survey of applications of fractional calculus in science and engineering.

Signal and Image Processing :
In the last decade, the use of fractional calculus in *signal processing* has tremendously increased. In *signal processing*, the fractional operators are used in the design of differentiator and integrator of fractional order, fractional order differentiator **FIR** (finite impulse response), **IIR** type digital fractional order differentiator (infinite impulse response), a new **IIR** (infinite impulse response)-type digital fractional order differentiator (**DFOD**) and for modeling the speech signal. The fractional calculus allows the edge detection, enhances the quality of images, with interesting possibilities in various image enhancement applications such as image restoration image denoising and the texture enhancement. He is used, in particularly, in satellite image classification, and astronomical image processing.

Electromagnetic Theory :
The use of fractional calculus in electromagnetic theory has emerged in the last two decades. In 1998, Engheta [72] introduced the concept of fractional curl operators and this concept is extended by Naqvi and Abbas [122]. Engheta's work gave birth to the newfield of research in Electromagnetics, namely, "*Fractional Paradigms in Electro-*

magnetic Theory". Nowadays fractional calculus is widely used in Electromagnetics to explore new results ; for example, Faryad and Naqvi [73] have used fractional calculus for the analysis of a Rectangular Waveguide.

Control Engineering :

In industrial environments, robots have to execute their tasks quickly and precisely, minimizing production time, and the robustness of control systems is becoming imperative these days. This requires flexible robots working in large workspaces, which means that they are influenced by nonlinear and fractional order dynamic effects.

Biological Population Model

The problems of the diffusion of biological populations occur nonlinearly and the fractional order differential equations appear more and more frequently in different research areas.

Reaction-Diffusion Equations

Fractional equations can be used to describe some physical phenomenon more accurately than the classical integer order differential equation. The reaction-diffusion equations play an important role in dynamical systems of mathematics, physics, chemistry, bioinformatics, finance, and other research areas. There has been a wide variety of analytical and numerical methods proposed for fractional equations ([116, 157]), for example, finite element method, Adomian decomposition method ([136]), and spectral technique ([117]). Interest in fractional reaction-diffusion equations has increased.

1.2 Notations and Definitions

Consider the complete metric space $C := \mathrm{C}(I, \mathbb{R}^m)$ of continuous functions from I, where $I = [0, T], T > 0$ into $\mathbb{R}^m$ equipped with the usual metric

$$d(u, v) := \max_{t \in I} \|u(t) - v(t)\|,$$

where $\|\cdot\|$ is a suitable norm on $\mathbb{R}^m$. Note that C is a Banach space with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

By $\mathcal{C} := C \times C$, we denote the complete metric space with the usual metric

$$D((u_1, v_1, (u_2, v_2)) := d(u_1, u_2) + d(v_1, v_2).$$

$\mathcal{C}$ is a Banach space with the norm

$$\|(u, v)\|_{\mathcal{C}} = \|u\|_\infty + \|v\|_\infty.$$

As usual, $\mathrm{AC}(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$, and $\mathrm{L}^1(I)$ denotes the space of Lebesgue-integrable functions $v : I \to \mathbb{R}^m$, with the norm

$$\|v\|_1 = \int_I \|v(t)\| \mathrm{d}t.$$

For any $n \in \mathbb{N}$, we denote by $\mathrm{AC}^n(I)$ the space defined by

$$\mathrm{AC}^n(I) := \left\{ w : I \to E : \ \frac{\mathrm{d}^n}{\mathrm{d}t^n} w(t) \in \mathrm{AC}(I) \right\}.$$

Let $\delta = t \frac{\mathrm{d}}{\mathrm{d}t}$, define the space

$$\mathrm{AC}^n_\delta := \left\{ u : I \to E : \ \delta^{n-1}[u(t)] \in \mathrm{AC}(I) \right\}.$$

Let $X := \mathrm{C}(\mathbb{R}_+, E)$ be the Fréchet space of all continuous functions u from $\mathbb{R}_+$ into E, equipped with the family of semi norms

$$\|u\|_n = \sup_{t \in [0,n]} \{\|u(t)\| : n \in \mathbb{N}\},$$

and the distance

$$d(u, v) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|u - v\|_n}{1 + \|u - v\|_n}; \ u, v \in X.$$

Definition 1.2.1 ([33]). *A nonempty subset $B \subset X$ is said to be bounded if*

$$\sup_{u \in B} \|u\|_n < \infty; \quad \textit{for} \quad n \in \mathbb{N}.$$

1.3 Fractional Calculus Theory

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [106] for a more detailed analysis.

Definition 1.3.1 (Hadamard fractional integral [106]). *The Hadamard fractional integral of order $q > 0$ for a function $u \in \mathrm{L}^1(I)$ is defined as*

$$\left({}^{\mathrm{H}}I_1^q u\right)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{u(s)}{s} \mathrm{d}s, \quad \textit{for a.e.} \quad x \in I = [1, X].$$

provided the integral exists.

Example 1.3.1. *Let $0 < q < 1$. Then*

$${}^{\mathrm{H}}I_1^q \ln t = \frac{(\ln t)^{1+q}}{\Gamma(2+q)} \quad \textit{for a.e.} \quad t \in [1, e].$$

Definition 1.3.2 (Hadamard fractional derivative [106]). *The Hadamard fractional derivative of order $q > 0$ applied to the function $u \in \mathrm{AC}_\delta^n(I)$ is defined as*

$$\left({}^{\mathrm{H}}D_1^q u\right)(x) = \delta^n \left({}^{\mathrm{H}}I_1^{n-q} u\right)(x).$$

In particular, if $q \in (0, 1]$ in Definition 1.3.2, then

$$\left({}^{\mathrm{H}}D_1^q u\right)(x) = \delta \left({}^{\mathrm{H}}I_1^{1-q} u\right)(x).$$

Example 1.3.2. *Let $0 < q < 1$. Then*

$${}^{\mathrm{H}}D_1^q \ln t = \frac{(\ln t)^{1-q}}{\Gamma(2-q)} \quad \textit{for a.e.} \quad t \in [1, e].$$

It has been proved (see e.g., Kilbas [104, Theorem 4.8]) that in the space $\mathrm{L}^1(I)$,with $x \in I = [1, \infty)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.,

$$\left({}^{\mathrm{H}}D_1^q\right)\left({}^{\mathrm{H}}I_1^q w\right)(x) = w(x).$$

From [106, Theorem 2.3], we have

$$\left({}^{\mathrm{H}}I_1^q\right)\left({}^{\mathrm{H}}D_1^q w\right)(x) = w(x) - \frac{\left({}^{\mathrm{H}}I_1^{1-q} w\right)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Analogous to Hadamard fractional calculus, the Caputo–Hadamard fractional derivative is defined in the following way.

Definition 1.3.3 (Caputo–Hadamard fractional derivative)**.** *The Caputo–Hadamard fractional derivative of order $q > 0$ applied to the function $u \in \mathrm{AC}_\delta^n$ is defined as*

$$\left({}^{\mathrm{HC}}D_1^q u\right)(x) = \left({}^{\mathrm{H}}I_1^{n-q}\delta^n u\right)(x).$$

In particular, if $q \in (0,1]$ in Definition 1.3.3, then

$$\left({}^{\mathrm{HC}}D_1^q u\right)(x) = \left({}^{\mathrm{H}}I_1^{1-q}\delta u\right)(x).$$

Lemma 1.3.1 ([90])**.** *Let $\alpha \geq 0$ and $n = [\alpha]+1$. If $u \in \mathrm{AC}_\delta^n[1,T]$, then the Caputo–Hadamard fractional differential equation*

$$\left({}^{\mathrm{HC}}D_1^\alpha u\right)(t) = 0$$

has the general solution

$$u(t) = \sum_{j=0}^{n-1} c_j (\ln t)^j,$$

and we have

$${}^{\mathrm{H}}I_1^\alpha\left({}^{\mathrm{HC}}D_1^\alpha u\right)(t) = u(t) + \sum_{j=0}^{n-1} c_j (\ln t)^j,$$

where $c_j \in \mathbb{R}$, $j = 0, 1, \ldots, n-1$.

Let us now recall some essential definitions on conformable derivatives that can be found in [16, 101].

Let $n < \alpha < n+1$, and set $\beta = \alpha - n$. For a function $f : [a,\infty) \to \mathbb{R}$, we denote by

$$\mathcal{I}_a^\alpha f(t) = \int_a^t (s-a)^{\alpha-1} f(s)\mathrm{d}s, n = 0,$$

and

$$\mathcal{I}_a^\alpha f(t) = \frac{1}{n!}\int_a^t (t-s)^n f(s) d\beta(s,a) = \frac{1}{n!}\int_a^t (t-s)^n (s-a)^{\beta-1} f(s)\mathrm{d}s; n \geq 1.$$

Definition 1.3.4 (conformable fractional derivative). *The conformable derivative of order $\alpha \in]0,1[$, of a function $f : [a, \infty) \to \mathbb{R}$ is defined by*

$$\mathcal{T}_a^\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-\alpha} - f(t)}{\epsilon}, t > a.$$

If $\mathcal{T}_a^\alpha f(t)$ exists on $(a,b), b > a$ and $\lim_{t \to a^+} T_a^\alpha f(t)$ exists, then we define

$$\mathcal{T}_a^\alpha f(a) = \lim_{t \to a^+} \mathcal{T}_a^\alpha f(t).$$

Definition 1.3.5. *The conformable derivative of order $\alpha \in]n, n+1[$ of a function $f : [a, \infty) \to \mathbb{R}$, when $f^{(n)}$ exists, is defined by $\mathcal{T}_a^\alpha f(t) = \mathcal{T}_a^\beta f^{(n)}(t)$, where $\beta = \alpha - n \in (0,1)$.*

Lemma 1.3.2. *For the properties of the conformable derivative, we mention the following :*
Let $n < \alpha < n+1$ and f be an $(n+1)$-differentiable at $t > a$, then we have

$$\mathcal{T}_a^\alpha f(t) = (t-a)^{n+1-\alpha} f^{(n+1)}(t),$$

and

$$\mathcal{I}_a^\alpha \mathcal{T}_a^\alpha f(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(n)}(a)(t-a)^k}{k!}.$$

In particular, if $0 < \alpha < 1$, then we have

$$\mathcal{I}_a^\alpha \mathcal{T}_a^\alpha f(t) = u(t) - u(a).$$

Remark 1.3.1. *We provide the following remarks :*

- *For $0 < \alpha < 1$, using Lemma 1.3.2 it follows that, if a function f is differentiable at $t > a$, then one has*

  $$\lim_{\alpha \to 1} \mathcal{T}_\alpha^a f(t) = f'(t)$$

  and

  $$\lim_{\alpha \to 0} \mathcal{T}_a^\alpha f(t) = (t-a) f'(t),$$

  i.e. the zero order derivative of a differentiable function does not return to the function itself.

– *Let $n < \alpha < n+1$, if f is $(n+1)$-differentiable on $(a,b), b > a$ and $\lim_{t\to a^+} f^{(n+1)}$ exists, then from Lemma 1.3.2, we get $\mathcal{T}_\alpha^a f(a) = \lim_{t\to a^+} T_\alpha^a f(t) = 0$.*
– *Let $n < \alpha < n+1$, if f is $(n+1)$-differentiable at $t > a$, then we can show that $\mathcal{T}_\alpha^a f(t) = \mathcal{T}_{\alpha-k}^a f^{(k)}(t)$ for all positive integer $k < \alpha$.*

Similarly to the classical case, we give a property on the extremum of a function that has a conformable derivative.

Proposition 1.3.1. *Let $1 < \alpha < 2$, if a function $f \in C^1[a,b]$ attains a global maximum (respectively minimum) at some point $\xi \in (a,b)$, then $\mathcal{T}_a^\alpha f(\xi) \le 0$ (respectively $\mathcal{T}_\alpha^a f(\xi) \ge 0$.*

Proof. The result follows from the fact that

$$\mathcal{T}_a^\alpha f(\xi) = \mathcal{T}_a^{\alpha-1} f'(\xi) = \lim_{\epsilon\to 0} \frac{f'(\xi + \epsilon(\xi-a)^{2-\alpha})}{\epsilon}.$$

Definition 1.3.6. *([99]) (The Katugampola fractional integral)*
The Katugampola fractional integrals of order $\alpha > 0$ of a function $h \in X_c^p(0,T)$, is defined by

$$\mathcal{I}_{0^+}^{\alpha,\rho} h(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1} h(s)}{(t^\rho - s^\rho)^{1-\alpha}} ds, t \in [0,T] \tag{1.3}$$

or $\rho > 0$. These integrals are called left-sided integrals. Similarly we can define right-sided integrals ([99]-[106])

Definition 1.3.7. *([100]) (The Katugampola fractional derivatives)*
The generalized fractional derivatives of order $\alpha > 0$, corresponding to the Katugampola fractional integrals ((3.22) defined for any $t \in [0,T]$, by

$${}^\rho D_{0^+}^\alpha h(t) = (t^{1-\rho}\frac{d}{dt})^n \mathcal{I}_{0^+}^{n-\alpha,\rho} h)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} (t^{1-\rho}\frac{d}{dt})^n \int_0^t \frac{s^{\rho-1} h(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds, t \in [0,T] \tag{1.4}$$

where $n = [\alpha] + 1$, and $\rho > 0$, provided the integrals exist.

Remark 1.3.2. *([100])*

$${}^\rho D_{0^+}^\alpha t^\mu = \frac{\rho^{\alpha-1}\Gamma(1+\frac{\mu}{\rho})}{\Gamma(1-\alpha+\frac{\mu}{\rho})} t^{\mu-\alpha\rho}.$$

Giving in particular

$$^{\rho}D^{\alpha}_{0+}t^{\rho(\alpha-m)} = 0, \ for \ each \ m = 1, 2, ...n.$$

$$\mathcal{I}^{\alpha,\rho}_{a^+}(t^{\rho} - a^{\rho})^{\gamma-1} = \frac{\rho^{-\alpha}\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(t^{\rho} - a^{\rho})^{\alpha+\gamma-1}.$$

Theorem 1.3.1. *([99],[100]). Let $\alpha, \rho, c \in \mathbb{R}$, be such that $\alpha, \rho > 0$. Then for any $f, g \in X^p_c[0, T]$, where $0 \leq p \leq \infty$, we have*
- Inverse property :

$$^{\rho}D^{\alpha}_{0+}(\mathcal{I}^{\alpha,\rho}_{0+}f)(t) = f(t), \ for \ all \ \alpha \in (0, 1). \tag{1.5}$$

- Linearity property : for all $\alpha \in (0, 1)$, we have

$$\begin{cases} ^{\rho}D^{\alpha}_{0+}(f+g)(t) =^{\rho} D^{\alpha}_{0+}f(t) +^{\rho} D^{\alpha}_{0+}g(t) \\ \mathcal{I}^{\alpha,\rho}_{0+}(f+g)(t) = \mathcal{I}^{\alpha,\rho}_{0+}f(t) + \mathcal{I}^{\alpha,\rho}_{0+}g(t) \end{cases} \tag{1.6}$$

Lemma 1.3.3. *([100]) Let $\alpha, \rho > 0$. If $u \in C([0, T], \mathbb{R})$, then*
(i)the fractional deferential equation $^{\rho}D^{\alpha}_{0+}u(t) = 0$, has a solutions

$$u(t) = C_1t^{\rho(\alpha-1)} + C_2t^{\rho(\alpha-2)} + ...C_nt^{\rho(\alpha-n)},$$

where $C_n \in \mathbb{R}, n = 0, 1, 2, 3, ...n - 1$ and $n = [\alpha] + 1$
(ii)if $^{\rho}D^{\alpha}_{0+}u(t) \in C([0, T], \mathbb{R})$ and $1 < \alpha \leq 2$, then

$$^{\rho}I^{\alpha}_{0+}(^{\rho}D^{\alpha}_{0+}u)(t) = u(t) + C_1t^{\rho(\alpha-1)} + C_2t^{\rho(\alpha-2)},$$

for some constant $C_1, C_2 \in \mathbb{R}$.

Lemma 1.3.4. *Let $\alpha, \rho > 0$. If $u \in C([a, b], \mathbb{R})$, then*
(i)The fractional deferential equation $^{c}D^{\rho,\alpha}_{0+}u(t) = 0$, has a solutions

$$u(t) = C_0 + C_1(\frac{t^{\rho} - a^{\rho}}{\rho}) + ...C_{n-1}(\frac{t^{\rho} - a^{\rho}}{\rho})^{n-1},$$

where $C_n \in \mathbb{R}, n = 0, 1, 2, 3, ...n - 1$. and $n = [\alpha] + 1$
(ii)If $^{c}D^{\rho,\alpha}_{0+}u(t) \in C([a, b], \mathbb{R})$ then
$^{\rho}\mathcal{I}^{c}_{a+}D^{\rho,\alpha}_{0+}u(t) = u(t) + C_0 + C_1(\frac{t^{\rho}-a^{\rho}}{\rho}) + ...C_{n-1}(\frac{t^{\rho}-a^{\rho}}{\rho})^{n-1}$, *where* $C_n \in \mathbb{R}, n = 0, 1, 2, 3, ...n - 1$.

Theorem 1.3.2. *[81](theorem of Ascoli-Arzela) Let $A \subset C(I, \mathbb{R})$, A is relatively compact (i.e $\overline{A}$ is compact) if :*

1. *A is uniformly bounded i.e, there exists $M > 0$ such that*

$$|f(x)| < M \text{ for every } f \in A \text{ and } x \in J.$$

2. *A is equicontinuous i.e, for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, \overline{x} \in J$, $|x - \overline{x}| \leq \delta$ implies $|f(x) - f(\overline{x})| \leq \epsilon$, for every $f \in A$.*

1.4 Some definitions and properties of the measure of non-compactness

In this section we define the Kuratowski (1896-1980) measures of non-compactness (**MNC** for short) and give their basic properties.

Definition 1.4.1. *[150] Let (X, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of X. For every $B \in \mathcal{B}$ we define the Kuratowski measure of non-compactness $\alpha(B)$ of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.*

Remark 1.4.1. *The diameter of a set B is the number $\sup\{d(x, y) : x \in B, y \in B\}$ denoted by $diam(B)$, with $diam(\emptyset) = 0$.*
It is clear that $0 \leq \alpha(B) \leq diam(B) < +\infty$ for each nonempty bounded subset B of X and that $diam(B) = 0$ if and only if B is an empty set or consists of exactly one point.

Definition 1.4.2. *[42] Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty]$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E,$$

where

$$diam(B_i) = sup\{||x - y|| : x, y \in B_i\}.$$

The Kuratowski measure of noncompactness satisfies the following properties :

Lemma 1.4.1. *([28, 42, 43, 150]) Let A and B bounded sets.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact), where $\overline{B}$ denotes the closure of B.

(b) nonsingularity : α is equal to zero on every one element-set.

(c) If B is a finite set, then $\alpha(B) = 0$.

(d) $\alpha(B) = \alpha(\overline{B}) = \alpha(convB)$, where $convB$ is the convex hull of B.

(e) monotonicity : $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(f) algebraic semi-additivity : $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where

$$A + B = \{x + y : x \in A, \quad y \in B\}.$$

(g) semi-homogencity : $\alpha(\lambda B) = |\lambda|\alpha(B)$; $\lambda \in R$, where $\lambda(B) = \{\lambda x : x \in B\}$.

(h) semi-additivity : $\alpha(A \bigcup B) = max\{\alpha(A), \alpha(B)\}$.

(i) $\alpha(A \bigcap B) = min\{\alpha(A), \alpha(B)\}$.

(j) invariance under translations : $\alpha(B + x_0) = \alpha(B)$ for any $x_0 \in E$.

Remark 1.4.2. *The a-measure of noncompactness was introduced by Kuratowski in order to generalize the Cantor intersection theorem*

Theorem 1.4.1. *[150] Let (X, d) be a complete metric space and $\{B_n\}$ be a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_{n\to\infty} \alpha(B_n) = 0$. Then the intersection B_∞ of all B_n is nonempty and compact.*

In [87], Horvath has proved the following generalization of the Kuratowski theorem :

Theorem 1.4.2. *[150] Let (X, d) be a complete metric space and $\{B_i\}_{i\in I}$ be a family of nonempty of closed and bounded subsets of X having the finite intersection property. If $\inf_{i\in I} \alpha(B_i) = 0$ then the intersection B_∞ of all B_i is nonempty and compact.*

Lemma 1.4.2. *[80] If $V \subset C(J, E)$ is a bounded and equicontinuous set, then*

(i) the function $t \to \alpha(V(t))$ is continuous on J, and

$$\alpha_c(V) = \sup_{0\leq t\leq T} \alpha(V(t)).$$

(ii) $\alpha\left(\int_0^T x(s)ds : x \in V\right) \leq \int_0^T \alpha(V(s))ds,$
where

$$V(s) = \{x(s) : x \in V\},\ \ s \in J.$$

In the definition of the Kuratowski measure we can consider balls instead of arbitrary sets.

Theorem 1.4.3. *([150]) Let $B(0,1)$ be the unit ball in a Banach space X. Then*

$$\alpha(B(0,1)) = \chi(B(0,1)) = 0$$

if X is finite dimensional, and $\alpha(B(0,1)) = 2$, $\chi(B(0,1)) = 1$ otherwise.

Theorem 1.4.4. *([150]) Let $S(0,1)$ be the unit sphere in a Banach space X. Then $\alpha(S(0,1)) = \chi(S(0,1)) = 0$ if X is finite dimensional, and $\alpha(S(0,1)) = 2$, $\chi(S(0,1)) = 1$ otherwise.*

Theorem 1.4.5. *([150]) The Kuratowski* **MNCs** *is related by the inequalities*

$$\chi(B) \leq \alpha(B) \leq 2\chi(B).$$

In the class of all infinite dimensional Banach spaces these inequalities are the best possible.

Example 1.4.1. *Let l^∞ be the space of all real bounded sequences with the supremum norm, and let A be a bounded set in l^∞. Then $\alpha(A) = 2\chi(A)$.*

For further facts concerning measures of non-compactness and their properties we refer to [28, 42, 43, 146, 150] and the references therein.

We recall the following definition of the notion of a sequence of measures of non-compactness [69, 70].

Definition 1.4.3. *Let $\mathcal{M}_F$ be the family of all nonempty and bounded subsets of a Fréchet space F. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n : \mathcal{M}_F \to [0,\infty)$ is said to be a family of measures of non-compactness in the real Fréchet space F if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_F$:*

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ *is full, that is :* $\mu_n(B)=0$ *for* $n\in\mathbb{N}$ *if and only if* B *is precompact,*

(b) $\mu_n(B_1)\leq\mu_n(B_2)$ *for* $B_1\subset B_2$ *and* $n\in\mathbb{N}$,

(c) $\mu_n(ConvB)=\mu_n(B)$ *for* $n\in\mathbb{N}$,

(d) If $\{B_i\}_{i=1,\cdots}$ *is a sequence of closed sets from* $\mathcal{M}_F$ *such that* $B_{i+1}\subset B_i;\ i=1,\cdots$ *and if* $\lim_{i\to\infty}\mu_n(B_i)=0$, *for each* $n\in\mathbb{N}$, *then the intersection set* $B_\infty:=\cap_{i=1}^{\infty}B_i$ *is nonempty.*

Some Properties :

(1) We call the family of measures of non-compactness $\{\mu_n\}_{n\in\mathbb{N}}$ to be homogeneous if $\mu_n(\lambda B)=|\lambda|\mu_n(B)$; for $\lambda\in\mathbb{R}$ and $n\in\mathbb{N}$.

(2) If the family $\{\mu_n\}_{n\in\mathbb{N}}$ satisfied the condition $\mu_n(B_1\cup B_2)\leq\mu_n(B_1)+\mu_n(B_2)$, for $n\in\mathbb{N}$, it is called subadditive.

(3) It is sublinear if both conditions (e) and (f) hold.

(4) We say that the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ has the maximum property if

$$\mu_n(B_1\cup B_2)=\max\{\mu_n(B_1),\mu_n(B_2)\},$$

(5) The family of measures of non-compactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if if the conditions (a), (g) and (h) hold ; (full sublinear and has maximum property).

Example 1.4.2. *[69, 124] For* $B\in\mathcal{M}_X,\ x\in B,\ n\in\mathbb{N}$ *and* $\epsilon>0$, *let us denote by* $\omega^n(x,\epsilon)$ *the modulus of continuity of the function* x *on the interval* $[0,n]$; *that is,*

$$\omega^n(x,\epsilon)=\sup\{\|x(t)-x(s)\|:t,s\in[0,n],|t-s|\leq\epsilon\}.$$

Further, let us put

$$\omega^n(B,\epsilon)=\sup\{\omega^n(x,\epsilon):x\in B\},$$

$$\omega_0^n(B)=\lim_{\epsilon\to0^+}\omega^n(B,\epsilon),$$

$$\bar{\alpha}^n(B)=\sup_{t\in[0,n]}\alpha(B(t)):=\sup_{t\in[0,n]}\alpha(\{x(t):x\in B\}),$$

and

$$\beta_n(B)=\omega_0^n(B)+\bar{\alpha}^n(B).$$

The family of mappings $\{\beta_n\}_{n\in\mathbb{N}}$ *where* $\beta_n:\mathcal{M}_X\to[0,\infty)$, *satisfies the conditions (a)-(d) fom Definition 1.4.3.*

Lemma 1.4.3. *[132] If Y is a bounded subset of a Fréchet space F, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\mu_n(Y) \leq 2\mu_n(\{y_k\}_{k=1}^{\infty}) + \epsilon; \ for\ n \in \mathbb{N}.$$

Lemma 1.4.4. *[132] If $\{u_k\}_{k=1}^{\infty} \subset L^1([0,n])$ is uniformly integrable, then $\mu_n(\{u_k\}_{k=1}^{\infty})$ is measurable for $n \in \mathbb{N}^*$, and*

$$\mu_n\left(\left\{\int_1^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \leq 2\int_1^t \mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0,n]$.

Definition 1.4.4. *Let Ω be a nonempty subset of a Fréchet space F, and let $A : \Omega \to F$ be a continuous operator which transforms bounded subsets of onto bounded ones. One says that A satisfies the Darbo condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of non-compactness $\{\mu_n\}_{n\in\mathbb{N}}$, if*

$$\mu_n(A(B)) \leq k_n\mu_n(B)$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$.
If $k_n < 1;\ n \in \mathbb{N}$ then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

1.5 Some fixed point theorems

Theorem 1.5.1. ***(Banach's fixed point theorem** [78]*

Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.

Definition 1.5.1. *([137]) A nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies one of the following conditions :*

(1) *For any $t > 0$ we have*

$$\lim_{n\to\infty} \phi^{(n)}(t) = 0,$$

where $\phi^{(n)}$ denotes the n-th iteration of ϕ.

(2) *The function ϕ is right-continuous and satisfies*

$$\forall t > 0 : \phi(t) < t.$$

Remark 1.5.1. *The choice $\phi(t) = kt$ with $0 < k < 1$ gives the classical Banach contraction mapping principle.*

For our purpose we will need the following fixed point theorem :

Theorem 1.5.2. *[61, 119] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$d(T(x), T(y)) \leq \phi(d(x, y)),$$

where ϕ is a comparison function. Then T has a unique fixed point in X.

Theorem 1.5.3. ***(Schauder fixed point theorem** [146]*

Let X be a Banach space, D be a bounded closed convex subset of X and $T : D \to D$ be a compact and continuous map. Then T has at least one fixed point in D.

Theorem 1.5.4. ***Mönch's Fixed Point Theorem**[21, 132]*
Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication

$$V = \overline{conv} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0 \tag{1.7}$$

holds for every subset V of D, then N has a fixed point.
Here α is the Kuratowski measure of noncompactness.

Theorem 1.5.5. ***(Darbo's Fixed Point Theorem)** [42, 78]*
Let X be a Banach space and C be a bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N : C \to C$ is such that for all closed subsets D of C,

$$\alpha(T(D)) \leq k\alpha(D), \tag{1.8}$$

where $0 \leq k < 1$, and α is the Kuratowski measure of noncompactness. Then T has a fixed point in C.

Remark 1.5.2. *Mappings satisfying the Darbo-condition (1.8) have subsequently been called k-set contractions.*

The following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 1.5.6. *[69, 70] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space F and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

For more details see [21, 33, 75, 78, 150, 156]

Theorem 1.5.7. *(**Nonlinear alternative of Leray-Schauder type**)[78]]*
Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to C$ continuous and compact operator.
Then,
(a) either T has fixed points,
(b) or there exist $u \in \partial U$ and $\lambda \in [0, 1]$ with $u = \lambda T(u)$.

Theorem 1.5.8. *(**Schaefer's fixed point theorem [78]** Let U be a Banach space and $T : U \to U$ be continuous and compact mapping (completely continuous mapping). Moreover, suppose*

$$S = \{u \in U : u = \lambda Tu, \quad \text{for some} \quad \lambda \in (0, 1)\}$$

be a bounded set. Then T has at least one fixed point in U.

Chapitre 2

Coupled Caputo-Hadamard fractional differential systems

2.1 A coupled Caputo-Hadamard fractional differential system

2.1.1 Introduction and Motivations

The purpose of this section, is to establish existence and uniqueness of solutions for the following of Caputo-Hadamard fractional differential system

$$\begin{cases} ({}^{HC}D^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ ({}^{HC}D^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in I := [1, T], \tag{2.1}$$

with the multipoint boundary conditions

$$\begin{cases} a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1) \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2) \\ a_3 v(1) - b_3 v'(1) = d_3 v(\xi_3) \\ a_4 v(T) + b_4 v'(T) = d_4 v(\xi_4), \end{cases} \tag{2.2}$$

where $T > 1$, $a_i, b_i, d_i \in R$, $\xi_i \in (1, T)$, $i = 1, 2, 3, 4$, $\alpha_j \in (1, 2]$, $f_j : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $j = 1, 2$, are given continuous functions, $\mathbb{R}^m$ for $m \in \mathbb{N}$ is the Banach space with a suitable norm $\|\cdot\|$, ${}^{HC}D^{\alpha_j}$ is the Caputo–Hadamard fractional derivative of order α_j, $j = 1, 2$.

In [10], the authors studied some existence results based on the Mönch's fixed point theorem associated with the technique of measure of noncompactness, for the following problem of Caputo-Hadamard fractional differential equation

$$\begin{cases} ({}^{Hc}D_1^r u)(t) = f(t, u(t)), \ t \in I := [1, T], \\ u(t)|_{t=1} = \phi, \end{cases} \tag{2.3}$$

and the problem of Caputo-Hadamard partial fractional differential equation

$$\begin{cases} ({}^{Hc}D_\sigma^r u)(t, x) = f(t, x, u(t, x)), \ (t, x) \in J := [1, T] \times [1, b], \\ u(t, 1) = \phi(t); \ t \in [1, T], \\ u(1, x) = \psi(x); \ x \in [1, b], \end{cases} \tag{2.4}$$

where $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $T, b > 1$, $\sigma = (1, 1)$, $f : J \times E \to E$ is a given continuous function, $\phi : [1, T] \to E$ and $\psi : [1, b] \to E$ are given absolutely continuous functions with $\phi(1) = \psi(1)$, and ${}^{Hc}D_1^r$ is the Caputo-Hadamard partial fractional derivative of order r.

In [76], the authors examined the multipoint boundary value problem for fractional integro-differential equations :

$$\begin{cases} ({}^{C}D_{0^+}^{\alpha} x)(t) = f\left(t, x(t), \int_0^t k_1(t, s) g(s, x(s)) ds,\right. \\ \left.\int_0^a k_2(t, s) h(s, x(s)) ds\right); \ t \in [0, 1], \ \alpha \in (1, 2], \\ a_1 x(0) - b_1 x'(0) = d_1 x(\xi_1), \\ a_2 x(1) - b_2 x'(1) = d_2 x(\xi_2). \end{cases} \tag{2.5}$$

They use the technique of measure of weak non compactness and the fixed point theory to discuss the existence of weak solutions.

2.1.2 Existence of solutions

Consider the complete metric space $C(I) := C(I, \mathbb{R}^m)$ of continuous functions from I into $\mathbb{R}^m$ equipped with the usual metric

$$d(u,v) := \max_{t\in I} \|u(t) - v(t)\|,$$

where $\|\cdot\|$ is a suitable norm on $\mathbb{R}^m$.

Notice that $C(I)$ is a Banach space with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t\in I} \|u(t)\|.$$

Let us defining what we mean by a solution of problem (2.1)-(2.2).

Definition 2.1.1. *By a solution of the problem (2.1)-(2.2) we mean a continuous function u that satisfies the equation (2.1) on I and the conditions (2.2).*

For the existence of solutions for the problem (2.1)-(2.2) ; we need the following auxiliary lemma :

Lemma 2.1.1. *Let $h \in C$ and $\alpha \in (1,2]$. Then the unique solution of the problem*

$$\begin{cases} ({}^{Hc}D_1^\alpha u)(t) = h(t); \;\; t \in I \\ a_1 u(1) - b_1 u'(1) = d_1 u(\xi_1) \\ a_2 u(T) + b_2 u'(T) = d_2 u(\xi_2) \end{cases}$$

is given by

$$u(t) = \int_1^T G(t,s)h(s)ds, \tag{2.6}$$

where G is the Green function with $G(t,s)$ given by

$$\begin{cases}
\frac{1}{s\Gamma(\alpha)}(ln\frac{t}{s})^{\alpha-1} \\
+\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(lnt)^{\alpha-1}}{s\Delta\Gamma(\alpha)}(ln\frac{\xi_1}{s})^{\alpha-1}[a_2(lnT)^{\alpha-2}+\frac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3}-d_2(ln\xi_2)^{\alpha-2}] \\
+\frac{d_1(lnt)^{\alpha-2}}{s\Delta\Gamma(\alpha)}(ln\frac{\xi_1}{s})^{\alpha-1}[a_2(lnT)^{\alpha-1}+\frac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2}-d_2(ln\xi_2)^{\alpha-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
;\ s\le\xi_1,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(lnt)^{\alpha-1}}{s\Delta\Gamma(\alpha)}(ln\frac{\xi_1}{s})^{\alpha-1}[a_2(lnT)^{\alpha-2}+\frac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3}-d_2(ln\xi_2)^{\alpha-2}] \\
+\frac{d_1(lnt)^{\alpha-2}}{s\Delta\Gamma(\alpha)}(ln\frac{\xi_1}{s})^{\alpha-1}[a_2(lnT)^{\alpha-1}+\frac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2}-d_2(ln\xi_2)^{\alpha-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
;\ s\le\xi_1,\ t\le s \\
\\
\frac{1}{s\Gamma(\alpha)}(ln\frac{t}{s})^{\alpha-1} \\
+\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
;\ \xi_1\le s\le\xi_2,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}-\frac{d_2}{\Gamma(\alpha)}(ln\frac{\xi_2}{s})^{\alpha-1}] \\
;\ \xi_1\le s\le\xi_2,\ t\le s \\
\\
\frac{1}{s\Gamma(\alpha)}(ln\frac{t}{s})^{\alpha-1} \\
+\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}];\ \xi_2\le s,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}] \\
-\frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{s\Delta}[\frac{a_2}{\Gamma(\alpha)}(ln\frac{T}{s})^{\alpha-1}+\frac{b_2}{\Gamma(\alpha-1)}(ln\frac{T}{s})^{\alpha-2}];\ \xi_2\le s,\ t\le s
\end{cases}$$

where

$$\begin{aligned}\Delta &= d_1(ln\xi_1)^{\alpha-1}[a_2(lnT)^{\alpha-2}+\tfrac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3}-d_2(ln\xi_2)^{\alpha-2}]\\ &- d_1(ln\xi_1)^{\alpha-2}[a_2(lnT)^{\alpha-1}+\tfrac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2}-d_2(ln\xi_2)^{\alpha-1}]\neq 0.\end{aligned}$$

Proof. From Lemma 1.3.1, the linear fractional differential equation

$$({}^{Hc}D_1^{\alpha}u)(t)=h(t),$$

gives

$$u(t)={}^{H}I_1^{\alpha}h(t)+c_1(lnt)^{\alpha-1}+c_2(lnt)^{\alpha-2}. \tag{2.7}$$

On the other hand, by the relation $D_1^{\beta}I_1^{\alpha}u(t)=I_1^{\alpha-\beta}u(t)$, we get

$$\begin{aligned}u'(t) &= \tfrac{1}{\Gamma(\alpha-1)}\int_1^t(ln\tfrac{t}{s})^{\alpha-2}h(s)\tfrac{ds}{s}\\ &+ \tfrac{\alpha-1}{t}c_1(lnt)^{\alpha-2}+\tfrac{\alpha-2}{t}c_2(lnt)^{\alpha-3}.\end{aligned}$$

From the boundary conditions, we have

$$\begin{cases}[d_1(ln\xi_1)^{\alpha-1}]c_1+[d_1(ln\xi_1)^{\alpha-2}]c_2=a_1^{H}I_1^{\alpha}h(1)-b_1^{H}I_1^{\alpha-1}h(1)-d_1^{H}I_1^{\alpha}h(\xi_1)\\ \\ [a_2(lnT)^{\alpha-1}+\tfrac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2}-d_2(ln\xi_2)^{\alpha-1}]c_1\\ +[a_2(lnT)^{\alpha-2}+\tfrac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3}-d_2(ln\xi_2)^{\alpha-2}]c_2\\ =d_2^{H}I_1^{\alpha}h(\xi_2)-a_2^{H}I_1^{\alpha}h(T)-b_2^{H}I_1^{\alpha-1}h(T).\end{cases}$$

Thus, we obtain

$$\begin{aligned}c_1 &= \tfrac{d_1(ln\xi_1)^{\alpha-2}}{\Delta}(a_2\int_1^T(ln\tfrac{T}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds\\ &+ b_2\int_1^T(ln\tfrac{T}{s})^{\alpha-2}\tfrac{h(s)}{s\Gamma(\alpha-1)}ds-d_2\int_1^{\xi_2}(ln\tfrac{\xi_2}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds\\ &- \tfrac{d_1}{\Delta\Gamma(\alpha)}(a_2(lnT)^{\alpha-2}+\tfrac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3}\\ &- d_2(ln\xi_2)^{\alpha-2})\int_1^{\xi_1}(ln\tfrac{\xi_1}{s})^{\alpha-1}\tfrac{h(s)}{s}ds,\end{aligned}$$

and

$$\begin{aligned}c_2 &= \tfrac{d_1}{\Delta\Gamma(\alpha)}(a_2(lnT)^{\alpha-1}+\tfrac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2}\\ &- d_2(ln\xi_2)^{\alpha-1})\int_1^{\xi_1}(ln\tfrac{\xi_1}{s})^{\alpha-1}\tfrac{h(s)}{s}ds\\ &- \tfrac{d_1(ln\xi_1)^{\alpha-1}}{\Delta}(a_2\int_1^T(ln\tfrac{T}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds\\ &+ b_2\int_1^T(ln\tfrac{T}{s})^{\alpha-2}\tfrac{h(s)}{s\Gamma(\alpha-1)}ds-d_2\int_1^{\xi_2}(ln\tfrac{\xi_2}{s})^{\alpha-1}\tfrac{h(s)}{s\Gamma(\alpha)}ds.\end{aligned}$$

Substituting the values ofc_1 and c_2 in (2.7), we get

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t (ln\frac{t}{s})^{\alpha-1}\frac{h(s)}{s}ds \\
&+ \frac{d_1(ln\xi_1)^{\alpha-2}(lnt)^{\alpha-1}}{\Delta}[a_2\int_1^T (ln\frac{T}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds \\
&+ b_2\int_1^T (ln\frac{T}{s})^{\alpha-2}\frac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(ln\frac{\xi_2}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds] \\
&- \frac{d_1(lnt)^{\alpha-1}}{\Delta\Gamma(\alpha)}[a_2(lnT)^{\alpha-2}+\frac{b_2}{T}(\alpha-2)(lnT)^{\alpha-3} \\
&- d_2(ln\xi_2)^{\alpha-2}]\int_1^{\xi_1}(ln\frac{\xi_1}{s})^{\alpha-1}\frac{h(s)}{s}ds \\
&+ \frac{d_1(lnt)^{\alpha-2}}{\Delta\Gamma(\alpha)}[a_2(lnT)^{\alpha-1}+\frac{b_2}{T}(\alpha-1)(lnT)^{\alpha-2} \\
&- d_2(ln\xi_2)^{\alpha-1}]\int_1^{\xi_1}(ln\frac{\xi_1}{s})^{\alpha-1}\frac{h(s)}{s}ds \\
&- \frac{d_1(ln\xi_1)^{\alpha-1}(lnt)^{\alpha-2}}{\Delta}[a_2\int_1^T (ln\frac{T}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds \\
&+ b_2\int_1^T (ln\frac{T}{s})^{\alpha-2}\frac{h(s)}{s\Gamma(\alpha-1)}ds - d_2\int_1^{\xi_2}(ln\frac{\xi_2}{s})^{\alpha-1}\frac{h(s)}{s\Gamma(\alpha)}ds] \\
&= \int_1^T G(t,s)h(s)ds.
\end{aligned}
$$

This completes the proof.

Remark 2.1.1. *Note that the function $G(\cdot,\cdot)$ is not continuous over $[1,T]\times[1,T]$, However, the function $t\mapsto\int_1^t G(t,s)ds$ is continuous on $[1,T]$.*

The following result follows now directly from Lemma 4.3.1.

Lemma 2.1.2. *Let $f_i : I\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m;\ i=1,2$ be such that $f_i(\cdot,u,v)\in C(I)$ for each $u,v,w\in C(I)$. Then the coupled system 2.1–2.2 is equivalent to the problem of obtaining the solution of the coupled system*

$$
\begin{cases}
u(t)=\int_1^T G_1(t,s)f_1(s,u(s),v(s))ds \\
v(t)=\int_1^T G_2(t,s)f_2(s,u(s),v(s))ds,
\end{cases}
$$

where G_1 is the Green function with $G_1(t,s)$ given by

$$
\begin{cases}
\frac{1}{s\Gamma(\alpha_1)}(ln\frac{t}{s})^{\alpha_1-1} \\
+\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
-\frac{d_1(lnt)^{\alpha_1-1}}{s\Delta_1\Gamma(\alpha_1)}(ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(lnT)^{\alpha_1-2}+\frac{b_2}{T}(\alpha_1-2)(lnT)^{\alpha_1-3}-d_2(ln\xi_2)^{\alpha_1-2}] \\
+\frac{d_1(lnt)^{\alpha_1-2}}{s\Delta_1\Gamma(\alpha_1)}(ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(lnT)^{\alpha_1-1}+\frac{b_2}{T}(\alpha_1-1)(lnT)^{\alpha_1-2}-d_2(ln\xi_2)^{\alpha_1-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
;\ s\le\xi_1,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
-\frac{d_1(lnt)^{\alpha_1-1}}{s\Delta_1\Gamma(\alpha_1)}(ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(lnT)^{\alpha_1-2}+\frac{b_2}{T}(\alpha_1-2)(lnT)^{\alpha_1-3}-d_2(ln\xi_2)^{\alpha_1-2}] \\
+\frac{d_1(lnt)^{\alpha_1-2}}{s\Delta_1\Gamma(\alpha_1)}(ln\frac{\xi_1}{s})^{\alpha_1-1}[a_2(lnT)^{\alpha_1-1}+\frac{b_2}{T}(\alpha_1-1)(lnT)^{\alpha_1-2}-d_2(ln\xi_2)^{\alpha_1-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
;\ s\le\xi_1,\ t\le s \\
\\
\frac{1}{s\Gamma(\alpha_1)}(ln\frac{t}{s})^{\alpha_1-1} \\
+\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
;\ \xi_1\le s\le\xi_2,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}-\frac{d_2}{\Gamma(\alpha_1)}(ln\frac{\xi_2}{s})^{\alpha_1-1}] \\
;\ \xi_1\le s\le\xi_2,\ t\le s \\
\\
\frac{1}{s\Gamma(\alpha_1)}(ln\frac{t}{s})^{\alpha_1-1} \\
+\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}];\ \xi_2\le s,\ s\le t \\
\\
\frac{d_1(ln\xi_1)^{\alpha_1-2}(lnt)^{\alpha_1-1}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}] \\
-\frac{d_1(ln\xi_1)^{\alpha_1-1}(lnt)^{\alpha_1-2}}{s\Delta_1}[\frac{a_2}{\Gamma(\alpha_1)}(ln\frac{T}{s})^{\alpha_1-1}+\frac{b_2}{\Gamma(\alpha_1-1)}(ln\frac{T}{s})^{\alpha_1-2}];\ \xi_2\le s,\ t\le s
\end{cases}
$$

and where G_2 is the Green function with $G_2(t,s)$ given by

$$\begin{cases}
\frac{1}{s\Gamma(\alpha_2)}(ln\frac{t}{s})^{\alpha_2-1} \\
+\frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
-\frac{d_3(lnt)^{\alpha_2-1}}{s\Delta_2\Gamma(\alpha_2)}(ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(lnT)^{\alpha_2-2}+\frac{b_4}{T}(\alpha_2-2)(lnT)^{\alpha_2-3}-d_4(ln\xi_4)^{\alpha_2-2}] \\
+\frac{d_3(lnt)^{\alpha_2-2}}{s\Delta_2\Gamma(\alpha_2)}(ln\frac{\xi_3}{s})^{\alpha_2-1}[a_2(lnT)^{\alpha_2-1}+\frac{b_4}{T}(\alpha_2-1)(lnT)^{\alpha_2-2}-d_4(ln\xi_4)^{\alpha_2-1}] \\
-\frac{d_3(ln\xi_1)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
;\ s\leq\xi_3,\ s\leq t \\
\\
\frac{d_1(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
-\frac{d_3(lnt)^{\alpha_2-1}}{s\Delta_2\Gamma(\alpha_2)}(ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(lnT)^{\alpha_2-2}+\frac{b_4}{T}(\alpha_2-2)(lnT)^{\alpha_2-3}-d_4(ln\xi_4)^{\alpha_2-2}] \\
+\frac{d_3(lnt)^{\alpha_2-2}}{s\Delta_2\Gamma(\alpha_2)}(ln\frac{\xi_3}{s})^{\alpha_2-1}[a_4(lnT)^{\alpha_2-1}+\frac{b_4}{T}(\alpha_2-1)(lnT)^{\alpha_2-2}-d_4(ln\xi_4)^{\alpha_2-1}] \\
-\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
;\ s\leq\xi_3,\ t\leq s \\
\\
\frac{1}{s\Gamma(\alpha_2)}(ln\frac{t}{s})^{\alpha_2-1} \\
+\frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
-\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
;\ \xi_3\leq s\leq\xi_4,\ s\leq t \\
\\
\frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_4)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
-\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_4)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}-\frac{d_4}{\Gamma(\alpha_2)}(ln\frac{\xi_4}{s})^{\alpha_2-1}] \\
;\ \xi_3\leq s\leq\xi_4,\ t\leq s \\
\\
\frac{1}{s\Gamma(\alpha_1)}(ln\frac{t}{s})^{\alpha_1-1} \\
+\frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}] \\
-\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}];\ \xi_4\leq s,\ s\leq t \\
\\
\frac{d_3(ln\xi_3)^{\alpha_2-2}(lnt)^{\alpha_2-1}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}] \\
-\frac{d_3(ln\xi_3)^{\alpha_2-1}(lnt)^{\alpha_2-2}}{s\Delta_2}[\frac{a_4}{\Gamma(\alpha_2)}(ln\frac{T}{s})^{\alpha_2-1}+\frac{b_4}{\Gamma(\alpha_2-1)}(ln\frac{T}{s})^{\alpha_2-2}];\ \xi_4\leq s,\ t\leq s
\end{cases}$$

where

$$\begin{aligned}\Delta_1 &= d_1(ln\xi_1)^{\alpha_1-1}[a_2(lnT)^{\alpha_1-2} + \tfrac{b_2}{T}(\alpha_1-2)(lnT)^{\alpha_1-3} - d_2(ln\xi_2)^{\alpha_1-2}]\\ &- d_1(ln\xi_1)^{\alpha_1-2}[a_2(lnT)^{\alpha_1-1} + \tfrac{b_2}{T}(\alpha_1-1)(lnT)^{\alpha_1-2} - d_2(ln\xi_2)^{\alpha_1-1}] \neq 0,\end{aligned}$$

and

$$\begin{aligned}\Delta_2 &= d_3(ln\xi_3)^{\alpha_2-1}[a_4(lnT)^{\alpha_2-2} + \tfrac{b_4}{T}(\alpha_2-2)(lnT)^{\alpha_2-3} - d_4(ln\xi_4)^{\alpha_2-2}]\\ &- d_3(ln\xi_3)^{\alpha_2-2}[a_4(lnT)^{\alpha_2-1} + \tfrac{b_4}{T}(\alpha_2-1)(lnT)^{\alpha_2-2} - d_4(ln\xi_4)^{\alpha_2-1}] \neq 0.\end{aligned}$$

Remark 2.1.2. *Notice that the function $G(\cdot,\cdot)$ is not continuous over $[1,T]\times[1,T]$, however the function $t \mapsto \int_1^t G(t,s)ds$ is continuous on $[1,T]$. Set*

$$G^* = \sup_{t\in[1,T]} \int_1^t |G(t,s)|ds.$$

The following hypotheses will be used in the sequel.

(H_1) The function $f_i;\ i=1,2$ satisfies the generalized Lipschitz condition :

$$\|f_i(t,u_1,v_1) - f_i(t,u_2,v_2)\| \leq \frac{1}{G_i^*}(\phi_i\|u_1-u_2\| + \psi_i\|v_1-v_2\|),$$

for $t\in I$ and $u_i, v_i \in \mathbb{R}^m$. where $\phi_i, \psi_i; i=1,2$ are comparison functions.

(H_2) There exist continuous functions$h_i, p_i, q_i : I \to \mathbb{R}_+;\ i=1,2$ such that

$$\|f_i(t,u,v)\| \leq h_i(t) + p_i(t)\|u\| + q_i(t)\|v\|;\ pourtout\ t\in I,\ et\ u,v\in\mathbb{R}^m.$$

Theorem 2.1.1. *Assume (H_1) Then (2.1)–(2.2) has a unique solution. .*

Proof.

Define the operator $N : C \to C$ by

$$(N(u,v))(t) = ((N_1u)(t), (N_2v)(t)), \tag{2.8}$$

where $N_1, N_2 : C \to C$ with

$$(N_1u))(t) = \int_1^T G_1(t,s)f_1(s,u(s),v(s))ds, \tag{2.9}$$

and

$$(N_2v)(t) = \int_1^T G_2(t,s)f_2(s,u(s),v(s))ds. \tag{2.10}$$

Clearly, the fixed points of the operator N are solutions of (2.1)–(2.2). For each $u_i, v_i \in C$, $i = 1, 2$, and $t \in I$, we have

$$\begin{array}{rcl}
\|(N_1u_1)(t) - (N_1u_2)(t)\| & = & \| \int_1^T G_1(t,s)[f_1(s,u_1(s),v_1(s)) - f_1(s,u_2(s),v_2(s))]ds\| \\
& \leq & \int_1^T \|G_1(t,s)[f_1(s,u_1(s),v_1(s)) - f_1(s,u_2(s),v_2(s))]\|ds \\
& \leq & \int_1^T |G_1(t,s)|\|f_1(s,u_1(s),v_1(s)) - f_1(s,u_2(s),v_2(s))\|ds \\
& \leq & \phi_1(\|u_1(s) - u_2(s)\|) + \psi_1(\|v_1(s) - v_2(s)\|) \\
& \leq & \phi_1(\|u_1(s) - u_2(s)\|) + \|v_1(s) - v_2(s)\|) \\
& + & \psi_1(\|u_1(s) - u_2(s)\|) + \|v_1(s) - v_2(s)\|) \\
& \leq & \phi_1(D((u_1,v_1),(u_2,v_2))) + \psi_1(D((u_1,v_1),(u_2,v_2))).
\end{array}$$

Also

$$\begin{array}{rcl}
\|(N_2v_1)(t) - (N_2v_2)(t)\| & = & \| \int_1^T G_2(t,s)[f_2(s,u_1(s),v_1(s)) - f_2(s,u_2(s),v_2(s))]ds\| \\
& \leq & \int_1^T \|G_2(t,s)[f_2(s,u_1(s),v_1(s)) - f_2(s,u_2(s),v_2(s))]\|ds \\
& \leq & \int_1^T |G_2(t,s)|\|f_2(s,u_1(s),v_1(s)) - f_2(s,u_2(s),v_2(s))\|ds \\
& \leq & \phi_2(\|u_1(s) - u_2(s)\|) + \psi_2(\|v_1(s) - v_2(s)\|) \\
& \leq & \phi_2(\|u_1(s) - u_2(s)\|) + \|v_1(s) - v_2(s)\|) \\
& + & \psi_2(\|u_1(s) - u_2(s)\|) + \|v_1(s) - v_2(s)\|) \\
& \leq & \phi_2(D((u_1,v_1),(u_2,v_2))) + \psi_2(D((u_1,v_1),(u_2,v_2))).
\end{array}$$

Thus, we get

$$D(N(u_1,v_1), N(u_2,v_2)) \leq \phi D((u_1,v_1),(u_2,v_2)).$$

where $\phi = \phi_1 + \phi_2 + \phi_1 + \psi_2$.

Consequently, from Theorem 1.5.2, the operator N has a unique fixed point, which is the unique solution of (2.1)–(2.2) on I.

Now, we prove an existence result by using Schauder fixed point theorem. Set

$$h_i^* := \sup_{t\in I} h(t), \quad p_i^* := \sup_{t\in I} p(t), \quad q_i^* := \sup_{t\in I} q(t), \quad i = 1, 2.$$

Theorem 2.1.2. *Assume* (H_2). *If*

$$G_1^* h_1^* + G_2^* h_2^* < 1,$$

then the coupled system (2.1)–(2.2) has at least one solution defined on I.

Proof. Let N be the operator defined in 2.8. Set

$$R \geq \frac{G_1^*(p_1^* + q_1^*) + G_2^*(p_2^* + q_2^*)}{1 - G_1^* h_1^* - G_2^* h_2^*}$$

and consider the closed and convex ball

$$B_R = \{(u,v) \in C : \ \|(u,v)\|_C \leq R\}.$$

Let $(u_1, u_2) \in B_R$. Then, for each $t \in I$ and any $i = 1, 2$, we have

$$\begin{array}{rcl} \|(N_i u_i)(t)\| & = & \int_1^T \|G_i(t,s) f_i(s, u(s), v(s)) ds\| \\ & \leq & \int_1^T |G_i(t,s)| \|f_1(s, u(s), v(s))\| ds \\ & \leq & \int_1^T |G_i(t,s)| [h_i(s) + p_i(s)\|u(s)\| + q_i(s)\|v(s)\|] ds \\ & = & G_i^*(h_i^* + R p_i^* + q_i^*). \end{array}$$

Thus,

$$\|N(u_1, u_2)\|_{\mathcal{C}} \leq R.$$

Hence N maps the ball B_R into itself. We shall show that the operator $N : B_R \to B_R$ satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1 : We show that N is continuous.
Let $\{(u_n, v_n)\}$ be a sequence such that $(u_n, v_n) \to (u, v)$ in B_r. Then, for each $t \in I$, we have

$$\begin{array}{rl} & \|(N(u_n, v_n)(t) - (N(u,v)(t)\| \\ = & \Sigma_{i=1}^2 \int_i^T \|G_1(t,s)[f_i(s, u_n(s), v_n(s)) - f_i(s, u(s), v(s))] ds\| \\ \leq & \Sigma_{i=1}^2 \int_1^T |G_1(t,s)| \|f_1(s, u_n(s), v_n(s)) - f_1(s, u(s), v(s))\| ds. \end{array}$$

Since $u_n \to u$, $v_n \to v$ as $n \to \infty$ et f_1,f_2 are continuous, by the Lebesgue dominated convergence theorem

$$\|N(u_n, v_n) - N(u, v)\|_{\mathcal{C}} \to 0 \ \ as \ \ n \to \infty.$$

Step 2 : We remark that $N(B_R)$ is bounded. This is clear since $N : B_R \to B_R$ and B_R is bounded.

Step 3 : We show that N maps bounded sets into equicontinuous sets in B_R. Let $t_1, t_2 \in I$ be such that $t_1 < t_2$ and let $(u_1, u_2) \in B_R$. Then, we have

$$\begin{array}{ll} & \|(N(u_1, u_2))(t_1) - (N(u_1, u_2))(t_2)\| \\ \leq & \| \int_1^{t_1} G_i(t_1, s) f_i(s, u_1(s), u_2(s))ds - \int_1^{t_2} G_i(t_2, s) f_i(s, u_1(s), u_2(s))ds\| \\ \leq & \int_1^{t_1} |G_i(t_1, s)| f_i(s, u_1(s), u_2(s))ds\| - \| \int_1^{t_2} |G_i(t_2, s)| f_i(s, u_1(s), u_2(s))ds \\ \leq & [(p_i^* + q_i^*)R + h_i^*] \left[\int_1^{t_1} |G_i(t_1, s)|ds - \int_{t_1}^{t_2} |G_i(t_2, s)|ds\right] \end{array}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzelà–Ascoli theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 1.5.3, we deduce that N has a fixed point u, which is a solution of problem (2.1)–(2.2).

2.1.3 Exemple

Consider the coupled system of Caputo–Hadamard fractional differential equations

$$\begin{cases} ({}^{Hc}D_1^{\frac{3}{2}} u)(t) = f_1(t, u(t), v(t)); \\ ({}^{Hc}D_1^{\frac{3}{2}} v)(t) = f_2(t, u(t), v(t)); \end{cases} \quad ; \ t \in [1, e], \tag{2.11}$$

with the multipoint boundary conditions

$$\begin{cases} u(1) - u'(1) = u(2) \\ 2u(T) + u'(T) = 2u(\frac{3}{2}) \\ 3v(1) - v'(1) = 3v(\frac{5}{4}) \\ v(T) + 2v'(T) = v(2), \end{cases} \tag{2.12}$$

where

$$
\begin{aligned}
f(t,u,v) &= \tfrac{t^{\frac{-1}{4}}(u(t)+1)\sin(t)}{4(1+\sqrt{t})(1+|u|+|v|)},\ t\in[1,e]\\
g(t,u,v) &= \tfrac{(v(t)+1)\cos(t)}{4(1+u(t)+v(t))},
\end{aligned}
$$

The hypothesis $(H1)$ is satisfied with

$$
\phi_1(x)=\frac{x}{4G_1^*},\quad \psi_2(x)=\frac{x}{4G_2^*},\quad \psi_1(x)=\phi_2(x)=0.
$$

Theorem 2.1.1 implies that the system 2.11–2.12 has a unique solution defined on $[1,e]$.

2.2 Implicit Coupled Caputo-Hadamard Fractional Differential Systems

2.2.1 Introduction and motivations

In recent years, fractional differential equations have found applications in diverse fields such as engineering, mathematics, and physics, as well as other applied sciences. There has been a significant focus on studying the existence of solutions for initial and boundary value problems related to fractional differential equations. To this end, several monographs [8, 12, 106, 140, 145, 161] and papers [63, 67, 110, 128, 139] have explored this area in depth.

In this section, we discuss the existence and uniqueness of solutions for the following coupled system of Caputo-Hadamard fractional differential equations

$$\begin{cases} ({}^{Hc}D_1^{\alpha_1}u_1)(t) = f_1(t, u_1(t), u_2(t), ({}^{Hc}D_1^{\alpha_1}u_1)(t)) \\ ({}^{Hc}D_1^{\alpha_2}u_2)(t) = f_2(t, u_1(t), u_2(t), ({}^{Hc}D_1^{\alpha_2}u_2)(t)) \end{cases} ; \ t \in I := [1, T], \tag{2.13}$$

with the multipoint boundary conditions

$$\begin{cases} a_1u_1(1) - b_1u_1'(1) = d_1u_1(\xi_1) \\ a_2u_1(T) + b_2u_1'(T) = d_2u_1(\xi_2) \\ a_3u_2(1) - b_3u_2'(1) = d_3u_2(\xi_3) \\ a_4u_2(T) + b_4u_2'(T) = d_4u_2(\xi_4) \end{cases} ; \ w \in \Omega, \tag{2.14}$$

where $T > 1$, $a_i, b_i, d_i \in \mathbb{R}$, $\xi_i \in (1, T)$; $i = 1, 2, 3, 4$, $\alpha_j \in (1, 2]$, $f_j : I \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$; $j = 1, 2$ are given continuous functions, $\mathbb{R}^m$; $m \in \mathbb{N}^*$ is the Euclidian Banach space with a suitable norm $\|\cdot\|$, ${}^{Hc}D_1^{\alpha_j}$ is the Caputo–Hadamard fractional derivative of order α_j; $j = 1, 2$.

In [56] ; the authors established the existence, uniqueness and stability results of solutions for the following initial value problem for imlicit fractional order differential equations

$$\begin{cases} {}^H D^\alpha y(t) = f(t, y(t), {}^H D^\alpha y(t)), \ t \in J, \ 0 < \alpha \leq 1, \\ y(1) = y_1, \end{cases}$$

where ${}^H D^\alpha$ is the Hadamard fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function space, $y_1 \in \mathbb{R}$ and $J = [1, T]$, $T > 1$.

In [47] ; the following classes of boundary value problems for the existence and stability of solutions for implicit fractional differential equations with Caputo fractional derivative :

$$\begin{cases} {}^c D^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t)), \ t \in J := [0, T], \ T > 0, \ 0 < \alpha \leq 1, \\ \alpha y(0) + by(T) = c, \end{cases}$$

where ${}^c D^\alpha$ is the fractional derivative of Caputo, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function, and a,b,c are real constants with $a + b \neq 0$, and

$$\begin{cases} {}^c D^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t)), \ t \in J := [0, T], \ T > 0, \ 0 < \alpha \leq 1, \\ y(0) + g(y) = y_0, \end{cases}$$

where $g : C([0, T], \mathbb{R}) \to \mathbb{R}$ a continuous function and y_0 a real constant ; are studied. This type of non-local Cauchy problem was introduced by Byszewski. The author observed that the non-local condition is more appropriate that the non-local condition(initial) to describe correctly some physics phenomenons and proved the existence and the uniqueness of weak solutions and also classical solutions for this type of problems. We take an example of non-local conditions as follows :

$$g(y) = \sum_{i=1}^{p} c_i y(t_i)$$

where c_i, $i = 1...p$ are constants and $0 < t_1 < ... < t_p \leq T$.

2.2.2 Existence of solutions

Consider the complete metric space $C(I) := C(I, \mathbb{R}^m)$ of continuous functions from I into $\mathbb{R}^m$ equipped with the usual metric

$$d(u, v) := \max_{t \in I} \|u(t) - v(t)\|,$$

where $\|\cdot\|$ is a suitable norm on $\mathbb{R}^m$. Notice that $C(I)$ is a Banach space with the supremum (uniform) norm

$$\|u\|_\infty := \sup_{t\in I} \|u(t)\|.$$

Definition 2.2.1. *By a solution of the implicit coupled system (2.13)-(2.14) we mean a coupled continuous functions $(u, v) \in C \times C$ satisfying the boundary conditions (2.14), and the equations (2.13) on I.*

The following hypotheses will be used in the sequel.

(H_3) The functions $f_i;\ i = 1, 2$ satisfy the generalized Lipschitz condition :

$$\|f_i(t, u_1, u_2, w_1)-f_i(t, v_1, v_2, w_2)\| \le \frac{1}{G_i^*}(\phi_i(\|u_1-v_1\|)+\psi_i(\|u_2-v_2\|))+\xi_i(\|w_1-w_2\|)).$$

Set

$$G^* = \sup_{t\in[1,T]} \int_1^t |G(t, s)|ds.$$

for $t \in I$ and $u_i, v_i, w_i \in \mathbb{R}^m$, where $\phi_i, \psi_i; .\xi_i;\ i = 1, 2$ are comparison functions.

(H_4) There exist continuous functions $h_i, p_i, q_i : I \to \mathbb{R}_+$ and $0 < k_i < 1;\ i = 1, 2$ such that

$$(1 + \|u_1\| + \|u_2\| + \|w_i\|)\|f_i(t, u_1, u_2, w_i)\|$$
$$\le h_i(t) + p_i(t)\|u_1\| + q_i(t)\|u_2\| + k_i(t)\|w_i\|;\ for\ t \in I,\ and\ u_i, w_i \in \mathbb{R}^m.$$

Theorem 2.2.1. *Assume that the hypothesis (H_1) holds. Then the coupled system (2.13)-(2.14) has a unique solution.*

Proof. Define the operator $N : \mathcal{C} \to \mathcal{C}$ by

$$(N(u_1, u_2))(t) = ((N_1u_1)(t), (N_2u_2)(t)), \tag{2.15}$$

where $N_1, N_2 : C \to C$ with

$$(N_1u_1))(t) = \mu_1(t) +^H I_1^\alpha g_1(t), \tag{2.16}$$

and

$$(N_2u_2)(t) = \mu_2(t) +^H I_1^\alpha g_2(t). \tag{2.17}$$

Clearly, the fixed points of the operator N are solutions of the coupled system (2.13)-(2.14).
For each $u_i, v_i \in C;\ i = 1, 2$ and $t \in I$, we have

$$\|(N_i(u_1,u_2))(t) - (N_i(v_1,v_2))(t)\| \leq \frac{1}{\Gamma(\alpha_i)} \int_1^T (ln\frac{t}{s})^{\alpha_i - 1} [\|g_i(s) - h_i(s)]\| \frac{ds}{s},$$

where $g_i, h_i \in C$ are given by
$g_i(t) = f_i(t, u_1, u_2, g_i(t))$, and $h_i(t) = f_i(t, v_1, v_2, h_i(t))$.
Then, from (H_1),

$$\|g_i(t) - h_i(t)\| \leq \frac{1}{G_i^*}(\phi_i(\|u_1(t) - v_1(t)\|) + \psi_i(\|u_2(t) - v_2(t)\|)) + \xi_i(\|g_i(t) - h_i(t)\|)).$$

Thus

$$\|g_i(t) - h_i(t)\| \leq \frac{\phi_i}{G_i^* - \xi_i}(\|u_1(t) - v_1(t)\|) + \frac{\psi_i}{G_i^* - \xi_i}(\|u_2(t) - v_2(t)\|),$$

for $i = 1, 2$. Hence,

$$\begin{aligned}
\|(N_1u_1)(t) - (N_1v_1)(t)\| &\leq \tfrac{1}{\Gamma(\alpha_1)} \int_1^T (ln\tfrac{t}{s})^{\alpha_1 - 1}[\|g_1(s) - h_1(s)]\| \tfrac{ds}{s} \\
&\leq \tfrac{\ln^{\alpha_1}(T)}{\Gamma(\alpha_1+1)}[\tfrac{\phi_1}{G_1^*-\xi_1}\|u_1(s) - v_1(s)\| + \tfrac{\psi_1}{G_1^*-\xi_1}\|u_2(s) - v_2(s)\|] \\
&\leq \tfrac{\ln^{\alpha_1}(T)\phi_1}{\Gamma(\alpha_1+1)(G_1^*-\xi_1)}(\|u_1(s) - v_1(s)\| + \|u_2(s) - v_2(s)\|) \\
&+ \tfrac{\ln^{\alpha_1}(T)\psi_1}{\Gamma(\alpha_1+1)(G_1^*-\xi_1)}(\|u_1(s) - v_1(s)\| + \|u_2(s) - v_2(s)\|) \\
&\leq \tfrac{\ln^{\alpha_1}(T)\phi_1}{\Gamma(\alpha_1+1)(G_1^*-\xi_1)}(D((u_1,u_2),(v_1,v_2))) \\
&+ \tfrac{\ln^{\alpha_1}(T)\psi_1}{\Gamma(\alpha_1+1)(G_1^*-\xi_1)}(D((u_1,u_2),(v_1,v_2))).
\end{aligned}$$

Also

$$\begin{aligned}
\|(N_1u_2)(t) - (N_2v_2)(t)\| &\leq \tfrac{1}{\Gamma(\alpha_2)} \int_1^T (ln\tfrac{t}{s})^{\alpha_2 - 1}[\|g_2(s) - h_2(s)]\| \tfrac{ds}{s} \\
&\leq \tfrac{\ln^{\alpha_1}(T)}{\Gamma(\alpha_2+1)}[\tfrac{\phi_2}{G_2^*-\xi_2}\|u_1(s) - v_1(s)\| + \tfrac{\psi_2}{G_2^*-\xi_2}\|u_2(s) - v_2(s)\|] \\
&\leq \tfrac{\ln^{\alpha_2}(T)\phi_2}{\Gamma(\alpha_2+1)(G_2^*-\xi_2)}(\|u_1(s) - v_1(s)\| + \|u_2(s) - v_2(s)\|) \\
&+ \tfrac{\ln^{\alpha_2}(T)\psi_2}{\Gamma(\alpha_2+1)(G_2^*-\xi_2)}(\|u_1(s) - v_1(s)\| + \|u_2(s) - v_2(s)\|) \\
&\leq \tfrac{\ln^{\alpha_2}(T)\phi_2}{\Gamma(\alpha_2+1)(G_2^*-\xi_2)}(D((u_1,u_2),(v_1,v_2))) \\
&+ \tfrac{\ln^{\alpha_2}(T)\psi_2}{\Gamma(\alpha_2+1)(G_2^*-\xi_2)}(D((u_1,u_2),(v_1,v_2))).
\end{aligned}$$

Thus, we get

$$D(N(u_1, u_2), N(v_1, v_2)) \leq \phi(D((u_1, u_2), (v_1, v_2))),$$

where

$$\phi = \frac{\ln^{\alpha_1}(T)(\phi_1 + \psi_1)}{\Gamma(\alpha_1 + 1)(G_1^* - \xi_1)} + \frac{\ln^{\alpha_2}(T)(\phi_2 + \psi_2)}{\Gamma(\alpha_2 + 1)(G_2^* - \xi_2)}. \tag{2.18}$$

Consequently, from Theorem (1.5.2), the operator N has a unique fixed point, which is the unique solution of our problem (2.13)-(2.14) on I.

Now, we prove an existence result by using Nonlinear alternative of Leray-Schauder fixed point theorem.
Set

$$h_i^* := \sup_{t \in I} h(t),\ \ p_i^* := \sup_{t \in I} p(t),\ \ q_i^* := \sup_{t \in I} q(t);\ \ i = 1, 2.$$

Theorem 2.2.2. *Assume that the hypothesis* (H_2) *holds. Then the problem (2.13)-(2.14) has at least one solution defined on* I.

Proof. Let $N : \mathcal{C} \to \mathcal{C}$ be the operator defined in (2.15). We need to show that N satisfies the conditions in Theorem (1.5.7). The proof will be given in several steps.

Step 1 : N is continuous.
Let $\{((u_1)_n, (u_2)_n)\}$ be a sequence such that $((u_1)_n, (u_2)_n) \to (u_1, u_2)$ in $\mathcal{C}$. Then, for each $t \in I$, we have

$$\|(N((u_1)_n, (u_2)_n))(t) - (N(u_1, u_2))(t)\| \leq \sum_{i=1}^{2} \frac{1}{\Gamma(\alpha_1)} \int_1^T (ln\frac{t}{s})^{\alpha_1 - 1} [\|(g_{in})(s) - (g_i)(s)]\| \frac{ds}{s}$$

where $g_{in}, g_i \in C$ such that
$g_{in}(t) = f_i(t, u_{1n}, u_{2n}, g_{in}(t))$ and $g_i(t) = f_i(t, u_1, u_2, g_i(t))$.
Since $(u_{1n}, u_{2n}) \to (u_1, u_2)$ as $n \to \infty$ and f_i ; i=1,2 are continuous, we get $g_{in}(t) \to g(t)$ as $n \to \infty$, then by the Lebesgue dominated convergence theorem ;

$$\|N(u_{1n}, u_{2n}) - N(u_1, u_2)\|_{\mathcal{C}} \to 0 \ \ as \ \ n \to \infty.$$

Step 2 : N maps bounded sets into bounded sets in $\mathcal{C}$.
Let $R > 0$ and set $B_R = \{(u_1, u_2) \in \mathcal{C} : \|(u_1, u_2)\|_{\mathcal{C}} \leq R\}$. Let $(u_1, u_2) \in B_R$. Then, for each $t \in I$,, and any $i = 1.2$, we have

$$(N_i u_i)(t) = \mu_i(t) + \frac{1}{\Gamma(\alpha_i)} \int_1^t (ln\frac{t}{s})^{\alpha_i - 1} g_i(s) \frac{ds}{s},$$

where $g_i \in C$ such that $g_i(t) = f_i(t, u_1, u_2, g_i(t))$.
From (H_2) we get

$$\|g_i(s)\| \leq \frac{h_i(s)}{1 - k_i} + \frac{p_i(s)}{1 - k_i} + \frac{q_i(s)}{1 - k_i}.$$

Then,

$$\begin{aligned}
\|(N_i u_i)(t)\| &\leq \|\mu_i(t)\| + \tfrac{1}{\Gamma(\alpha_i)} \int_1^T (ln\tfrac{t}{s})^{\alpha_i - 1} \|g_i(s)\| \tfrac{ds}{s} \\
&\leq \|\mu_i(t)\| + \tfrac{1}{\Gamma(\alpha_i)} \int_1^T (ln\tfrac{t}{s})^{\alpha_i - 1} \left[\tfrac{h_i(s)}{1-k_i} + \tfrac{p_i(s)}{1-k_i} + \tfrac{q_i(s)}{1-k_i}\right] \tfrac{ds}{s} \\
&\leq \|\mu_i\|_\infty + \tfrac{\ln^{\alpha_i}(T)}{\Gamma(\alpha_i+1)} \left[\tfrac{p_i^*}{1-k_i} + \tfrac{q_i^*}{1-k_i} + \tfrac{h_i^*}{1-k_i}\right].
\end{aligned}$$

Thus,

$$\|N(u_1, u_2)\|_{\mathcal{C}} \leq \sum_{i=1}^{2} \left(\|\mu_i\|_\infty + \frac{\ln^{\alpha_i}(T)}{\Gamma(\alpha_i + 1)} \left[\frac{p_i^*}{1 - k_i} + \frac{q_i^*}{1 - k_i} + \frac{h_i^*}{1 - k_i}\right] \right) := M^*.$$

Step 3 : N maps bounded sets into equicontinuous sets in B_R.
Let $t_1, t_2 \in I$, such that $t_1 < t_2$ and let $(u_1, u_2) \in B_R$. Then, we have

$$\begin{aligned}
\|(N(u_1, u_2))(t_1) - (N(u_1, u_2))(t_2)\| &\leq \|\mu_i(t_1) - \mu_i(t_2)\| \\
&+ \tfrac{1}{\Gamma(\alpha_i)} \int_1^T ((ln\tfrac{t_2}{s})^{\alpha_i - 1} - (ln\tfrac{t_1}{s})^{\alpha_i - 1}) \|g_i(s)\| \tfrac{ds}{s} \\
&\leq \|\mu_i(t_1) - \mu_i(t_2)\| \\
&+ [\tfrac{h_i^*}{1-k_i} + \tfrac{p_i^*}{1-k_i} + \tfrac{q_i^*}{1-k_i}] \\
&\times \tfrac{1}{\Gamma(\alpha_i)} \int_1^T ((ln\tfrac{t_2}{s})^{\alpha_i - 1} - (ln\tfrac{t_1}{s})^{\alpha_i - 1}) \tfrac{ds}{s}.
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3, together with the Arzelá–Ascoli theorem, we can conclude that $N : B_R \to B_R$ is continuous and completely continuous.

Step 4 : We now show there exists an open set $U \subseteq \mathcal{C}$ with $(u_1, u_2) \neq \lambda N(u_1, u_2)$, for $\lambda \in [0, 1]$ and $(u_1, u_2) \in \partial U$. Let $(u_1, u_2) \in \mathcal{C}$ and $(u_1, u_2) = \lambda N(u_1, u_2)$, $\lambda \in [0, 1]$

. Then $u_1(t) = \lambda(N_1u_1)(t)$, and $u_2(t) = \lambda(N_2u_2)(t)$. Thus, as in step 2, for each $t \in I$, we have

$$\begin{array}{rcl} \|u_i(t)\| & \leq & \|\mu_i(t)\| + \frac{1}{\Gamma(\alpha_i)} \int_1^T (ln\frac{t}{s})^{\alpha_i-1} \|g_i(s)\| \frac{ds}{s} \\ & \leq & \|\mu_i(t)\| + \frac{1}{\Gamma(\alpha_i)} \int_1^T (ln\frac{t}{s})^{\alpha_i-1} \left[\frac{h_i(s)}{1-k_i} + \frac{p_i(s)}{1-k_i} + \frac{q_i(s)}{1-k_i} \right] \frac{ds}{s} \\ & \leq & \|\mu_i\|_\infty + \frac{\ln^{\alpha_i}(T)}{\Gamma(\alpha_i+1)} \left[\frac{p_i^*}{1-k_i} + \frac{q_i^*}{1-k_i} + \frac{h_i^*}{1-k_i} \right]. \end{array}$$

Hence,

$$\|(u_1, u_2)\|_{\mathcal{C}} \leq M^*.$$

Set

$$U = \{(u_1, u_2) \in \mathcal{C} : \|(u_1, u_2)\|_{\mathcal{C}} \leq M^* + 1\}.$$

By our choice of U, there is no $(u_1, u_2) \in \partial U$ such that $(u_1, u_2) = \lambda N(u_1, u_2)$, for $\lambda \in [0, 1]$. As a consequence of Theorem 1.5.7, we deduce that N has a fixed point (u_1, u_2) in $\overline{U}$ which is a solution of problem (2.13)- (2.14).

2.2.3 Example

Consider the following implicit coupled system of Caputo–Hadamard fractional differential equations

$$\begin{cases} ({}^{Hc}D_1^{\frac{3}{2}}u_1)(t) = f(t, u_1(t), u_2(t), ({}^{Hc}D_1^{\frac{3}{2}}u_1)(t)); \\ ({}^{Hc}D_1^{\frac{3}{2}}u_2)(t) = g(t, u_1(t), u_2(t), ({}^{Hc}D_1^{\frac{3}{2}}u_2)(t)); \end{cases} \quad ; \ t \in [1, e], \tag{2.19}$$

with the multipoint boundary conditions

$$\begin{cases} u_1(1) - u_1'(1) = u_1(2) \\ 2u_1(T) + u_1'(T) = 2u_1(\frac{3}{2}) \\ 3u_2(1) - u_2'(1) = 3u_2(\frac{5}{4}) \\ u_2(T) + 2u_2'(T) = u_2(2), \end{cases} \tag{2.20}$$

where

$$f(t, u_1, u_2, w) = \frac{t^{\frac{-1}{4}} u(t) \sin(t)}{24(1 + \sqrt{t})(1 + |u_1| + |u_2| + |w|)}; \ t \in [1, e],$$

$$g(t, u_1, v_2, w) = \frac{u_1(t) \cos(t)}{24(1 + |u_1| + |u_2| + |w|)}; \ t \in [1, e].$$

The hypothesis (H_1) is satisfied with

$$\phi_1(x) = \frac{x}{24G_1^*}, \psi_2(x)\frac{x}{24G_2^*},$$

$$\xi_1(x) = \psi_1(x) = \phi_2(x) = \xi_2(x) = 0.$$

Hence, Theorem 2.2.2 implies that the system (2.19)-(2.20) has a unique solution defined on $[1, e]$.

Chapitre 3

Coupled Systems of Conformable fractioonal differential equations

3.1 Coupled conformable fractional differential system

3.1.1 Introduction and motivations

In [109], the authors considered the following conformable impulsive problem :

$$\begin{cases} \mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta) = \aleph\left(\zeta, \chi_\zeta, \mathcal{T}^{\vartheta}_{\jmath}\chi(\zeta)\right), & \zeta \in \Omega_\jmath; \jmath = 0, 1, \ldots, \beta, \\ \Delta\chi|_{\zeta=\zeta_\jmath} = \Upsilon_\jmath(\chi_{\zeta_\jmath^-}), & \jmath = 1, 2, \ldots, \beta, \\ \chi(\zeta) = \mu(\zeta), & \zeta \in (-\infty, \varkappa], \end{cases}$$

where $0 \leq \varkappa = \zeta_0 < \zeta_1 < \cdots < \zeta_\beta < \zeta_{\beta+1} = \bar{\varkappa} < \infty$, $\mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $\aleph : \Omega \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\Omega := [\varkappa, \bar{\varkappa}]$, $\Omega_0 := [\varkappa, \zeta_1]$, $\Omega_\jmath := (\zeta_\jmath, \zeta_{\jmath+1}]; \jmath = 1, 2, \ldots, \beta$, $\mu : (-\infty, \varkappa] \to \mathbb{R}$ and $\Upsilon_\jmath : \mathcal{Q} \to \mathbb{R}$ are given continuous functions, and $\mathcal{Q}$ is called a phase space.

In this section, we investigate the existence of solutions for the following coupled conformable fractional differential system :

$$\begin{cases} (\mathcal{T}_{0^+}^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{0^+}^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in I, \tag{3.1}$$

with the following coupled boundary conditions :

$$(u(0), v(0)) = (\delta_1 v(T), \delta_2 u(T)), \tag{3.2}$$

where $T > 0$, $I := [0, T]$, $\alpha_i \in (0, 1]$; $i = 1, 2$ $f_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $i = 1, 2$ are given continuous functions, $\mathcal{T}_0^{\alpha_i}$ is the conformable fractional derivative of order $\alpha_i; i = 1, 2$, and δ_1, δ_2 are real numbers with $\delta_1\delta_2 \neq 1$.

Next, we investigate the following coupled conformable fractional differential system :

$$\begin{cases} (\mathcal{T}_{a^+}^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{a^+}^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in [a, \infty), \tag{3.3}$$

with the coupled initial conditions :

$$(u(a), v(a)) = (u_a, v_a), \tag{3.4}$$

where $a > 0$, $\alpha_i \in (0, 1]$; $i = 1, 2$, $(E, \|\cdot\|)$ is a (real or complex) Banach space, $u_a, v_a \in E$ and $f_i : \mathbb{R}_+ \times E \times E \to E$; $i = 1, 2$ are given continuous functions.

3.1.2 Existence Results in Banach spaces

In this section, we are concerned with the existence and uniqueness of solutions of the problem 3.1-3.2

Lemma 3.1.1. *Let $x, y \in C$, and $\delta_1\delta_2 \neq 1$ Then the unique solution (u, v) of problem*

$$\begin{cases} \mathcal{T}_a^{\alpha_1}u(t) = x(t); \ t \in I := [0, T], \ \alpha_1 \in (0, 1], \\ \mathcal{T}_a^{\alpha_2}v(t) = y(t); \ t \in I := [0, T], \ \alpha_2 \in (0, 1], \\ u(0) = \delta_1 v(T), \\ v(0) = \delta_2 u(T), \end{cases} \tag{3.5}$$

is given by

$$u(t) = \frac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\int_0^T s^{\alpha_1-1}x(s)ds + \int_0^T s^{\alpha_2-1}y(s)\mathrm{d}s\right] + \int_0^t s^{\alpha_1-1}x(s)\mathrm{d}s,$$

$$v(t) = \frac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\int_0^T s^{\alpha_2-1}y(s)ds + \int_0^T s^{\alpha_1-1}x(s)\mathrm{d}s\right] + \int_0^t s^{\alpha_2-1}y(s)\mathrm{d}s.$$

Proof. By Lemma 1.3.2, solving the linear fractional differential equation

$$\mathcal{T}_0^{\alpha_1}u(t) = x(t),$$

we find that

$$\mathcal{J}_0^{\alpha_1}\mathcal{T}_0^{\alpha_1}u(t) = \mathcal{J}_0^{\alpha_1}x(t).$$

Hence,

$$u(t) = u(0) + \int_0^t s^{\alpha_1-1}x(s)\mathrm{d}s, \tag{3.6}$$

$$v(t) = v(0) + \int_0^t s^{\alpha_2-1}y(s)\mathrm{d}s. \tag{3.7}$$

By using the boundary conditions $u(0) = \delta_1 v(T)$, and $v(0) = \delta_2 u(T)$, we obtain

$$u(0) = \delta_1\left[v(0) + \int_0^T s^{\alpha_2-1}y(s)\mathrm{d}s\right], \tag{3.8}$$

and

$$v(0) = \delta_2\left[u(0) + \int_0^T s^{\alpha_1-1}x(s)\mathrm{d}s\right]. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$u(0) = \frac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\int_0^T s^{\alpha_1-1}x(s)\mathrm{d}s + \int_0^T s^{\alpha_2-1}y(s)\mathrm{d}s\right],$$

and

$$v(0) = \frac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\int_0^T s^{\alpha_2-1}y(s)\mathrm{d}s + \int_0^T s^{\alpha_1-1}x(s)\mathrm{d}s\right].$$

Thus,

$$\begin{cases} u(t) = \dfrac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\displaystyle\int_0^T s^{\alpha_1-1}x(s)\mathrm{d}s + \int_0^T s^{\alpha_2-1}y(s)\mathrm{d}s\right] + \int_0^t s^{\alpha_1-1}x(s)\mathrm{d}s, \\ \\ v(t) = \dfrac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\displaystyle\int_0^T s^{\alpha_2-1}y(s)\mathrm{d}s + \int_0^T s^{\alpha_1-1}x(s)\mathrm{d}s\right] + \int_0^t s^{\alpha_2-1}y(s)\mathrm{d}s. \end{cases}$$

The following lemma is a direct conclusion of Lemma 3.1.1.

Lemma 3.1.2. *Let $f_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, be such that $f_i(\cdot, u, v) \in \mathrm{C}(I)$ for each $u, v \in \mathrm{C}(I)$. Then the coupled system (3.1)-(3.2) is equivalent to the coupled system of integral equations*

$$\begin{cases} u(t) = \dfrac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2 \displaystyle\int_0^T s^{\alpha_1-1} f_1(s,u(s),v(s))ds + \int_0^T s^{\alpha_2-1} f_2(s,u(s),v(s))\mathrm{d}s\right] \\ \qquad + \displaystyle\int_0^t s^{\alpha_1-1} f_1(s,u(s),v(s))\mathrm{d}s, \\ v(t) = \dfrac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1 \displaystyle\int_0^T s^{\alpha_2-1} f_2(s,u(s),v(s))\mathrm{d}s + \int_0^T s^{\alpha_1-1} f_1(s,u(s),v(s))\mathrm{d}s\right] \\ \qquad + \displaystyle\int_0^t s^{\alpha_2-1} f_2(s,u(s),v(s))\mathrm{d}s. \end{cases}$$

Now, we shall prove the main results concerning the existence of solutions of our first problem by applying Schaefer's fixed point theorem.
Let us introduce the following hypothesis :
(H) there exist real constants $L_i, K_i, M_i > 0$; $i = 1, 2$, such that

$$|f_i(t,u_1,u_2)| \leq L_i + K_i|u_1| + M_i|u_2|; \quad \text{for} \quad t \in I \quad \text{and} \quad u_i \in \mathbb{R}.$$

Set

$$W_1 = \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|} + 1\right]\frac{\mathcal{T}^{\alpha_1}}{\alpha_1},\ W_2 = \left[\frac{|\delta_1|}{|1-\delta_1\delta_2|}\right]\frac{\mathcal{T}^{\alpha_2}}{\alpha_2},$$
$$W_3 = \left[\frac{|\delta_2|}{|1-\delta_1\delta_2|}\right]\frac{\mathcal{T}^{\alpha_1}}{\alpha_1},\ W_4 = \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|} + 1\right]\frac{\mathcal{T}^{\alpha_2}}{\alpha_2}.$$

Theorem 3.1.1. *Assume that the hypothesis (H) is satisfies. If*

$$(W_1 + W_3)(K_1 + M_1) + (W_2 + W_4)(K_2 + M_2) < 1, \tag{3.10}$$

then the problem (3.1)-(3.2) has at least one solution.

Proof. Define the operator $N : \mathcal{C} \to \mathcal{C}$ by

$$(N(u,v))(t) = ((N_1 u)(t), (N_2 v)(t)), \tag{3.11}$$

where $N_1, N_2 : C \to C$ are given by

$$
\begin{aligned}
(N_1u)(t) &= \tfrac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\int_0^T s^{\alpha_1-1}f_1(s,u(s),v(s))ds + \int_0^T s^{\alpha_2-1}f_2(s,u(s),v(s))ds\right]\\
&+ \int_0^t s^{\alpha_1-1}f_1(s,u(s),v(s))ds,
\end{aligned}
$$

and

$$
\begin{aligned}
(N_2v)(t) &= \tfrac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\int_0^T s^{\alpha_2-1}f_2(s,u(s),v(s))ds + \int_0^T s^{\alpha_1-1}f_1(s,u(s),v(s))ds\right]\\
&+ \int_0^t s^{\alpha_2-1}f_2(s,u(s),v(s))ds.
\end{aligned}
$$

Set

$$R \geq \frac{(W_1+W_3)L_1+(W_2+W_4)L_2}{1-(W_1+W_3)(K_1+M_1)-(W_2+W_4)(K_2+M_2)},$$

and consider the closed and convex ball

$$B_R = \{(u,v)\in\mathcal{C}:\ \|(u,v)\|_{\mathcal{C}} \leq R\}.$$

Let $(u,v)\in B_R$. Then, for each $t\in I$ and any $i=1,2$, we have

$$
\begin{aligned}
|(N_1u)(t)| &\leq \left|\frac{\delta_1\delta_2}{1-\delta_1\delta_2}\right|\int_0^T s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s\\
&+\left|\frac{\delta_1}{1-\delta_1\delta_2}\right|\int_0^T s^{\alpha_2-1}|f_2(s,u(s),u(s))|\mathrm{d}s\\
&+\int_0^T s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s\\
&\leq\left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1\right]\int_0^T s^{\alpha_1-1}(L_1+K_1|u(s)|+M_1|v(s)|)\mathrm{d}s\\
&+\frac{|\delta_1|}{|1-\delta_1\delta_2|}\int_0^T s^{\alpha_2-1}(L_2+K_2|u(s)|+M_2|v(s)|)\mathrm{d}s\\
&\leq\left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1\right]\frac{\mathcal{T}^{\alpha_1}}{\alpha_1}(L_1+(K_1+M_1)R)\\
&+\left[\frac{|\delta_1|}{|1-\delta_1\delta_2|}\right]\frac{\mathcal{T}^{\alpha_2}}{\alpha_2}(L_2+(K_2+M_2)R)\\
&\leq W_1(L_1+(K_1+M_1)R)+W_2(L_2+(K_2+M_2)R).
\end{aligned}
$$

Also,

$$\begin{aligned}
|(N_2 v)(t)| &= \left| \frac{\delta_2\delta_1}{1-\delta_2\delta_1}\int_0^T s^{\alpha_2-1} f_2(s,u(s),v(s))\mathrm{d}s \right.\\
&\quad + \frac{\delta_2}{1-\delta_2\delta_1}\int_0^T s^{\alpha_1-1} f_1(s,u(s),u(s))\mathrm{d}s\\
&\quad \left. + \int_0^t s^{\alpha_2-1} f_2(s,u(s),v(s))\mathrm{d}s \right|\\
&\le \left| \frac{\delta_2\delta_1}{1-\delta_2\delta_1}\int_0^T s^{\alpha_2-1} f_2(s,u(s),v(s))\mathrm{d}s \right|\\
&\quad + \left| \frac{\delta_2}{1-\delta_2\delta_1}\int_0^T s^{\alpha_2-1} f_1(s,u(s),u(s))\mathrm{d}s \right|\\
&\quad + \left| \int_0^T s^{\alpha_2-1} f_2(s,u(s),v(s))\mathrm{d}s \right|\\
&\le W_3(L_1+(K_1+M_1)R) + W_4(L_2+(K_2+M_2)R).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
|N(u,v)(t)| \le{}& ((W_1+W_3)(K_1+M_1)+(W_2+W_4)(K_2+M_2))R\\
&+ (W_1+W_3)L_1 + (W_2+W_4)L_2.
\end{aligned}$$

Thus,

$$\|N(u,v)\|_{\mathcal{C}} \le R.$$

Hence, N maps the ball B_R into itself. We shall show that the operator $N : B_R \to B_R$ satisfies the assumptions of Schaefer's fixed point theorem. The proof will be given in several steps.

Step 1. We show that N is continuous. Let $\{(u_n, v_n)\}$ be a sequence such that $(u_n, v_n) \to (u, v)$ in B_R. Then, for each $t \in I$, we have

$$\begin{aligned}
&|N_1(u_n,v_n)(t) - N_1(u,v)(t)|\\
&\quad \le [\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1]\int_0^T s^{\alpha_1-1}|[f_1(s,u_n(s),v_n(s)) - f_1(s,u(s),v(s))]|\mathrm{d}s\\
&\qquad + \frac{|\delta_1|}{|1-\delta_1\delta_2|}\int_0^T s^{\alpha_2-1}|[f_2(s,u_n(s),v_n(s)) - f_2(s,u(s),v(s))]|\mathrm{d}s.
\end{aligned}$$

Analogously, we get

$$\begin{aligned}
&|N_2(u_n,v_n)(t)-N_2(u,v)(t)| \\
&\quad\le [\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1]\int_0^T s^{\alpha_1-1}|[f_2(s,u_n(s),v_n(s))-f_2(s,u(s),v(s))]|\mathrm{d}s \\
&\quad\quad+\frac{|\delta_2|}{|1-\delta_1\delta_2|}\int_0^T s^{\alpha_2-1}|[f_1(s,u_n(s),v_n(s))-f_1(s,u(s),v(s))]|\mathrm{d}s.
\end{aligned}$$

Since $(u_n,v_n)\to(u,v)$ as $n\to\infty$ and f_i, $i=1,2$, are continuous, by the Lebesgue dominated convergence theorem

$$\|N(u_n,v_n)-N(u,v)\|_{\mathcal{C}}\to 0 \quad \text{as} \quad n\to\infty.$$

Step 2. We show that N maps bounded sets into bounded and equicontinuous sets in B_R. $N(B_R)$ is bounded. This is clear since $N:B_R\to B_R$ and B_R is bounded.

Now, let $t_1,t_2\in[0,T]$ be such that $t_1<t_2$. and let $(u_1;u_2)\in B_R$. Then, we have

$$\begin{aligned}
|(N_1u)(t_2)-(N_1u)(t_1)| &\le \int_0^{t_2} s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s \\
&\quad-\int_0^{t_1} s^{\alpha_1-1}|f_1(s,u(s),v(s)|\mathrm{d}s \\
&\le \int_{t_1}^{t_2} s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s \\
&\le \frac{L_1+K_1R+M_1R}{\alpha_1}(t_2^{\alpha_1}-t_1^{\alpha_1}).
\end{aligned}$$

Thus,

$$|(N_1u)(t_2)-(N_1u)(t_1)|\le\frac{L_1+K_1R+M_1R}{\alpha_1}(t_2^{\alpha_1}-t_1^{\alpha_1}). \tag{3.12}$$

In a similar manner, we can easily get

$$|(N_2v)(t_2)-(N_2v)(t_1)|\le\frac{L_1+K_2R+M_2R}{\alpha_2}(t_2^{\alpha_2}-t_1^{\alpha_2}). \tag{3.13}$$

The right-hand sides of the inequalities (3.12) and (3.13) tend to zero as $t_2\to t_1$. Therefore, the operator $N(u,v)$ is equicontinuous. By collecting the above steps along with the Arzela-Ascoli theorem, we deduce that $N:B_R\to B_R$ is completely continuous mapping.

Step 3. The set $\Omega = \{(u,v) \in \mathcal{C} : (u,v) = \lambda N(u,v);\ 0 \le \lambda \le 1\}$ is bounded.
Let $(u,v) \in \Omega$ such that $(u,v) = \lambda N(u,v)$. Then for any $t \in I$, we have

$$u(t) = \lambda(N_1 u)(t),\ \textit{and}\ v(t) = \lambda(N_2 v)(t).$$

Hence,

$$\begin{aligned} u(t) = {} & \frac{\lambda\delta_1}{1-\delta_1\delta_2}\left[\delta_2 \int_0^T s^{\alpha_1-1} f_1(s,u,v)ds + \int_0^T s^{\alpha_2-1} f_2(s,u(s),v(s))\mathrm{d}s\right] \\ & + \lambda \int_0^t s^{\alpha_1-1} f_1(s,u(s),v(s))\mathrm{d}s. \end{aligned}$$

From the assumption (H), we obtain

$$|u(t)| \le W_1(L_1 + (K_1+M_1)(|u(t)|+|v(t)|)) + W_2(L_2 + (K_2+M_2)(|u(t)|+|v(t)|)).$$

By the same approach, we have

$$|v(t)| \le W_3(L_1 + (K_1+M_1)(|u(t)|+|v(t)|)) + W_4(L_2 + (K_2+M_2)(|u(t)|+|v(t)|)).$$

Thus, we obtain

$$\begin{aligned} |u(t)|+|v(t)| \quad \le & \quad ((W_1+W_3)(K_1+M_1) + (W_2+W_4)(K_2+M_2))(|u(t)|+|v(t)|) \\ + & \quad (W_1+W_3)L_1 + (W_2+W_4)L_2. \end{aligned}$$

This gives

$$|u(t)|+|v(t)| \le \frac{(W_1+W_3)L_1 + (W_2+W_4)L_2}{1 - ((W_1+W_3)(K_1+M_1) + (W_2+W_4)(K_2+M_2))} := \nu.$$

Hence,

$$\|(u,v)\|_{\mathcal{C}} \le \nu.$$

Therefore, the set Ω is bounded.
As a consequence of Theorem 1.5.8, we conclude that N has at least one fixed point.
This confirms that there exists at least one solution of the coupled system (3.1)-(3.2).

3.1.3 Existence results in Fréchet spaces

Now, we shall prove the main results concerning the existence of solutions of our problems.

Let us introduce the following hypotheses :

(H_1) The functions $f_i;\ i=1,2$ are measurable on $\mathbb{R}_+$; for each $t \in I$ *and* $u_i, v_i \in E$, and the the functions $(u,v) \to f_i(t,u,v)$ are continuous on E for a.e.$t \in \mathbb{R}_+;\ i = 1,2$.

(H_2) There exist continuous functions $h_i, p_i, q_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $0 < k_i < 1;\ i = 1,2$, such that

$$\|f_i(t,u_1,u_2)\| \leq h_i(t) + p_i(t)\|u_1\| + q_i(t)\|u_2\|; \quad \text{for} \quad t \in \mathbb{R}_+, \quad \text{and} \quad u_i, v_i \in E.$$

(H_3) For each bounded sets $B_i \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f_i(t,B_1,B_2)) < p_i(t)\mu(B_1) + q_i(t)\mu(B_2),$$

where μ is a measure of noncompactness on the Banach space E.

For $n \in \mathbb{N}$, set

$$p_i^* = \sup_{t\in[0,n]} p_i(t),\ q_i^* = \sup_{t\in[0,n]} q_i(t),\ h_i^* = \sup_{t\in[0,n]} h_i(t).$$

Theorem 3.1.2. *Assume that* (H_1)*-*(H_3) *are satisfied. If*

$$(p_1^*+q_1^*)\frac{(n-a)^{\alpha_1}}{\alpha_1} + (p_2^*+q_2^*)\frac{(n-a)^{\alpha_2}}{\alpha_2} < 1,$$

for each $n \in \mathbb{N}^*$, *then the problem (3.3)-(3.4) has at least one solution.*

Proof. Define the operator $N : \mathcal{C} \to \mathcal{C}$ by

$$(N(u,v))(t) = ((N_1u)(t),(N_2v)(t)), \tag{3.14}$$

where $N_1, N_2 : C \to C$ with

$$(N_1u))(t) = u_a + \int_1^t (s-a)^{\alpha_1-1} f_1(s,u(s),v(s))ds, \tag{3.15}$$

and

$$(N_2v))(t) = v_a + \int_1^t (s-a)^{\alpha_2 - 1} f_2(s, u(s), v(s))ds. \tag{3.16}$$

Clearly, the fixed points of the operator N are solutions of the coupled system (3.3)-(3.4).

For any $n \in \mathbb{N}^*$, we set

$$R_n \geq \frac{\|u_a\| + \|v_a\| + h_1^* \frac{(n-a)^{\alpha_1}}{\alpha_1} + h_2^* \frac{(n-a)^{\alpha_2}}{\alpha_2}}{1 - ((p_1^* + q_1^*) \frac{(n-a)^{\alpha_1}}{\alpha_1} + (p_2^* + q_2^*) \frac{(n-a)^{\alpha_2}}{\alpha_2})}.$$

Consider the ball

$$B_{R_n} := B(0, R_n) = \{(u, v) \in X : \|u\|_n \leq R_n, \|v\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}^*$, and each $u, v \in B_{R_n}$ and $t \in [0, n]$ we have

$$\begin{aligned}
\|(N_1u)(t)\| &\leq \|u_a\| + \int_1^t (s-a)^{\alpha_1 - 1} \|f_1(s, u(s), v(s))\| \mathrm{d}s \\
&\leq \|u_a\| + \int_1^t (s-a)^{\alpha_1 - 1} (h_1(s) + p_1(s) \|u_1\| + q_1(s) \|u_2\|) \mathrm{d}s \\
&\leq \|u_a\| + (h_1^* + (p_1^* + q_1^*)R_n) \int_1^t (s-a)^{\alpha_1 - 1} \mathrm{d}s \\
&\leq \|u_a\| + (h_1^* + (p_1^* + q_1^*)R_n) \frac{(n-a)^{\alpha_1}}{\alpha_1},
\end{aligned}$$

and

$$\begin{aligned}
\|(N_2v)(t)\| &\leq \|v_a\| + \int_1^t (s-a)^{\alpha_2 - 1} \|f_2(s, u(s), v(s))\| \mathrm{d}s \\
&\leq \|v_a\| + \int_1^t (s-a)^{\alpha_2 - 1} (h_2(s) + p_2(s) \|u_1\| + q_2(s) \|u_2\|) \mathrm{d}s \\
&\leq \|v_a\| + (h_2^* + (p_2^* + q_2^*)R_n) \int_1^t (s-a)^{\alpha_2 - 1} \mathrm{d}s \\
&\leq \|v_a\| + (h_2^* + (p_2^* + q_2^*)R_n) \frac{(n-a)^{\alpha_2}}{\alpha_2}.
\end{aligned}$$

Then,

$$\begin{aligned}\|(N(u,v))(t)\| &\leq \|u_a\|+\|v_a\|+h_1^*\frac{(n-a)^{\alpha_1}}{\alpha_1}+h_2^*\frac{(n-a)^{\alpha_2}}{\alpha_2}+((p_1^*+q_1^*)\frac{(n-a)^{\alpha_1}}{\alpha_1}\\ &\quad+(p_2^*+q_2^*)\frac{(n-a)^{\alpha_2}}{\alpha_2})R_n\\ &\leq R_n.\end{aligned}$$

Thus,

$$\|(N(u,v)\|_n \leq R_n. \tag{3.17}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 1.5.6. The proof will be given in two steps.

Step 1 : $N(B_{R_n})$ is bounded and $N : N(B_{R_n}) \to N(B_{R_n})$ is continuous.
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, $N(B_{R_n})$ is bounded. Let $\{(u_k, v_k)\}_{k\in N}$ be a sequence such that $(u_k, v_k) \to (u, v)$ in B_{R_n}. Then, for each $t \in [0, n]$, we have

$$\begin{aligned}&\|(N(u_n,v_n))(t)-(N(u,v))(t)\|\\ &\quad\leq \sum_{i=1}^{2}\int_a^t \left\|(s-a)^{\alpha_i-1}[f_i(s,u_n(s),v_n(s)-f_i(s,(u(s),v(s))]\right\| \mathrm{d}s\\ &\quad\leq \sum_{i=1}^{2}\int_a^t (s-a)^{\alpha_i-1}\,\|[f_i(s,u_n(s),v_n(s)-f_i(s,(u(s),v(s))]\|\,\mathrm{d}s.\end{aligned}$$

Since $(u_k, v_k) \to (u, v)$ as $k \to \infty$ and f_i, $i = 1, 2$, are continuous, by the Lebesgue dominated convergence theorem

$$\|N(u_n,v_n)-N(u,v)\|_n \to 0 \quad \text{as} \quad k\to\infty.$$

Step 2 : For each bounded equicontinuous subset D of B_{R_n}, $\mu_n(N(D)) < \ell_n\mu_n(D)$.
From Lemmas 1.4.3 and 1.4.4, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence

$\{u_k, v_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [a, n]$, we have

$$\begin{aligned}
\mu((ND)(t)) &= \sum_{i=1}^{2} \mu(\{u_{ia} + \int_a^t (s-a)^{\alpha_i - 1} f_i(s, (u(s), v(s))ds; (u,v) \in D\}) \\
&\leq \sum_{i=1}^{2} \mu(\{\int_a^t (s-a)^{\alpha_i - 1} f_i(s, (u_k(s), v_k(s))ds\}_{k=1}^{\infty}) + \epsilon \\
&\leq \sum_{i=1}^{2} \int_a^t (s-a)^{\alpha_i - 1} \mu(\{f_i(s, (u_k(s), v_k(s))\}_{k=1}^{\infty}) ds + \epsilon \\
&\leq \sum_{i=1}^{2} \int_a^t (s-a)^{\alpha_i - 1} p_i(s) \mu(\{(u_k(s))\}_{k=1}^{\infty}) + q_i(s) \mu(\{v_k(s))\}_{k=1}^{\infty}) ds + \epsilon \\
&\leq ((p_1^* + q_1^*) \frac{(n-a)^{\alpha_1}}{\alpha_1} + (p_2^* + q_2^*) \frac{(n-a)^{\alpha_2}}{\alpha_2}) \mu_n(D).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((ND)(t)) \leq ((p_1^* + q_1^*) \frac{(n-a)^{\alpha_1}}{\alpha_1} + (p_2^* + q_2^*) \frac{(n-a)^{\alpha_2}}{\alpha_2}) \mu_n(D).$$

Thus,

$$\mu_n(ND) \leq ((p_1^* + q_1^*) \frac{(n-a)^{\alpha_1}}{\alpha_1} + (p_2^* + q_2^*) \frac{(n-a)^{\alpha_2}}{\alpha_2}) \mu_n(D).$$

As a consequence of steps 1 and 2 together with Theorem 1.5.6, we can conclude that N has at least one fixed point in B_{R_n} which is a solution of problem (3.3)-(3.4).

3.1.4 Examples

Example 3.1.1. *Consider the coupled system of Conformable fractional differential equations*

$$\begin{cases} (\mathcal{T}_{0^+}^{\frac{1}{2}} u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{0^+}^{\frac{1}{2}} v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \quad t \in [0, 1], \tag{3.18}$$

with the following coupled boundary conditions :

$$(u(0), v(0)) = (1, 2), \tag{3.19}$$

where

$$f_1(t,u,v) = \frac{\sin(u+v)}{40(e^t+1)},$$
$$f_2(t,u,v) = \frac{\tan u}{10+|u|+|v|}, \quad t\in[0,1]; \quad u,v\in\mathbb{R}.$$

The hypothesis (H) and the condition (3.10) are satisfied with

$$M_1 = K_1 = \frac{1}{80}, \quad K_2 = \frac{1}{10}, \quad \delta_1=\delta_2=\frac{1}{2},$$
$$W_1 = W_4 = \frac{8}{3}, W_2 = W_3 = \frac{4}{3}.$$

Hence, Theorem 3.1.1 implies that the system (3.18)–(3.19) *has at least one solution defined on* $[0,1]$.

Example 3.1.2. *Let*

$$l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{k=1}^{\infty} |u_k| < \infty \right\}$$

be the Banach space with the norm

$$\|u\| = \sum_{k=1}^{\infty} |u_k|,$$

and $C(\mathbb{R}_+, l^1)$ *be the Fréchet space of all continuous functions* v *from* $\mathbb{R}_+$ *into* l^1, *equipped with the family of seminorms*

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|; \ n\in\mathbb{N}.$$

Consider the coupled system of Conformable fractional differential equations

$$\begin{cases} (\mathcal{T}_{0^+}^{\frac{1}{5}} u_k)(t) = f_k(t,u(t),v(t)) \\ (\mathcal{T}_{0^+}^{\frac{1}{5}} v_k)(t) = g_k(t,u(t),v(t)) \end{cases} ; \quad t\in[1,\infty), \ k=1,2,\ldots, \tag{3.20}$$

with the following initial coupled conditions :

$$(u_k(1), v_k(1)) = (0,0), \tag{3.21}$$

where

$$f_k(t,u,v) = \frac{c}{1+\|u\|+\|v|}(e^{-7}+\frac{1}{e^{t+5}})(2^{-k}+u_k(t)), \quad t\in[1,\infty),$$
$$g_k(t,u,v) = \frac{c}{e^{t+5}(1+\|u\|+\|v\|)}(2^{-k}+v_k(t)), \quad t\in[1,\infty),\ k=1,2,\cdots,\ c>0,$$

for each $t \in [1,n];\ n\in\mathbb{N}$, *with*

$$f=(f_1,f_2,\ldots,f_k,\ldots),\ g=(g_1,g_2,\ldots,g_k,\ldots),\ \textit{and}\ u=(u_1,u_2,\ldots,u_k,\ldots).$$

We can show that all hypotheses of Theorem 3.1.2 are satisfied with

$$h_1(t)=p_1(t)=c(e^{-7}+\frac{1}{e^{t+5}}), \quad q_1(t)=p_2(t)=0, \quad h_2(t)=q_2(t)=\frac{c}{e^{t+5}}.$$

So,

$$h_1^*=p_1^*=c(e^{-7}+e^{-6}), \quad h_2^*=q_2^*=ce^{-6}.$$

Therefore, Theorem 3.1.2 implies that the system (3.20)–(3.21) *has at least one solution defined on* $[1,\infty)$.

3.2 Stability and Attractivity Results For Coupled Fractional Conformable System

3.2.1 Introduction and motivations

In this section, we investigate the existence and stability of solutions for the following coupled Conformable fractional differential system :

$$\begin{cases} (\mathcal{T}_{0^+}^{\alpha_1}u)(t)=f_1(t,u(t),v(t)) \\ (\mathcal{T}_{0^+}^{\alpha_2}v)(t)=f_2(t,u(t),v(t)) \end{cases} ;\ t\in I, \tag{3.22}$$

with the following coupled boundary conditions :

$$(u(0),v(0))=(\delta_1 v(T),\delta_2 u(T)), \tag{3.23}$$

where $T>0,\ I:=[0,T],\ \alpha_i\in(0,1];\ i=1,2,\ f_i: I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R};\ i=1,2$ are given continuous functions, $\mathcal{T}_a^{\alpha_i,\rho}$ is the conformable fractional derivative of order

α_i ; $i = 1, 2$, and δ_1, δ_2 are real numbers with $\delta_1\delta_2 \neq 1$.

Next, we investigate the attractivity of solutions for the following coupled conformable fractional differential system :

$$\begin{cases} (\mathcal{T}_{a^+}^{\alpha_1} u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{a^+}^{\alpha_2} v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \ t \in [a, \infty), \tag{3.24}$$

with the following coupled initial conditions :

$$(u(a), v(a)) = (u_a, v_a), \tag{3.25}$$

where $a > 0$, $\alpha_i \in (0, 1]$; $i = 1, 2$, $(\mathbb{R}, \|.\|)$ is a Banach space, $u_a, v_a \in \mathbb{R}$ and $f_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $i = 1, 2$ are given continuous functions.
The rest of this paper is organized in the following manner : In Section 3.2.2, we briefly review some of the relevant definitions from fractional calculus and prove an auxiliary lemma that will be used later. We discusses the Hyers-Ulam stability of solutions to the considered problem (3.22)-(3.23) and presents sufficient conditions for the stability, and Section 3.2.3 deals with proving the existence and attractivity of solutions for the given problem (3.24)-(3.25) using the Schauder's fixed point theorem.

3.2.2 Existence and Ulam stability of solutions

First, let us introduce some basic lemmas and definitions that are needed throughout all the manuscript.
Let $C := \mathrm{C}(I, \mathbb{R})$ be the Banach space equipped with the norm defined by

$$\|u\|_\infty := \sup_{t \in I} |u(t)|.$$

By $\mathcal{C} := C \times C$, we denote the complete metric space with the usual metric

$$D((u_1, v_1), (u_2, v_2)) := d(u_1, u_2) + d(v_1, v_2).$$

$\mathcal{C}$ is a Banach space with the norm

$$\|(u, v)\|_{\mathcal{C}} = \|u\|_\infty + \|v\|_\infty.$$

Let $BC := BC(\mathbb{R}_+)$ be the Banach space of all bounded and continuous functions from $\mathbb{R}_+$*into* $\mathbb{R}$, equipped with the norm

$$\|u\|_{BC} := \sup_{t\in\mathbb{R}_+} |u(t)|.$$

It is clair that the product $\mathcal{BC} := BC \times BC$ turns out to be a Banach space if equipped with the norm

$$\|(u,v)\|_{\mathcal{BC}} = \|u\|_{BC} + \|v\|_{BC}.$$

Definition 3.2.1. *By a solution of problem (3.22)-(3.23), we mean a coupled function $(u,v) \in \mathcal{C}$ that satisfies the system*

$$\begin{cases} (\mathcal{T}_{0^+}^{\alpha_1} u)(t) = f_1(t, u(t), v(t)), \\ (\mathcal{T}_{0^+}^{\alpha_2} v)(t) = f_2(t, u(t), v(t)), \end{cases}$$

on I and the following coupled boundary conditions :

$$(u(0), v(0)) = (\delta_1 v(T), \delta_2 u(T)).$$

Now, we consider the Ulam stability for system (3.22)-(3.23). Let $\varepsilon > 0$ and $\Phi : I \to \mathbb{R}_+$ be a continuous function. We consider the following inequalities :

$$\begin{cases} |(\mathcal{T}_{a^+}^{\alpha_1} u)(t) - f_1(t, u(t), v(t))| \leq \frac{\varepsilon}{2}, \\ |(\mathcal{T}_{a^+}^{\alpha_2} v)(t) - f_2(t, u(t), v(t))| \leq \frac{\varepsilon}{2}, \end{cases} \quad t \in [0, T); \tag{3.26}$$

$$\begin{cases} |(\mathcal{T}_{a^+}^{\alpha_1} u)(t) - f_1(t, u(t), v(t))| \leq \frac{1}{2}\Phi(t) \\ |(\mathcal{T}_{a^+}^{\alpha_2} v)(t) - f_2(t, u(t), v(t))| \leq \frac{1}{2}\Phi(t), \end{cases} \quad t \in [0, T); \tag{3.27}$$

$$\begin{cases} |(\mathcal{T}_{a^+}^{\alpha_1} u)(t) - f_1(t, u(t), v(t))| \leq \frac{\varepsilon}{2}\Phi(t) \\ |(\mathcal{T}_{a^+}^{\alpha_2} v)(t) - f_2(t, u(t), v(t))| \leq \frac{\varepsilon}{2}\Phi(t), \end{cases} \quad t \in [0, T). \tag{3.28}$$

Set

$$|(u(t), v(t))| := |u(t)| + |v(t)|.$$

Definition 3.2.2. *[8, 135] System (3.22)-(3.23) is Ulam-Hyers stable if there exists a real number $c_{f_1,f_2} > 0$ such that, for each $\varepsilon > 0$ and for each solution $(u^*, v^*) \in \mathcal{C}$ of inequalities (3.26) there exists a solution $(u, v) \in C(I)$ of (3.22)-(3.23) with*

$$|(u^*(t) - u(t), v^*(t) - v(t))| \leq \varepsilon c_{f_1,f_2}, t \in I.$$

Definition 3.2.3. *[8, 135] System (3.22)-(3.23) is generalized Ulam-Hyers stable if there exists $c_{f_1,f_2} : C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_{f_i}(0) = 0, i = 1, 2,$ such that, for each $\varepsilon > 0$ and for each solution $(u^*, v^*) \in \mathcal{C}$ of inequalities (3.26)) there exists a solution $(u, v) \in \mathcal{C}$ of (3.22)-(3.23) with*

$$|(u^*(t) - u(t), v^*(t) - v(t))| \leq c_{f_1,f_2}(\varepsilon), t \in I.$$

Definition 3.2.4. *[8, 135] System (3.22)-(3.23) is Ulam.Hyers.Rassias stable with respect to Φ if there exists a real number $c_{f_1,f_2,\Phi} > 0$ such that, for each $\varepsilon > 0$ and for each solution $(u^*, v^*) \in \mathcal{C}$ of inequalities (3.28)) there exists a solution $(u, v) \in \mathcal{C}$ of (3.22)-(3.23) with*

$$|(u^*(t) - u(t), v^*(t) - v(t))| \leq \varepsilon c_{f_1,f_2,\Phi}\Phi(t), t \in I.$$

Definition 3.2.5. *[8, 135] System (3.22)-(3.23) is generalized Ulam.Hyers.Rassias stable with respect to Φ if there exists a real number $c_{f_1,f_2,\Phi} > 0$ such that, for each solution $(u^*, v^*) \in \mathcal{C}$ of inequalities (3.28)) there exists a solution $(u, v) \in \mathcal{C}$ of (3.22)-(3.23) with*

$$|(u^*(t) - u(t), v^*(t) - v(t))| \leq c_{f_1,f_2,\Phi}\Phi(t), t \in I.$$

Remark 3.2.1. *It is clear that*

1. *Definition (3.2.2) $\Rightarrow$ Definition (3.2.3),*
2. *Definition (3.2.4)$\Rightarrow$ Definition (3.2.5),*
3. *Definition (3.2.4) for $\Phi(.) = 1 \Rightarrow$ Definition (3.2.2).*

Let us introduce the following hypotheses :

(H_4) There exist real constants $M_i, K_i > 0$; $i = 1, 2$, such that

$$|f_i(t, u_1, v_1) - f_i(t, u_2, v_2)| \leq K_i|u_1 - u_2| + M_i|v_1 - v_2|,$$

for each $t \in I$ and each $v_i, u_i \in \mathbb{R}$.

(H_5) There exists $\lambda_\Phi > 0$ such that, for each $t \in I$, we have

$$(I_0^{\alpha_i}\Phi)(t) \leq \lambda_\Phi \Phi(t), i = 1, 2.$$

Set

$$W_1 = \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|} + 1\right]\frac{\mathcal{T}^{\alpha_1}}{\alpha_1}, W_2 = \left[\frac{|\delta_1|}{|1-\delta_1\delta_2|}\right]\frac{\mathcal{T}^{\alpha_2}}{\alpha_2}$$
$$W_3 = \left[\frac{|\delta_2|}{|1-\delta_1\delta_2|}\right]\frac{\mathcal{T}^{\alpha_1}}{\alpha_1}, W_4 = \left[\frac{|\delta_2\delta_1|}{|1-\delta_1\delta_2|} + 1\right]\frac{\mathcal{T}^{\alpha_2}}{\alpha_2},$$
$$f_i^* = \sup_{t\in I}|f_i(t, 0, 0)| < \infty \quad \text{for all} \quad i = 1, 2.$$

Theorem 3.2.1. *Assume that hypothesis (H_1) holds with*

$$(W_1 + W_3)(K_1 + M_1) + (W_2 + W_4)(K_2 + M_2) < 1, \tag{3.29}$$

then system (3.22)-(3.23) has at least one solution defined on I. Moreover, if hypotheses (H_1)-(H_2) hold, then system (3.22)-(3.23) is generalized Ulam–Hyers–Rassias stable.

Proof. Define the operator $N : \mathcal{C} \to \mathcal{C}$ by

$$(N(u, v))(t) = ((N_1u)(t), (N_2v)(t)), \tag{3.30}$$

where $N_1, N_2 : C \to C$ are given by

$$\begin{aligned}(N_1u)(t) &= \tfrac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\int_0^T s^{\alpha_1-1}f_1(s, u(s), v(s))ds + \int_0^T s^{\alpha_2-1}f_2(s, u(s), v(s))ds\right] \\ &+ \int_0^t s^{\alpha_1-1}f_1(s, u(s), v(s))ds,\end{aligned}$$

and

$$\begin{aligned}(N_2v)(t) &= \tfrac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\int_0^T s^{\alpha_2-1}f_2(s,u(s),v(s))ds+\int_0^T s^{\alpha_1-1}f_1(s,u(s),v(s))ds\right]\\ &+ \int_0^t s^{\alpha_2-1}f_2(s,u(s),v(s))ds.\end{aligned}$$

Set

$$R\geq\frac{(W_1+W_3)f_1^*+(W_2+W_4)f_2^*}{1-(W_1+W_3)(K_1+M_1)-(W_2+W_4)(K_2+M_2)},$$

and consider the closed and convex ball

$$B_R=\{(u,v)\in\mathcal{C}:\ \|(u,v)\|_{\mathcal{C}}\leq R\}.$$

Remark 3.2.2. *From $H1$ for each $u,v\in\mathbb{R}$ and $t\in I$, we have that*

$$\begin{aligned}&|f_i(t,u,v)|\\ \leqslant&|f_i(t,u,v)-f_i(t,0,0)|+|f_i(t,0,0)|\\ \leqslant&K_i|u|+M_i|v|+f_i^*\\ \leqslant&(K_i+M_i)R+f_i^*.\end{aligned}$$

Let $(u,v)\in B_R$. Then, for each $t\in I$ and any $i=1,2$, we have

$$\begin{aligned}|(N_1u)(t)|\leq&\left|\frac{\delta_1\delta_2}{1-\delta_1\delta_2}\right|\int_0^T s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s\\ &+\left|\frac{\delta_1}{1-\delta_1\delta_2}\right|\int_0^T s^{\alpha_2-1}|f_2(s,u(s),u(s))|\mathrm{d}s\\ &+\int_0^T s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s\\ \leq&\left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2}|+1\right]\int_0^T s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s\\ &+\left[\frac{|\delta_1|}{|1-\delta_1\delta_2|}\right]\int_0^T s^{\alpha_2-1}|f_2(s,u(s),v(s))|\mathrm{d}s\\ \leq&\left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1\right]\frac{\mathcal{T}^{\alpha_1}}{\alpha_1}((K_1+M_1)R+f_1^*)\\ &+\left[\frac{|\delta_1|}{|1-\delta_1\delta_2|}\right]\frac{\mathcal{T}^{\alpha_2}}{\alpha_2}((K_2+M_2)R+f_2^*)\\ \leq&W_1((K_1+M_1)R+f_1^*)+W_2((K_2+M_2)R+f_2^*).\end{aligned}$$

Also

$$\begin{aligned}
|(N_2v)(t)| =& \left|\frac{\delta_2\delta_1}{1-\delta_2\delta_1}\int_0^T s^{\alpha_2-1}f_2(s,u(s),v(s))\mathrm{d}s\right.\\
&+\frac{\delta_2}{1-\delta_2\delta_1}\int_0^T s^{\alpha_1-1}f_1(s,u(s),u(s))\mathrm{d}s\\
&\left.+\int_0^t s^{\alpha_2-1}f_2(s,u(s),v(s))\mathrm{d}s\right|\\
\leq&\left|\frac{\delta_2\delta_1}{1-\delta_2\delta_1}\int_0^T s^{\alpha_2-1}f_2(s,u(s),v(s))\mathrm{d}s\right|\\
&+\left|\frac{\delta_2}{1-\delta_2\delta_1}\int_0^T s^{\alpha_2-1}f_1(s,u(s),u(s))\mathrm{d}s\right|\\
&+\left|\int_0^T s^{\alpha_2-1}f_2(s,u(s),v(s))\mathrm{d}s\right|\\
\leq& W_3((K_1+M_1)R+f_1^*)+W_4((K_2+M_2)R+f_2^*).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
|N(u,v)(t)| \leq&((W_1+W_3)(K_1+M_1)+(W_2+W_4)(K_2+M_2))R\\
&+(W_1+W_3)f_1^*+(W_2+W_4)f_2^*.
\end{aligned}$$

Thus

$$\|N(u,v)\|_{\mathcal{C}} \leq R.$$

Hence N maps the ball B_R into itself. We shall show that the operator $N : B_R \to B_R$ satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1 N is continuous.

Let $\{(u_n, v_n)\}$ be a sequence such that $(u_n, v_n) \to (u, v)$ in B_R. Then, for each $t \in I$, we have

$$
\begin{aligned}
&|N_1(u_n,v_n)(t)-N_1(u,v)(t)|\\
&\le \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1\right]\int_0^T s^{\alpha_1-1}|[f_1(s,u_n(s),v_n(s))-f_1(s,u(s),v(s))]|\mathrm{d}s\\
&+\frac{|\delta_1|}{|1-\delta_1\delta_2|}\int_0^T s^{\alpha_2-1}|[f_2(s,u_n(s),v_n(s))-f_2(s,u(s),v(s))]|\mathrm{d}s.
\end{aligned}
$$

Analogously, we get

$$
\begin{aligned}
&|N_2(u_n,v_n)(t)-N_2(u,v)(t)|\\
&\le \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1\right]\int_0^T s^{\alpha_1-1}|[f_2(s,u_n(s),v_n(s))-f_2(s,u(s),v(s))]|\mathrm{d}s\\
&+\frac{|\delta_2|}{|1-\delta_1\delta_2|}\int_0^T s^{\alpha_2-1}|[f_1(s,u_n(s),v_n(s))-f_1(s,u(s),v(s))]|\mathrm{d}s.
\end{aligned}
$$

Since $(u_n,v_n)\to(u,v)$ as $n\to\infty$ and f_i, $i=1,2$, are continuous, by the Lebesgue dominated convergence theorem

$$
\|N(u_n,v_n)-N(u,v)\|_{\mathcal{C}}\to 0 \quad \text{as} \quad n\to\infty.
$$

Step 2 $N(B_R)$ is bounded. This is clear since $N(B_R)\subset B_R$ and B_R is bounded.

Step 3 We show that N maps bounded sets into equicontinuous sets in B_R.
Let $t_1,t_2\in[0,T]$ such that $t_1<t_2$ and let $(u,v)\in B_R$. Then, we have

$$
\begin{aligned}
&|(N_1u)(t_2)-(N_1u)(t_1)|\\
&\le \int_0^{t_2} s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s-\int_0^{t_1} s^{\alpha_1-1}|f_1(s,u(s),v(s)|\mathrm{d}s\\
&\le \int_{t_1}^{t_2} s^{\alpha_1-1}|f_1(s,u(s),v(s))|\mathrm{d}s\\
&\le \frac{K_1R+M_1R+f_1^*}{\alpha_1}(t_2^{\alpha_1}-t_1^{\alpha_1}).
\end{aligned}
$$

Thus, we get

$$
|(N_1u)(t_2)-(N_1u)(t_1)|\le \frac{K_1R+M_1R+f_1^*}{\alpha_1}(t_2^{\alpha_1}-t_1^{\alpha_1}). \tag{3.31}
$$

In a similar manner, we can easily get

$$|(N_2v)(t_2) - (N_2v)(t_1)| \leq \frac{K_2R + M_2R + f_2^*}{\alpha_2}(t_2^{\alpha_2} - t_1^{\alpha_2}). \tag{3.32}$$

The right-hand sides of the inequalities (3.31) and (3.32) tend to zero as $t2 \to t1$. There- fore, the operator $N(u,v)$ is equicontinuous. As a consequence of the above three steps with the Arzela-Ascoli theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 1.5.3, we deduce that N has at least a fixed point (u,v) which is a solution of our system (3.22)-(3.23).

Step 4 Generalized Ulam–Hyers–Rassias stability.
Let us assume that (u,v) is a solution of system (3.22)-(3.23), let (u^*,v^*) be a solution of inequality (3.27) if and only if there is $(g_1,g_2) \in C(I,\mathbb{R})$ (where g_1 depends on solution u^* and g_2 depends on solution v^*) such that

(i) $|g_1(t)| \leq \frac{1}{2}\Phi(t)$ and $|g_2(t)| \leq \frac{1}{2}\Phi(t)$ for all $t \in [0,T]$.

(ii) For all $t \in [0,T]$

$$\begin{cases} (\mathcal{T}_{a^+}^{\alpha_1}u^*)(t) - f_1(t,u^*(t),v^*(t)) = g_1(t), \\ (\mathcal{T}_{a^+}^{\alpha_2}v^*)(t) - f_2(t,u^*(t),v^*(t)) = g_2(t). \end{cases}$$

So

$$\begin{aligned} u^*(t) &= Z_{u^*} + \tfrac{1}{2}\left[\tfrac{\delta_1}{1-\delta_1\delta_2}\right]\left[\delta_2\int_0^T s^{\alpha_1-1}g_1(s)ds + \int_0^T s^{\alpha_2-1}g_2(s)ds\right] \\ &+ \int_0^t s^{\alpha_1-1}f_1(s,u^*(s),v^*(s))ds + \tfrac{1}{2}\int_0^t s^{\alpha_1-1}g_1(s)ds, \end{aligned}$$

and

$$\begin{aligned} v^*(t) &= Z_{v^*} + \tfrac{1}{2}\left[\tfrac{\delta_2}{1-\delta_1\delta_2}\right]\left[\delta_1\int_0^T s^{\alpha_2-1}g_2(s)ds + \int_0^T s^{\alpha_1-1}g_1(s)ds\right] \\ &+ \int_0^t s^{\alpha_2-1}f_2(s,u^*(s),v^*(s))ds + \tfrac{1}{2}\int_0^t s^{\alpha_2-1}g_2(s)ds, \end{aligned}$$

where

$$Z_{u^*} = \frac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\int_0^T s^{\alpha_1-1}f_1^*(s,u(s),v(s))ds + \int_0^T s^{\alpha_2-1}f_2^*(s,u(s),v(s))ds\right],$$

and

$$Z_{v^*} = \frac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\int_0^T s^{\alpha_2-1}f_2^*(s,u(s),v(s))ds + \int_0^T s^{\alpha_1-1}f_1^*(s,u(s),v(s))ds\right].$$

It follows that

$$\begin{aligned}
&|u^*(t) - Z_{u^*} - \int_0^t s^{\alpha_1 - 1} f_1(s, u^*(s), v^*(s))ds| \\
\leq &\frac{1}{2}\left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}\int_0^T s^{\alpha_1-1}\Phi(s)ds + \frac{|\delta_1|}{|1-\delta_1\delta_2|}\int_0^T s^{\alpha_2-1}\Phi(s)ds \right.\\
&\left. + \int_0^T s^{\alpha_1-1}\Phi(s)ds\right] \\
\leq &\frac{1}{2}\left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1\right](\mathcal{I}_0^{\alpha_1}\Phi)(t) + \frac{1}{2}\left[\frac{|\delta_1|}{|1-\delta_1\delta_2|}\right](\mathcal{I}_0^{\alpha_2}\Phi)(t) \\
\leq &\frac{1}{2}\left[\frac{|\delta_1\delta_2|+|\delta_1|}{|1-\delta_1\delta_2|}+1\right]\lambda_\Phi\Phi(t).
\end{aligned}$$

Similarly

$$|v^*(t) - Z_{v^*} - \int_0^t s^{\alpha_2-1} f_2(s, u^*(s), v^*(s))ds| \leq \frac{1}{2}\left[\frac{|\delta_2\delta_1|+|\delta_2|}{|1-\delta_1\delta_2|}+1\right]\lambda_\Phi\Phi(t).$$

From hypotheses $H1$ and $H2$, for each $t \in I$, we have

$$\begin{aligned}
|u^*(t) - u(t)| = &\left|u^*(t) - Z_{u^*} - \int_0^t s^{\alpha_1-1} f_1(s, u(s), v(s))ds\right| \\
\leq &\left|u^*(t) - Z_{u^*} - \int_0^t s^{\alpha_1-1} f_1(s, u^*(s), v^*(s))ds\right. \\
&\left. + \int_0^t s^{\alpha_1-1} f_1(s, u^*(s), v^*(s)) - f_1(s, u(s), v(s))\mathrm{d}s\right| \\
\leq &\left|u^*(t) - Z_{u^*} - \int_0^t s^{\alpha_1-1} f_1(s, u^*(s), v^*(s))ds\right| \\
&+ \int_0^T s^{\alpha_1-1}|f_1(s, u^*(s), v^*(s)) - f_1(s, u(s), v(s))|\mathrm{d}s \\
\leq &\frac{1}{2}\left[\frac{|\delta_1\delta_2|+|\delta_1|}{|1-\delta_1\delta_2|}+1\right]\lambda_\Phi\Phi(t) \\
&+ \int_0^T s^{\alpha_1-1}[K_1|u^*(t)-u(t)| + M_1|v^*(t)-v(t)|]\mathrm{d}s \\
\leq &\frac{1}{2}\left[\frac{|\delta_1\delta_2|+|\delta_1|}{|1-\delta_1\delta_2|}+1\right]\lambda_\Phi\Phi(t) \\
&+ \frac{\mathcal{T}^{\alpha_1}}{\alpha_1}(K_1+M_1)(|u^*(t)-u(t)| + |v^*(t)-v(t)|).
\end{aligned}$$

Also, we get

$$\begin{aligned}
|v(t)-v^*(t)| =& \left|v^*(t) - Z_{v^*} - \int_0^t s^{\alpha_2-1} f_2(s,u(s),v(s))ds\right| \\
\leq& \left|v^*(t) - Z_{v^*} - \int_0^t s^{\alpha_2-1} f_2(s,u^*(s),v^*(s))ds\right| \\
&+ \int_0^T s^{\alpha_2-1}|f_2(s,u^*(s),v^*(s)) - f_2(s,u(s),v(s))|\mathrm{d}s \\
\leq& \frac{1}{2}\left[\frac{|\delta_1\delta_2|+|\delta_2|}{|1-\delta_1\delta_2|}+1\right]\lambda_\Phi \Phi(t) \\
&+ \int_0^T s^{\alpha_2-1}[K_2|u^*(t)-u(t)| + M_2|v^*(t)-v(t)|]\mathrm{d}s \\
\leq& \frac{1}{2}\left[\frac{|\delta_1\delta_2|+|\delta_2|}{|1-\delta_1\delta_2|}+1\right]\lambda_\Phi \Phi(t) \\
&+ \frac{\mathcal{T}^{\alpha_2}}{\alpha_2}(K_2+M_2)(|u^*(t)-u(t)|+|v^*(t)-v(t)|).
\end{aligned}$$

Thus

$$\begin{aligned}
|(u^*(t),v^*(t))-(u(t),v(t))| =& |u^*(t)-u(t)|+|v^*(t)-v(t)| \\
\leq& \left[\frac{T^{\alpha_1}}{\alpha_1}(K_1+M_1)+\frac{T^{\alpha_2}}{\alpha_2}(K_2+M_2)\right] \\
&\times |(u^*(t),v^*(t))-(u(t),v(t))| \\
&+ \frac{1}{2}\left[\frac{|\delta_1\delta_2|+|\delta_1|}{|1-\delta_1\delta_2|}+\frac{|\delta_1\delta_2|+|\delta_2|}{|1-\delta_1\delta_2|}+2\right]\lambda_\Phi\Phi(t) \\
\leq& \left[\frac{\frac{1}{2}\left[\frac{|\delta_1\delta_2|+|\delta_1|}{|1-\delta_1\delta_2|}+\frac{|\delta_1\delta_2|+|\delta_2|}{|1-\delta_1\delta_2|}+2\right]}{1-\left(\frac{\mathcal{T}^{\alpha_1}}{\alpha_1}(K_1+M_1)+\frac{\mathcal{T}^{\alpha_2}}{\alpha_2}(K_2+M_2)\right)}\right]\lambda_\Phi\Phi(t) \\
\leq& c_{f_1,f_2,\Phi}\Phi(t).
\end{aligned}$$

Hence, problem (3.22)-(3.23) is generalized Ulam–Hyers–Rassias stable.

3.2.3 Attractivity results

Let $\emptyset \neq \Omega \subset BC$ and let $N : \Omega \to \Omega$, and consider the solution of the equation

$$(Nu)(t) = u(t). \tag{3.33}$$

We introduce the following concept of attractivity of solutions for equation (3.33).

Definition 3.2.6. *Solutions of equation (3.33) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for any solutions $v = v(t)$ and $w = w(t)$ of equations (3.33) belonging to $B(u_0, \eta) \cap \Omega$; we can write*

$$\lim_{t\to\infty}(u(t) - v(t)) = 0. \tag{3.34}$$

If limit (3.34) is uniform with respect to $B(u_0, \eta) \cap \Omega$; then the solutions of equation (3.33) are said to be uniformly locally attractive (or, equivalently, that the solutions of (3.33) are locally asymptotically stable.

Lemma 3.2.1. *[106]. Let $D \subset BC$. Then D is relatively compact in BC if the following conditions are satisfied :*
(a) D is uniformly bounded in BC;
(b) the functions belonging to D are almost equicontinuous in $\mathbb{R}_+$; i.e., equicontinuous on every compact set in $\mathbb{R}_+$;
(c) the functions from D are equiconvergent, i.e., given $\varepsilon > 0$; there exists $T(\varepsilon) > 0$ such that

$$|u(t) - \lim_{t\to\infty} u(t)| < \varepsilon$$

for any $t \geq T(\varepsilon)$ and $u \in D$.

Let us introduce the following hypotheses.
(H_6) The functions $f_i : [a, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous for a.e. $i = 1, 2$.
(H_7) There exist continuous functions $h_i, p_i, q_i : [a, \infty) \to \mathbb{R}_+$; $i = 1, 2$, such that

$$|f_i(t, u_1, u_2)| \leq h_i(t) + p_i(t)|u_1| + q_i(t)|u_2|,$$

for $t \in [a, \infty)$, and $u_i, v_i \in BC$.

Moreover, assume that

$$\lim_{t\to\infty}(\mathcal{I}_a^{\alpha_i} h_i)(t) = \lim_{t\to\infty}(\mathcal{I}_a^{\alpha_i} p_i)(t) = \lim_{t\to\infty}(\mathcal{I}_a^{\alpha_i} q_i)(t) = 0.$$

Set

$$p_i^* = \sup_{t\in[a,\infty)}(\mathcal{I}_a^{\alpha_i} p_i)(t), q^* = \sup_{t\in[a,\infty)}(\mathcal{I}_a^{\alpha_i} q_i)(t), h^* = \sup_{t\in[a,\infty)}(\mathcal{I}_a^{\alpha_i} h_i)(t), \psi_i^* = h_i^* + (p_i^* + q_i^*)R.$$

Now, we shall prove the following theorem concerning the existence and the attractivity of solutions of our problem (3.24)-(3.25).

Theorem 3.2.2. *Assume that (H_3)-(H_4) hold. Then the problem (3.24)-(3.25) has at least one solution defined on $[a,\infty)$. Moreover, the solutions of problem (3.24)-(3.25) are uniformly locally attractive.*

Proof. Define the operator $N:\mathcal{BC}\to\mathcal{BC}$ by

$$(N(u,v))(t) = ((N_1u)(t),(N_2v)(t)), \tag{3.35}$$

where $N_1, N_2 : BC \to BC$ with

$$(N_1u)(t) = u_a + \int_1^t (s-a)^{\alpha_1-1} f_1(s,u(s),v(s))ds, \tag{3.36}$$

and

$$(N_2v)(t) = v_a + \int_1^t (s-a)^{\alpha_2-1} f_2(s,u(s),v(s))ds. \tag{3.37}$$

Clearly, the fixed points of the operator N are solutions of the coupled system (3.24)-(3.25).

Set

$$R \geq \frac{|u_a|+|v_a|+h_1^*+h_2^*}{1-((p_1^*+q_1^*)+(p_2^*+q_2^*))},$$

and consider the ball

$$B_R := B(0,R) = \{(u,v)\in\mathcal{BC} : \|(u,v)\|_{\mathcal{BC}} \leq R\}.$$

The operator N maps $\mathcal{BC}$ into $\mathcal{BC}$. Indeed the map $N(u,v)$ is continuous on $\mathbb{R}_+$ for any $(u,v)\in\mathcal{BC}$ and for each $t\in[a,\infty)$; we have

$$\begin{aligned}
|(N_1u)(t)| &\leq |u_a| + \textstyle\int_a^t (s-a)^{\alpha_1-1}|f_1(s,u(s),v(s))|ds\\
&\leq |u_a| + \textstyle\int_a^t (s-a)^{\alpha_1-1}(h_1(s)+p_1(s)|u(s)|+q_1(s)|v(s)|)ds\\
&\leq |u_a| + h_1^* + (p_1^*+q_1^*)R,
\end{aligned}$$

and

$$\begin{aligned}
|(N_2v)(t)| &\leq |v_a| + \textstyle\int_a^t (s-a)^{\alpha_2-1}|f_2(s,u(s),v(s))|ds\\
&\leq |v_a| + \textstyle\int_a^t (s-a)^{\alpha_2-1}(h_2(s)+p_2(s)|u(s)|+q_2(s)|v(s)|)ds\\
&\leq |v_a| + h_2^* + (p_2^*+q_2^*)R.
\end{aligned}$$

Thus, we get

$$|N(u,v)(t)| \leq |u_a| + |v_a| + h_1^* + h_2^* + (p_1^* + q_1^* + p_2^* + q_2^*)R.$$

Thus

$$\|(N(u,v)\|_{\mathcal{BC}} \leq R. \tag{3.38}$$

This proves that N transforms the ball B_R into itself. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 1.5.3. The proof will be given in several steps.

Step 1. N is continuous.

Let $\{(u_n, v_n)\}_{n\in N}$ be a sequence such that $(u_n, v_n) \to (u, v)$ in B_R.
Then, for each $t \in [a, \infty)$, we have

$$\begin{aligned} |(N(u_n,v_n))(t) - (N(u,v))(t)| \;\; & \leqslant \;\; \textstyle\sum_{i=1}^{2} \int_a^t (s-a)^{\alpha_i - 1} |f_i(s, u_n(s), v_n(s) \\ & \quad - f_i(s, (u(s), v(s))| ds. \end{aligned} \tag{3.39}$$

Case 1. If $t \in [a, T]; T > a$, since $(u_n, v_n) \to (u, v)$ as $n \to \infty$ and f_i, $i = 1, 2$, are continuous, by the Lebesgue dominated convergence theorem, equation (3.39) implies

$$\|N(u_n, v_n) - N(u, v)\|_{\mathcal{BC}} \to 0 \quad \text{as} \quad n \to \infty.$$

Case 2. If $t \in (T, \infty); T > a$, then, from the accepted hypotheses and (3.39), we get

$$\begin{aligned} |(N(u_n,v_n))(t) - (N(u,v))(t)| \;\; & \leqslant \;\; 2\textstyle\sum_{i=1}^{2} \int_a^t (s-a)^{\alpha_i - 1} [h(s) + p(s)|u(s)| \\ & \quad + q(s)|v(s)|] \mathrm{d}s. \end{aligned} \tag{3.40}$$

Since $(u_n, v_n) \to (u, v)$ as $n \to \infty$ and $(\mathcal{I}_a^\alpha h_i)(t) = (\mathcal{I}_a^\alpha p_i)(t) = (\mathcal{I}_a^\alpha q_i)(t) \to 0$ as $t \to \infty$, then (3.40) gives

$$\|N(u_n, v_n) - N(u, v)\|_{\mathcal{BC}} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2.$N(B_R)$ is uniformly bounded.

This is clear because $N(B_R) \subset B_R$ and B_R is bounded.

Step 3. $N(B_R)$ is equicontinuous on every compact subset $[a, T]$ of $\mathbb{R}_+; T > 0$.

Let $t_1, t_2 \in [a, T], t_1 < t_2$ and let $(u, v) \in B_R$. Thus we have

$$\begin{aligned} |(N(u,v))(t_2) - (N(u,v))(t_1)| &\leq \sum_{i=1}^{2} \int_a^{t_2} (s-a)^{\alpha_i - 1} |f_i(s, u(s), v(s)| ds \\ &\quad - \int_a^{t_1} (s-a)^{\alpha_i - 1} |f_i(s, u(s), v(s)| ds \\ &\leq \sum_{i=1}^{2} \int_{t_1}^{t_2} (s-a)^{\alpha_i - 1} |f_i(s, u(s), v(s)| ds. \end{aligned}$$

As $t_1 \to t_2$ and the continuity of the function f_i ; the right hand side of the above inequality tends to zero.

Step 4. $N(B_R)$ is equiconvergent.

Let $t \in [a, \infty)$ and $(u, v) \in B_R$, then we have

$$\begin{aligned} |(Nu)(t)| &\leq |u_a| + \int_a^t (s-a)^{\alpha_1 - 1} |f_1(s, u(s), v(s))| ds \\ &\leq |u_a| + \int_a^t (s-a)^{\alpha_1 - 1} (h_1(s) + p_1(s)|u(s)| + q_1(s)|v(s)|) ds \\ &\leq |u_a| + (I_a^{\alpha_1} h_1)(t) + [(I_a^{\alpha_1} p_1)(t) + (I_a^{\alpha_1} q_1)(t)]R. \end{aligned}$$

Since $(\mathcal{I}_a^{\alpha_1} h_1)(t) = (\mathcal{I}_a^{\alpha_1} p_1)(t) = (\mathcal{I}_a^{\alpha_1} q_1)(t) \to 0$ as $t \to \infty$, we get

$$|(Nu)(t)| \to |u_a| \quad \text{as} \quad t \to \infty.$$

Hence,

$$|(Nu)(t) - (Nu)(\infty)| \to 0 \quad \text{as} \quad t \to \infty,$$

and

$$\begin{aligned} |(Nv)(t)| &\leq |v_a| + \int_a^t (s-a)^{\alpha_2 - 1} |f_2(s, u(s), v(s))| ds \\ &\leq |v_a| + \int_a^t (s-a)^{\alpha_2 - 1} (h_2(s) + p_2(s)|u(s)| + q_2(s)|v(s)|) ds \\ &\leq |v_a| + (I_a^{\alpha_2} h_2)(t) + [(I_a^{\alpha_2} p_2)(t) + (I_a^{\alpha_2} q_2)(t)]R. \end{aligned}$$

Since $(\mathcal{I}_a^{\alpha_2} h_2)(t) = (\mathcal{I}_a^{\alpha_2} p_2)(t) = (\mathcal{I}_a^{\alpha_2} q_2)(t) \to 0$ as $t \to \infty$, we get

$$|(Nv)(t)| \to |v_a| \quad \text{as} \quad t \to \infty.$$

Hence,

$$|(Nv)(t) - (Nv)(\infty)| \to 0 \quad \text{as} \quad t \to \infty.$$

Thus

$$|(N(u,v))(t) - (N(u,v))(+\infty)| \to 0 \quad \text{as} \quad t \to +\infty.$$

As a consequence of steps 1 to 4 together with the Lemma 3.2.1, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 1.5.3, we deduce that N has a fixed point (u,v) which is a solution of the problem (3.24)-(3.25) on $\mathbb{R}_+$.

Step 5. The uniform local attractivity of solutions.
let us assume that (u_0, v_0) is a solution of problem (3.24)-(3.25) with the conditions of this theorem.
Taking $(u,v) \in B((u_0,v_0), \overline{R})$ with $\overline{R} = 2(\psi_1^* + \psi_2^*)$; we have

$$\begin{aligned}
|(N_1u)(t) - u_0(t)| &= |(N_1u)(t) - (N_1u_0)(t)| \\
&\leq \int_a^t (s-a)^{\alpha_1 - 1} |f_1(s,u(s),v(s)) - f_1(s,u_0(s),v_0(s))| ds \\
&\leq \int_a^t (s-a)^{\alpha_1 - 1} |f_1(s,u(s),v(s))| + |f_1(s,u_0(s),v_0(s))| ds \\
&\leq 2\int_a^t (s-a)^{\alpha_1 - 1} (h_1(s) + p_1(s)|u(s)| + q_1(s)|v(s)|) ds \\
&\leq 2(h_1^* + (p_1^* + q_1^*)R) \\
&\leq 2\psi_1^*,
\end{aligned}$$

and similarly

$$\begin{aligned}
|(N_2v)(t) - v_0(t)| &= |(N_2v)(t) - (N_2v_0)(t)| \\
&\leq \int_a^t (s-a)^{\alpha_2 - 1} |f_2(s,u(s),v(s)) - f_2(s,u_0(s),v_0(s))| ds \\
&\leq \int_a^t (s-a)^{\alpha_1 - 1} |f_2(s,u(s),v(s))| + |f_2(s,u_0(s),v_0(s))| ds \\
&\leq 2\int_a^t (s-a)^{\alpha_1 - 1} (h_2(s) + p_2(s)|u(s)| + q_2(s)|v(s)|) ds \\
&\leq 2(h_2^* + (p_2^* + q_2^*)R) \\
&\leq 2\psi_2^*.
\end{aligned}$$

Thus, we get

$$|N(u,v)(t) - (u_0,v_0)(t)| \leq 2(\psi_1^* + \psi_2^*).$$

Thus

$$\|N(u,v)-(u_0,v_0)\|_{\mathcal{BC}} \leq \overline{R}.$$

Hence, we conclude that N is a continuous function such that

$$N(B((u_0,v_0),\overline{R})) \subset B((u_0,v_0),\overline{R})$$

Moreover, if (u,v) is a solution of problem (**??**)-(**??**), then

$$\begin{array}{rcl} |u(t)-u_0(t)| & = & |(N_1u)(t)-(N_1u_0)(t)| \\ & \leq & \int_a^t (s-a)^{\alpha_1-1}|f_1(s,u(s),v(s))-f_1(s,u_0(s),v_0(s))|ds \\ & \leq & \int_a^t (s-a)^{\alpha_1-1}[|f_1(s,u(s),v(s))|+|f_1(s,u_0(s),v_0(s))|]ds \\ & \leq & 2\int_a^t (s-a)^{\alpha_1-1}(h_1(s)+p_1(s)|u(s)|+q_1(s)|v(s)|)ds, \end{array}$$

and

$$\begin{array}{rcl} |v(t)-v_0(t)| & = & |(N_2v)(t)-(N_2v_0)(t)| \\ & \leq & \int_a^t (s-a)^{\alpha_2-1}|f_2(s,u(s),v(s))-f_1(s,u_0(s),v_0(s))|ds \\ & \leq & \int_a^t (s-a)^{\alpha_2-1}[|f_2(s,u(s),v(s))|+|f_2(s,u_0(s),v_0(s))|]ds \\ & \leq & 2\int_a^t (s-a)^{\alpha_2-1}(h_2(s)+p_2(s)|u(s)|+q_2(s)|v(s)|)ds. \end{array}$$

Thus

$$|(u,v)(t)-(u_0,u_0)(t)| \leq 2\sum_{i=1}^{2}\int_a^t (s-a)^{\alpha_i-1}(h_i(s)+p_i(s)|u(s)|+q_i(s)|v(s)|)ds. \quad (3.41)$$

By using (3.41) and the fact that $\lim_{t\to\infty}(\mathcal{I}_a^{\alpha_i}h_i)(t) = \lim_{t\to\infty}(\mathcal{I}_a^{\alpha_i}p_i)(t)\lim_{t\to\infty}(\mathcal{I}_a^{\alpha_i}q_i)(t) = 0$, we deduce that

$$\lim_{t\to\infty}|(u,v)(t)-(u_0,v_0)(t)| = 0.$$

Consequently, all solutions of problem (3.24)-(3.25) are uniformly locally attractive.

3.2.4 Examples

Example 3.2.1. *Consider the coupled system of Conformable fractional differential equations*

$$\begin{cases} (\mathcal{T}_{0^+}^{\frac{1}{2}} u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{0^+}^{\frac{1}{2}} v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \quad t \in [0,1], \tag{3.42}$$

with the following coupled boundary conditions :

$$u(0) = \frac{1}{2}v(1), v(0) = \frac{1}{2}u(1), \tag{3.43}$$

where

$$\begin{aligned} f_1(t,u,v) &= \frac{1}{4(t+2)^2}\frac{u(t)}{1+u(t)} + \frac{1}{\sqrt{t^2+2}}, \quad t \in [0,1], \\ f_2(t,u,v) &= \frac{1}{32\pi}\sin(2\pi v(t)) + \frac{1}{2}, \quad t \in [0,1]; \quad u, v \in \mathbb{R}. \end{aligned}$$

The hypothesis (H_4) is satisfied with

$$L_1 = \sup_{t\in[0,1]} f_1(t,0,0) = \frac{1}{\sqrt{2}} < \infty, L_2 = \sup_{t\in[0,1]} f_2(t,0,0) = \frac{1}{2} < \infty,$$

$$K_1 = M_2 = \frac{1}{16}, K_2 = M_1 = 0,$$

$$W_1 = W_4 = \frac{8}{3}, W_2 = W_3 = \frac{4}{3}.$$

The hypothesis (H_5) is satisfied with $\phi(t) = t^2$. With the obvious elementary computation, we have

$$\begin{aligned} \mathcal{I}_0^{\frac{1}{2}}\phi(t) &= \int_0^t s^{\alpha-1}\phi(s)\mathrm{d}s \\ &= \int_0^t s^{\alpha-1}s^2\mathrm{d}s \\ &\leq \frac{t^2}{\alpha+2} \\ &= \frac{\phi(t)}{\alpha+2}. \end{aligned}$$

Thus

$$\mathcal{I}_0^{\alpha}\phi(t) \leq \frac{t^2}{\alpha+2} := \lambda_\phi \phi(t).$$

Hence, Theorem 3.2.1 implies that the system (3.42)–(3.43) is generalized Ulam–Hyers–Rassias stable.

Example 3.2.2. *Consider the coupled system of Conformable fractional differential equations*

$$\begin{cases} (\mathcal{T}_{0^+}^{\frac{1}{2}} u)(t) = g_1(t, u(t), v(t)) \\ (\mathcal{T}_{0^+}^{\frac{1}{2}} v)(t) = g_2(t, u(t), v(t)) \end{cases} ; \quad t \in [1, \infty), \tag{3.44}$$

with the following coupled boundary conditions :

$$(u(1), v(1)) = (0, 0), \tag{3.45}$$

where

$$g_1(t, u, v) = \frac{t^{-2}(1 - \frac{5}{3}t^{-1}) \cos t}{64(1 + \sqrt{t})(1 + |u| + |v|)}, \quad t \in [1, \infty),$$
$$g_2(t, u, v) = \frac{\sqrt{t}(t-2)e^{-t} \sin t}{(1 + t^2 + |u| + |v|)}(1 + v(t)), \quad t \in [1, \infty); \quad u, v \in \mathbb{R}.$$

Clearly, the function g_1, g_2 are continuous.
The hypothesis (H_6) is satisfied with

$$h_1(t) = \frac{t^{-2}|(1 - \frac{5}{3}t^{-1})||\cos t|}{64(1 + \sqrt{t})}, h_2(t) = q_2(t) = \sqrt{t}|(t-2|)e^{-t}|\sin t|,$$
$$p_1(t) = q_1(t) = p_2(t) = 0; t \in [1, \infty).$$

Also, for $t > 2$, we have

$$\sqrt{t}|(t-2)|e^{-t}|\sin t| \leq \sqrt{t}(t-2)e^{-t},$$
$$\frac{t^{-2}|(1 - \frac{5}{3}t^{-1})||\cos t|}{64(1 + \sqrt{t})} \leq \frac{t^{-2}(1 - \frac{5}{3}t^{-1})}{64}.$$

In addition, we have

$$\int_1^t s^{-1/2} h_1(s) ds \leq \frac{1}{64} \int_1^t s^{-1/2} s^{-2} (1 - \frac{5}{3} s^{-1}) ds \leq -\frac{1}{96} t^{-3/2} (1 - t^{-1}) \to 0 \quad as \quad t \to \infty,$$

$$\int_1^t s^{-1/2} h_2(s) ds \leq \int_1^t s^{-1/2} s^{1/2} (s-2) e^{-s} ds \leq (-t+1)e^{-t} \to 0 \quad as \quad t \to \infty.$$

$$\int_1^t s^{-1/2} q_2(s) ds \leq \int_1^t s^{-1/2} s^{1/2} (s-2) e^{-s} ds \leq (-t+1) e^{-t} \to 0 \quad as \quad t \to \infty.$$

Therefore, Theorem 3.2.2 implies that the system (3.44)–(3.45) has at least one solution defined on $[1, \infty)$ *and moreover, the solutions of this problem are uniformly locally attractive.*

Chapitre 4

Coupled Katugambola fractional differential systems

4.1 A Coupled Katugampola fractional differential system with Boundary Conditions

4.1.1 Introduction and motivations

In this chapter we investigate the existence of solutions for the following coupled Katugampola fractional differential system

$$\begin{cases} ({}^{\rho}D_0^{\alpha_1}u)(t) = f_1(t,u(t),v(t)) \\ ({}^{\rho}D_0^{\alpha_2}v)(t) = f_2(t,u(t),v(t)) \end{cases} ;\ t \in I := [0,T], \tag{4.1}$$

with the boundary conditions

$$\begin{cases} \mathcal{I}_0^{2-\alpha_1,\rho}u(0) = a_1; \mathcal{I}_{0^+}^{2-\alpha_1,\rho}u(T) = b_1 \\ \\ \mathcal{I}_0^{2-\alpha_2,\rho}v(0) = a_2; \mathcal{I}_{0^+}^{2-\alpha_2,\rho}v(T) = b_2, \end{cases} \tag{4.2}$$

where $T > 0,\ ,\ t \in (0,T);\ ,\ \alpha_i \in (1,2],\ f_i : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m;\ i = 1,2$ are given continuous functions, $\mathbb{R}^m;\ m \in \mathbb{N}^*$ is the Euclidian Banach space with a suitable norm $\|\cdot\|$, $\mathcal{I}_0^{2-\alpha_i,\rho}$ is Katugampola fractional integral of order $2-\alpha_i$.

4.1.2 Existence Results in Banach spaces

In this section, we are concerned with the existence and uniqueness results of the coupled system (4.1)-(4.2).

Lemma 4.1.1. *Let $h \in C$, and $\alpha \in (1, 2]$. Then the unique solution $u \in C_{2-\alpha,\rho}(I)$ of problem*

$$\begin{cases} ({}^{\rho}D_{0^+}^{\alpha_1}u)(t) = h(t); \; t \in I \\ I_0^{2-\alpha_1,\rho}u(0_+) = a_1; I_0^{2-\alpha_1,\rho}u(T) = b_1 \end{cases}$$

is given by

$$\begin{array}{rcl} u(t) & = & \frac{2\rho^{2-\alpha}T^{-\rho}}{\Gamma(\alpha_1)}(b_1 - a_1 - I_{0^+}^{2,\rho}h(T))t^{\rho(\alpha_1-1)} \\ & + & \frac{\rho^{2-\alpha}}{\Gamma(\alpha_1-1)}a_1 t^{\rho(\alpha_1-2)} \\ & + & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}h(s)ds \end{array}.$$

Proof. Solving the linear equation

$$({}^{\rho}D_0^{\alpha}u)(t) = h(t),$$

From Lemma 1.3.3, we find easily :

$$u(t) = I_{0^+}^{\alpha,\rho}h(t) + C_1 t^{\rho(\alpha-1)} + C_2 t^{\rho(\alpha-2)} \tag{4.3}$$

From the boundary conditions and from (1.3.3), we have

$$\begin{array}{rcl} I_{0^+}^{2-\alpha,\rho}u(0) & = & I_0^{2-\alpha,\rho}I_0^{\alpha,\rho}h(t) + C_1 I_{0^+}^{2-\alpha,\rho}t^{\rho(\alpha-1)} \\ & + & C_2 I_{0^+}^{2-\alpha,\rho}t^{\rho(\alpha-2)} \\ & = & I_{0^+}^{2,\rho}h(0) + C_1 \lim_{t\to 0^+} \dfrac{\rho^{\alpha-2}\Gamma(\alpha)}{\Gamma(2-\alpha+\alpha)}t^{2-\alpha+\alpha-1} \\ & + & C_2 \lim_{t\to 0^+} \dfrac{\rho^{\alpha-2}\Gamma(\alpha-1)}{\Gamma(2-\alpha+\alpha-1)}t^{2-\alpha+\alpha-2} \\ & = & C_2 \frac{\rho^{\alpha-2}\Gamma(\alpha-1)}{\Gamma(1)} \\ & = & a_1 \\ & \Rightarrow & C_2 = \frac{\rho^{2-\alpha}a_1}{\Gamma(\alpha-1)} \end{array}$$

$$C_2 = \frac{\rho^{2-\alpha}a_1}{\Gamma(\alpha-1)}.$$

and

$$\begin{aligned} I_{0^+}^{2-\alpha,\rho}u(T) &= I_0^{2-\alpha,\rho}I_0^{\alpha,\rho}h(T) + C_1 I_0^{2-\alpha,\rho}T^{\rho(\alpha-1)} \\ &+ C_2 I_{0^+}^{2-\alpha,\rho}T^{\rho(\alpha-2)} \\ &= I_0^{2,\rho}h(0) + C_1\tfrac{\rho^{\alpha-2}\Gamma(\alpha)}{\Gamma(2-\alpha+\alpha)}T^{\rho(2-\alpha+\alpha-1)} \\ &+ C_2\tfrac{\rho^{\alpha-2}\Gamma(\alpha-1)}{\Gamma(2-\alpha+\alpha-1)}T^{\rho(2-\alpha+\alpha-2)} \\ &= I_0^{2,\rho}h(T) + C_1\tfrac{\rho^{\alpha-2}\Gamma(\alpha)}{\Gamma(2-\alpha+\alpha)}T^{\rho} \\ &+ C_2\rho^{\alpha-2}\Gamma(\alpha-1) \\ &= b_1 \\ &\Rightarrow\ C_1 = \tfrac{\rho^{2-\alpha}\Gamma(2)T^{-\rho}}{\Gamma(\alpha)}(b_1 - a_1 - I_{0^+}^{2,\rho}h(T)) \end{aligned}$$

$$C_1 = \frac{\rho^{2-\alpha}\Gamma(2)T^{-\rho}}{\Gamma(\alpha)}(b_1 - a_1 - I_{0^+}^{2,\rho}h(T)).$$

Substituting the values of c_1 and c_2 in (4.3), we get

$$\begin{aligned} u(t) &= \tfrac{2\rho^{2-\alpha}T^{-\rho}}{\Gamma(\alpha)}(b_1 - a_1 - I_{0^+}^{2,\rho}h(T))t^{\rho(\alpha-1)} \\ &+ \tfrac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}a_1 t^{\rho(\alpha-2)} \\ &+ \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}h(s)ds \end{aligned}.$$

We concluded the following lemma.

Lemma 4.1.2. *Let $f_i : I \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m;\ i = 1,2$ such that $f_i(\cdot,u,v) \in C_{2-\alpha_i,t^\rho}(I)$ for each $u, v \in C_{2-\alpha_i,\rho}(I)$. Then the coupled system (4.1)-(4.2) is equivalent to the problem of obtaining the solution of the coupled system*

$$\begin{cases} u(t) = \frac{2\rho^{2-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}(b_1 - a_1 - I_{0^+}^{2,\rho}h(T))t^{\rho(\alpha_1-1)} \\ +\frac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1 t^{\rho(\alpha_1-2)} \\ +\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1}h(s)ds \\ v(t) = \frac{2\rho^{2-\alpha_2}T^{-\rho}}{\Gamma(\alpha_2)}(b_2 - a_2 - I_{0^+}^{2,\rho}h(T))t^{\rho(\alpha_2-1)} \\ +\frac{\rho^{2-\alpha}}{\Gamma(\alpha_1-1)}a_2 t^{\rho(\alpha_2-2)} \\ +\frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_2-1}h(s)ds, \end{cases}$$

Definition 4.1.1. *By a solution of the problem (4.1)-(4.2) we mean a coupled continuous functions $(u, v) \in C_{2-\alpha_1,\rho}(I) \times C_{2-\alpha_2,\rho}(I)$ satisfying the boundary conditions (4.2), and the equations (4.1) on I.*

The following hypotheses will be used in the sequel.

(H_1') There exist constants $K_i > 0$ and $0 < L_i < 1$ such that The functions $f_i;\ i = 1,2$ satisfy the generalized Lipschitz condition :

$$\|f_i(t,u_1,v_1) - f_i(t,u_2,v_2)\| \leq K_i t^{\rho(2-\alpha_i)}\|u_1 - u_2\| + L_i t^{\rho(2-\alpha_i)}\|v_1 - v_2\|,$$

for $t \in I$ and $u_i, v_i \in \mathbb{R}^m$.

We are now in a position to state and prove our existence result for the problem(4.1)-(4.2) based on concept of measures of noncompactness and Darbo's fixed point theorem.

Remark 4.1.1. *[35] Condition (H_1) is equivalent to the inequality*

$$\alpha(f_i(t,B_1,B_2)) \leq K_i t^{\rho(2-\alpha_i)}\alpha(B_1) + L_i t^{\rho(2-\alpha_i)}\alpha(B_2),$$

for any bounded sets $B_1, B_2 \subseteq \mathcal{C}$ and for each $t \in I$.

Theorem 4.1.1. *Assume (H_1'). If*

$$\frac{\rho^{-\alpha_1}}{2\Gamma(\alpha_1)}T^{\rho(\alpha_1)}+\frac{\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}T^{\rho(\alpha_1)})(K_1+L_1)+\frac{\rho^{-\alpha_2}}{2\Gamma(\alpha_2)}T^{\rho(\alpha_2)}+\frac{\rho^{-\alpha_2}}{\Gamma(\alpha_2+1)}T^{\rho(\alpha_2)})(K_2+L_2) < 1, \tag{4.4}$$

then the coupled system (4.1)- (4.2) has at least one solution defined on I.

Proof. Define the operator $N : \mathcal{C} \to \mathcal{C}$ by

$$(N(u,v))(t) = ((N_1 u)(t), (N_2 v)(t)), \tag{4.5}$$

where $N_1 : C_{2-\alpha_1,\rho} \to C_{2-\alpha_1,\rho}$ and $N_2 : C_{2-\alpha_2,\rho} \to C_{2-\alpha_2,\rho}$ with

$$\begin{aligned}(N_1 u))(t) &= \tfrac{2\rho^{2-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}(b_1 - a_1 - \tfrac{\rho^{-1}}{\Gamma(2)}\int_0^T s^{\rho-1}(T^\rho - s^\rho)f_1(s,u(s);v(s))ds)t^{\rho(\alpha_1-1)}\\ &+ \tfrac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1 t^{\rho(\alpha_1-2)}\\ &+ \tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1}f_1(s,u(s);v(s))ds.\end{aligned}$$

and

$$\begin{aligned}(N_2 v))(t) &= \tfrac{2\rho^{2-\alpha_2}T^{-\rho}}{\Gamma(\alpha_2)}(b_2 - a_2 - \tfrac{\rho^{-1}}{\Gamma(2)}\int_0^T s^{\rho-1}(T^\rho - s^\rho)f_2(s,u(s);v(s))ds)t^{\rho(\alpha_2-1)}\\ &+ \tfrac{\rho^{\alpha_2-2}}{\Gamma(\alpha_2-1)}a_2 t^{\rho(\alpha_2-2)}\\ &+ \tfrac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_2-1}f_2(s,u(s);v(s))ds.\end{aligned}$$

Clearly, the fixed points of the operator N are solutions of the coupled system (4.1)-(4.2).

For each $u_i, v_i \in C_{2-\alpha_i,\rho};\ i = 1,2$ and $t \in I$,

$$R \geq \frac{\sum_{i=1}^{2}\left[\frac{\rho^{2-\alpha_i}}{\Gamma(\alpha_i-1)}a_i + \frac{2\rho^{2-\alpha_i}}{\Gamma(\alpha_i)}(b_i - a_i) + (\frac{\rho^{-\alpha_i}}{2\Gamma(\alpha_i)} + \frac{\rho^{-\alpha_i}}{\Gamma(\alpha_i+1)})T^{2\rho}f_i^*\right]}{1 - \sum_{i=1}^{2}\left[(\frac{\rho^{-\alpha_i}}{2\Gamma(\alpha_i)} + \frac{\rho^{-\alpha_i}}{\Gamma(\alpha_i+1)})T^{\rho(\alpha_1)}(K_i + L_i)\right]}, \tag{4.6}$$

and consider the closed and convex ball

$$B_R = \{(u,v) \in C_{2-\alpha_i,\rho} : \|(u,v)\|_{\mathcal{C}} \leq R\}.$$

Remark 4.1.2. (H_1),*we have*

$$\begin{aligned} \|f_i(t,u,v)\| &\leq \|f_i(t,u,v) - f_i(t,0,0)\| + \|f_i(t,0,0)\| \\ &\leq K_i t^{\rho(2-\alpha_i)}\|u\| + L_i t^{\rho(2-\alpha_i)}\|v\| + \|f_i(t,0,0)\| \\ &\leq K_i\|u\|_{C_{2-\alpha_1,\rho}} + L_i\|v\|_{C_{2-\alpha_2,\rho}} + \|f_i(t,0,0)\| \\ &\leq (K_i + L_i)R + f_i^*. \end{aligned}$$

Let $(u,v) \in B_R$. Then, for each $t \in I$, and any $i = 1,2$, we have

$$\begin{aligned} \|t^{\rho(2-\alpha_1)}(N_1u)(t)\| &= \|\tfrac{2\rho^{2-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}(b_1 - a_1 - \tfrac{\rho^{-1}}{2}\int_0^T s^{\rho-1}(T^\rho - s^\rho)f_1(s,u(s);v(s))ds)t^\rho \\ &+ \tfrac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1 \\ &+ \tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_1)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1}f_1(s,u(s);v(s))ds\| \\ &\leq \tfrac{2\rho^{2-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}(b_1 - a_1 - \tfrac{\rho^{-1}}{2}\int_0^T s^{\rho-1}(T^\rho - s^\rho)\|f_1(s,u(s);v(s))\|ds)t^\rho \\ &+ \tfrac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1 \\ &+ \tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_1)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1}\|f_1(s,u(s);v(s))\|ds \\ &\leq \tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_0^T s^{\rho-1}(T^\rho - s^\rho)\|f_1(s,u(s);v(s))\|ds \\ &+ \tfrac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1 + \tfrac{2\rho^{2-\alpha_1}}{\Gamma(\alpha_1)}(b_1 - a_1) \\ &+ \tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_1)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1}\|f_1(s,u(s);v(s))\|ds \\ &\leq \tfrac{\rho^{-\alpha_1}}{2\Gamma(\alpha_1)}T^{2\rho}[(K_1 + L_1)R + f_1^*] \\ &+ \tfrac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1 + \tfrac{2\rho^{2-\alpha_1}}{\Gamma(\alpha_1)}(b_1 - a_1) \\ &+ \tfrac{\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}T^{2\rho}[(K_1 + L_1)R + f_1^*] \\ &\leq (\tfrac{\rho^{-\alpha_1}}{2\Gamma(\alpha_1)}T^{2\rho} + \tfrac{\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}T^{2\rho})R(K_1 + L_1) \\ &+ \tfrac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1 + \tfrac{2\rho^{2-\alpha_1}}{\Gamma(\alpha_1)}(b_1 - a_1) + (\tfrac{\rho^{-\alpha_1}}{2\Gamma(\alpha_1)} + \tfrac{\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)})T^{2\rho}f_1^*. \end{aligned}$$

Similarly,

$$\begin{aligned}\|(N_2 v)\|_{C_{2-\alpha_2,\rho}} &\leq (\tfrac{\rho^{-\alpha_2}}{2\Gamma(\alpha_2)}T^{2\rho} + \tfrac{\rho^{-\alpha_2}}{\Gamma(\alpha_2+1)}T^{2\rho})(R(K_2+L_2)) + \tfrac{\rho^{2-\alpha_2}}{\Gamma(\alpha_2-1)}a_2 + \tfrac{2\rho^{2-\alpha_2}}{\Gamma(\alpha_1)}(b_2-a_2)\\ &+ (\tfrac{\rho^{-\alpha_2}}{2\Gamma(\alpha_2)} + \tfrac{\rho^{-\alpha_2}}{\Gamma(\alpha_2+1)})T^{2\rho}f_2^*.\end{aligned}$$

Thus

$$\|N(u,v)\|_{\mathcal{C}} \leq R. \tag{4.7}$$

Hence N maps the ball B_R into it self. We shalls how that N satisfies the assumption of Darbo's fixed point Theorem. The proof will be given in several steps.

Step 1 : We show that N is continuous. Let$\{(u_n, v_n)\}$ be a sequence such that $(u_n, v_n) \to (u, v)$ in B_R. Then, for each $t \in I$, we have

$$\begin{aligned}& \|t^{\rho(2-\alpha_i)}(N(u_n,v_n)(t) - (N(u,v)(t)\|\\ = \quad & \Sigma_{i=1}^2 \|(\frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}T^{-\rho}t^{\rho}\int_0^T s^{\rho-1}(T^\rho - s^\rho)[f_i(s,u_n(s);v_n(s)) - f_i(s,u(s);v(s)]ds\\ & +\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_i)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_i-1}[f_i(s,u_n(s);v_n(s)) - f_i(s,u(s);v(s)]ds\|\\ \leq \quad & \Sigma_{i=1}^2\left(\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}T^{-\rho}t^{\rho}\int_0^T s^{\rho-1}(T^\rho - s^\rho)\|[f_i(s,u_n(s);v_n(s)) - f_i(s,u(s);v(s)]\|ds\right.\\ & \left.+\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_i)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_i-1}\|[f_i(s,u_n(s);v_n(s)) - f_i(s,u(s);v(s)]\|ds\right)\\ \leq \quad & \Sigma_{i=1}^2\left(\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}T^{-\rho}t^{\rho}\int_0^T s^{\rho-1}(T^\rho - s^\rho)(K_i s^{\rho(2-\alpha_1)}\|u_n - u\| + L_i s^{\rho(2-\alpha_1)}\|v_n - v\|)ds\right.\\ & \left.+\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_i)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_i-1}\|(K_i\|u_n - u\| + L_i\|v_n - v\|)ds\right)\\ \leq \quad & \Sigma_{i=1}^2\left(\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}(K_i\|u_n - u\|_{C_{\alpha_i,\rho}} + L_i\|v_n - v\|_{C_{\alpha_i,\rho}})\int_0^T s^{\rho-1}(T^\rho - s^\rho)ds\right.\\ & \left.+\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}T^{\rho(2-\alpha_i)}(K_i\|u_n - u\|_{C_{\alpha_i,\rho}} + L_i\|v_n - v\|_{C_{\alpha_i,\rho}})\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_i-1}ds\right).\end{aligned}$$

Since $u_n \to u$, $v_n \to v$ as $n \to \infty$ et f_1,f_2 are continuous , then by the Lebesgue dominated convergence theorem ;

$$\|N(u_n,v_n) - N(u,v)\|_{\mathcal{C}} \to 0 \ \ as \ \ n \to \infty.$$

Step 2 : We remark that $N(B_R)$ is bounded. This is clear since $N : B_R \to B_R$ and B_R is bounded.

Step 3 : We show that$N$ maps bounded sets into equicontinuous sets in B_R. Let

$t_1, t_2 \in I$, such that $t_1 < t_2$ and let $(u_1, u_2) \in B_R$. Then, we have

$$\begin{array}{rl} & \|t_2^{\rho(2-\alpha_i)}(N(u_1,u_2))(t_2) - t_1^{\rho(2-\alpha_i)}(N(u_1,u_2))(t_1)\| \\ \leq & \|\frac{\rho^{1-\alpha_i}T^{-\rho}}{\Gamma(\alpha_i)}(t_2^\rho - t_1^\rho)\int_0^T s^{\rho-1}(T^\rho - s^\rho)f_i(s,u(s);v(s))ds \\ & +\frac{\rho^{1-\alpha_i}t_2^{\rho(2-\alpha_i)}}{\Gamma(\alpha_i)}\int_0^{t_2} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha_i-1}f_i(s,u_1(s);u_2(s))ds \\ & -\frac{\rho^{1-\alpha_i}t_1^{\rho(2-\alpha_i)}}{\Gamma(\alpha_i)}\int_0^{t_1} s^{\rho-1}(t_1^\rho - s^\rho)^{\alpha_i-1}f_i(s,u_1(s);u_2(s))ds\| \\ \leq & \|\frac{\rho^{1-\alpha_i}T^{-\rho}}{\Gamma(\alpha_i)}(t_2^\rho - t_1^\rho)\int_0^T s^{\rho-1}(T^\rho - s^\rho)f_i(s,u(s);v(s))ds \\ & +\frac{\rho^{1-\alpha_i}t_2^{\rho(2-\alpha_i)}}{\Gamma(\alpha_i)}\int_{t_1}^{t_2} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha_i-1}f_i(s,u_1(s);u_2(s))ds \\ & -\frac{\rho^{1-\alpha_i}t_1^{\rho(2-\alpha_i)}}{\Gamma(\alpha_i)}\int_0^{t_1} s^{\rho-1}(t_1^\rho - s^\rho)^{\alpha_i-1}f_i(s,u_1(s);u_2(s))ds \\ & +\frac{\rho^{1-\alpha_i}t_2^{\rho(2-\alpha_i)}}{\Gamma(\alpha_i)}\int_0^{t_1} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha_i-1}f_i(s,u_1(s);u_2(s))ds\| \\ \leq & \frac{(t_2^\rho - t_1^\rho)}{\rho^{\alpha_i}\Gamma(\alpha_i)}T^\rho(f_i^* + K_i + L_i)R \\ & +\frac{t_1^\rho + t_2^\rho + 2(t_2^\rho - t_1^\rho)^{\alpha_i}}{\rho^{\alpha_i}\Gamma(\alpha_i+1)}T^{\rho(2-\alpha_i)}(f_i^* + K_i + L_i)R. \end{array}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Step 4 : The operator $N : B_R \to B_R$ is a strict set contraction.
Let $V \in B_R$ and $t \in I$, then we have

$$\begin{array}{rl} & \alpha(t^{\rho(2-\alpha_1)}N_1(V)(t)) \\ = & \alpha((N_1(u,v)(t),(u,v) \in V \\ \leq & \frac{2\rho^{2-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}(b_1 - a_1 - \frac{\rho^{-1}}{2}\{\int_0^T s^{\rho-1}(T^\rho - s^\rho)\alpha f_1(s,u(s);v(s))ds,(u,v) \in V\})t^\rho \\ & +\frac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1 + \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_1)}\{\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1}\alpha f_1(s,u(s);v(s))ds,(u,v) \in V\}. \end{array}$$

Then Remark 4.1.1 implies that, for each $s \in I$

$$\alpha(\{f_1(s,u(s);v(s))ds,(u,v) \in V\}) \leq K_1\alpha(\{u(s),(u) \in V\}) + L_1\alpha(\{v(s),(v) \in V\}).$$

Then

$$\begin{aligned}
\alpha(t^{\rho(2-\alpha_1)}N_1(V)(t)) &\leq K_1\frac{\rho^{1-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}t^{\rho}\{\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})\{\alpha(u(s))\}ds,(u)\in V\}\\
&+ L_1\frac{\rho^{1-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}t^{\rho}\{\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})\{\alpha(v(s))\}ds,(v)\in V\}\\
&+ \frac{2\rho^{2-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}(b_1-a_1)t^{\rho}\\
&+ \frac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1\\
&+ K_1\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_1)}\{\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-1}\{\alpha(u(s))\}ds,(u)\in V\}\\
&+ L_1\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_1)}\{\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-1}\{\alpha(v(s))\}ds,(v)\in V\}\\
&\leq \frac{(K_1+L_1)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V)\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}t^{\rho(\alpha_1-1)}T^{-\rho}\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})ds\\
&+ \frac{2\rho^{2-\alpha_1}T^{-\rho}}{\Gamma(\alpha_1)}(b_1-a_1)t^{\rho}\\
&+ \frac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1\\
&+ \frac{(K_1+L_1)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V)\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_0^t s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-1}ds\\
&\leq \frac{2\rho^{2-\alpha_1}}{\Gamma(\alpha_1)}(b_1-a_1)\\
&+ \frac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1\\
&+ \frac{(K_1+L_1)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V)\rho^{-\alpha_1}}{2\Gamma(\alpha_1)}(T^{\rho})^{\alpha_1}\\
&+ \frac{(K_1+L_1)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V)\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}(T^{\rho})^{\alpha_1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha_{C_{2-\alpha_1,\rho}}(N_1V) &\leq (\frac{(K_1+L_1)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V)\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{(K_1+L_1)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V))\rho^{-\alpha_1}}{2\Gamma(\alpha_1)})(T^{\rho\alpha_1})+(\frac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1\\
&+ \frac{2\rho^{2-\alpha_1}}{\Gamma(\alpha_1)}(b_1-a_1)).
\end{aligned}$$

Similarly

$$\begin{aligned}
\alpha_{C_{2-\alpha_2,\rho}}(N_2V) &\leq (\frac{(K_2+L_2)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V)\rho^{-\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{(K_2+L_2)\alpha_{C_{2-\alpha_2,t^{\rho}}}(V)\rho^{-\alpha_2}}{2\Gamma(\alpha_2)})(T^{\rho\alpha_2})+(\frac{\rho^{2-\alpha_2}}{\Gamma(\alpha_2-1)}a_2\\
&+ \frac{2\rho^{2-\alpha_2}}{\Gamma(\alpha_2)}(b_2-a_2)).
\end{aligned}$$

Thus

$$\begin{aligned}
\alpha_C(NV) &\leq (\frac{(K_1+L_1)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V)\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{(K_1+L_1)\alpha_{C_{2-\alpha_1,t^{\rho}}}(V)\rho^{-\alpha_1}}{2\Gamma(\alpha_1)})(T^{\rho\alpha_1})\\
&+ (\frac{\rho^{2-\alpha_1}}{\Gamma(\alpha_1-1)}a_1+\frac{2\rho^{2-\alpha_1}}{\Gamma(\alpha_1)}(b_1-a_1))\\
&+ (\frac{(K_2+L_2)\alpha_{C_{2-\alpha_2,t^{\rho}}}(V)\rho^{-\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{(K_2+L_2)\alpha_{C_{2-\alpha_2,t^{\rho}}}(V)\rho^{-\alpha_2}}{2\Gamma(\alpha_2)})(T^{\rho\alpha_2})\\
&+ (\frac{\rho^{2-\alpha_2}}{\Gamma(\alpha_2-1)}a_2+\frac{2\rho^{2-\alpha_2}}{\Gamma(\alpha_2)}(b_2-a_2)).
\end{aligned}$$

So, the operator N is a set contraction. As a consequence of Theorem (1.5.5),we deduce that N has a fixed point which is solution to the problem (4.1)-(4.2). This completes the proof.

Now, Our next existence result for the problem (4.1)-(4.2) is based on concept of measures of noncompactness and Mönch's fixed point theorem

Theorem 4.1.2. *Assume that the hypothesis* (H_1),(H_2) *holds.*
then the coupled system (4.1)- (4.2) has at least one solution defined on I.

Proof.$N : \mathcal{C} \to \mathcal{C}$ be the operator defined in (4.5). We shall show that N satisfies the assumption of Mönch's fixed point theorem. We know that $N : B_R \to B_R$ is bounded and continuous, we need to prove that the implication

$$V = \overline{conv}N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of B_R.Now let V be a subset of B_R such that $V \subset \overline{conv}N(V) \cup \{0\}$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on I.

$$\begin{aligned}
t^{\rho(2-\alpha_i)}w_i(t) &\leq \alpha(t^{\rho(2-\alpha_1)}N_i(V)(t) \cup \{0\}) \\
&\leq \alpha(t^{\rho(2-\alpha_i)}N_i(V)(t)) \\
&\leq \alpha(t^{\rho(2-\alpha_i)}N_i(u_i)(t), u_i \in V) \\
&\leq K_1\tfrac{\rho^{1-\alpha_1}T^{-\rho}}{\Gamma(\alpha_i)}\{\textstyle\int_0^t s^{\rho-1}(t^\rho - s^\rho)\{\alpha(u_1(s))\}ds, (u_1) \in V\}t^\rho \\
&+ L_1\tfrac{\rho^{1-\alpha_i}T^{-\rho}}{\Gamma(\alpha_i)}\{\textstyle\int_0^t s^{\rho-1}(t^\rho - s^\rho)\{\alpha(u_2(s))\}ds, (u_2) \in V\}t^\rho \\
&+ \tfrac{2\rho^{2-\alpha_i}T^{-\rho}}{\Gamma(\alpha_1)}(b_i - a_i)t^\rho \\
&+ \tfrac{\rho^{2-\alpha_i}}{\Gamma(\alpha_i-1)}a_i \\
&+ K_i\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)}t^{\rho(2-\alpha_i)}\{\textstyle\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_i-1}\{\alpha(u_1(s))\}ds, (u_1) \in V\} \\
&+ L_i\tfrac{\rho^{1-\alpha_i}}{\Gamma(\alpha_1)}t^{\rho(2-\alpha_i)}\{\textstyle\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_i-1}\{\alpha(u_2(s))\}ds, (u_2) \in V\} \\
&\leq \tfrac{(K_i+L_i)\rho^{1-\alpha_i}}{\Gamma(\alpha_i)}t^{\rho(\alpha_i-1)}\textstyle\int_0^t s^{\rho-1}(t^\rho - s^\rho)(t^{\rho(2-\alpha_i)}w_i(s))ds. \\
&+ \tfrac{2\rho^{2-\alpha_i}T^{-\rho}}{\Gamma(\alpha_i)}(b_i - a_i)t^\rho \\
&+ \tfrac{\rho^{2-\alpha_i}}{\Gamma(\alpha_i-1)}a_i \\
&+ \tfrac{(K_i+L_i)\rho^{1-\alpha_i}}{\Gamma(\alpha_i)}\textstyle\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_i-1}(t^{\rho(2-\alpha_i)}w_i(s))ds.
\end{aligned}$$

implies that $w_i(t) = 0$ for each $t \in I$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzel'a theorem,Vis relatively compact in B_R. Applying now Theorem

(1.5.4) we conclude that N has a fixed point $(u_1, u_2) \in B_R$.. Hence N has a fixed point which is solution to the problem (4.1)- (4.2).This completes the proof.

4.2 Examples

Let

$$l^1 = \left\{ u = (u_1, u_2, \ldots, u_m, \ldots), \sum_{m=1}^{\infty} |u_m| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{m=1}^{\infty} |u_m|.$$

Consider the coupled system of Caputo–Katugampola fractional differential equations

$$\begin{cases} ({}^1D_0^{\frac{3}{2}} u_n)(t)(t) = f_n(t, u(t), v(t)) \\ ({}^1D_0^{\frac{3}{2}} v_n)(t) = g_n(t, u(t), v(t)), \end{cases} \quad t \in [0,1] \tag{4.8}$$

with the boundary conditions

$$\begin{cases} \mathcal{I}_0^{\frac{1}{2},1} u(0) = 1 = \mathcal{I}_{0^+}^{\frac{1}{2},1} u(1) \\ \\ \mathcal{I}_0^{\frac{1}{2},1} v(0) = 2 = \mathcal{I}_{0^+}^{\frac{1}{2},1} v(1) \end{cases} \tag{4.9}$$

where

$$f_n(t, u(t)) = \frac{e^{-t-5} u_n(t))}{1 + \|u(t)\|_{l^1} + \|v(t)\|_{l^1}}, \quad t \in [0,1],$$
$$g_n(t, u, v) = \frac{(2^{-n} + v_n(t)) \sin t}{64(\|v(t)\|_{l^1} + 1)(1 + \|u(t)\|_{l^1} + \|v(t)\|_{l^1})}, \quad t \in [0,1].$$

with $f = (f_1, f_2, \ldots, f_n, \ldots), g = (g_1, g_2, \ldots, g_n, \ldots)$ *and* $u = (u_1, u_2, \ldots, u_n, \ldots)$.
For each $t \in [0,1]$, we have

$$\begin{aligned} \|f(t, u(t), v(t))\|_{l^1} &= \sum_{n=1}^{\infty} |f_n(s, u_n(s), v_n(s))| \\ &\leq e^{-6} \|u\|_{l^1}. \end{aligned}$$

and

$$\begin{aligned}\|g(t,u(t),v(t))\|_{l^1} &= \sum_{n=1}^{\infty}|g_n(s,u_n(s),v_n(s))|\\ &\leq \frac{1}{64}.\end{aligned}$$

The hypothesis [(H_2)]is satisfied with

$$K_1 \leq e^{-6}.$$

$$L_1^* = K_2^* = L_2^* = 0.$$

In addition, with good choice of the constants $d_i; i = 1,2$, a simple computation show that all conditions of Theorem 4.1.1 are satisfied. Hence, the problem (4.8)-(4.9) has at least one solution defined on $[0,1]$.

4.3 A Coupled Caputo-Katugampola Fractional Differential System with Boundary Conditions

4.3.1 Introduction and motivations

In this chapter we investigate the existence of solutions for the following coupled Katugampola fractional differential system

$$\begin{cases}({}^cD_{a^+}^{\alpha_1,\rho}u)(t) = f_1(t,u(t),v(t))\\ ({}^cD_{a^+}^{\alpha_2,\rho}v)(t) = f_2(t,u(t),v(t))\end{cases} ;\ t \in I := [a,b], \tag{4.10}$$

with the boundary conditions

$$\begin{cases}u(a) = \lambda_1 v(b);{}^c D_{a^+}^{\gamma_1,\rho}u(b) = \lambda_2 \sum_{i=1}^{N}({}^cD_{a^+}^{\delta_1,\rho}v)(\eta_i)\\ \\ v(a) = \mu_1 u(b);{}^c D_{a^+}^{\gamma_2,\rho}v(b) = \mu_2 \sum_{i=1}^{M}({}^cD_{a^+}^{\delta_2,\rho}u)(\xi_i)\end{cases} ; \tag{4.11}$$

where $a,b > 0,\ ,\ t \in (a,b);\ ,\ \alpha_i \in (1,2], \gamma_1, \delta_1 \in (0,1], \eta_i \in R$ for $i = 1,2.....N(N \in \mathbb{N})\xi_i \in \mathbb{R}$ for $i = 1,2.....M(M \in \mathbb{N})a < \xi_1 < \xi_2.... < b, \lambda_i, \mu_i, i = 1,2$ are real positive constants $f_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R};\ i = 1,2$ are given continuous functions and ${}^cD_a^{\alpha_i,\rho}$ is caputo- Katugampola fractional derivative of order $\alpha_i\,;\, i = 1,2$.

4.3.2 Existence Results in Banach spaces

In this section, we are concerned with the existence and uniqueness results of the coupled system (4.10)-(4.11).

Lemma 4.3.1. *Let*

$$\Delta = \frac{4\left(\frac{b^\rho - a^\rho}{\rho}\right)^{2-\gamma_1-\gamma_2}}{\Gamma(2-\gamma_1)\Gamma(2-\gamma_2)} - \frac{4\lambda_2\mu_2}{\Gamma(2-\delta_2)\Gamma(2-\delta_1)} \sum_{i=1}^{M} \left(\frac{\xi_i^\rho - a^\rho}{\rho}\right)^{1-\delta_2} \sum_{i=1}^{N} \left(\frac{\eta_i^\rho - a^\rho}{\rho}\right)^{1-\delta_1} \neq 0,$$

and $\mu_1\lambda_1 \neq 1$ *Let* $x, y \in C$, *and* $\alpha \in (1,2]$. *Then the unique solution of problem*

$$\begin{cases} ({}^cD_{a^+}^{\alpha_1,\rho}u)(t) = x(t);\ t \in I := [a,b], \\ ({}^cD_{a^+}^{\alpha_2,\rho}v)(t) = y(t);\ t \in I := [a,b], \\ u(a) = \lambda_1 v(b); {}^c D_{a^+}^{\gamma_1,\rho}u(b) = \lambda_2 \sum_{i=1}^{N} ({}^cD_{a^+}^{\delta_1,\rho}v)(\eta_i) \\ v(a) = \mu_1 u(b); {}^c D_{a^+}^{\gamma_2,\rho}v(b) = \mu_2 \sum_{i=1}^{M} ({}^cD_{a^+}^{\delta_2,\rho}u)(\xi_i) \end{cases} \tag{4.12}$$

is given by

$$\begin{aligned} u(t) \ &= \ \frac{2\lambda_1}{\Delta(1-\lambda_1\mu_1)} \Bigg[\Bigg(\frac{\mu_1\lambda_2\mu_2}{\Gamma(2-\delta_1)} \left(\frac{b^\rho-a^\rho}{\rho}\right) \textstyle\sum_{i=1}^{N} (\frac{\eta_i^\rho-a^\rho}{\rho})^{1-\delta_1} + \frac{\mu_2}{\Gamma(2-\gamma_1)} \left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1} \\ &+ \ \frac{\lambda_2\mu_2}{\Gamma(2-\delta_1)} \left(\frac{t^\rho-a^\rho}{\rho}\right) \textstyle\sum_{i=1}^{N} \left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1} \Bigg) B_3 \\ &- \ \Bigg(\frac{\mu_1\lambda_2}{\Gamma(2-\delta_1)} \left(\frac{b^\rho-a^\rho}{\rho}\right) \textstyle\sum_{i=1}^{N} \left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1} + \frac{1}{\Gamma(2-\gamma_1)} \left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1} \\ &- \ \frac{\lambda_2}{\Gamma(2-\delta_1)} \left(\frac{t^\rho-a^\rho}{\rho}\right) \textstyle\sum_{i=1}^{N} \left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1} \Bigg) A_3 \\ &+ \ \Bigg(\frac{\mu_1\lambda_2}{\Gamma(2-\gamma_2)} \left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{\mu_2\lambda_2}{\Gamma(2-\delta_2)} \textstyle\sum_{i=1}^{M} \left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2} \left(\frac{b^\rho-a^\rho}{\rho}\right) \\ &+ \ \frac{\lambda_2}{\Gamma(2-\gamma_2)} \left(\frac{t^\rho-a^\rho}{\rho}\right) \left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2} \Bigg) A_2 \\ &- \ \Bigg(\frac{\mu_1}{\Gamma(2-\gamma_2)} \left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{\mu_2}{\Gamma(2-\delta_2)} \textstyle\sum_{i=1}^{M} \left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2} \left(\frac{b^\rho-a^\rho}{\rho}\right) \\ &- \ \frac{1}{\Gamma(2-\gamma_2)} \left(\frac{t^\rho-a^\rho}{\rho}\right) \left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2} \Bigg) B_2 \Bigg] \\ &+ \ \frac{\lambda_1}{1-\lambda_1\mu_1} (\mu_1 B_1 + A_1) + \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)} \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1} x(s) ds. \end{aligned}$$

$$
\begin{aligned}
v(t) &= \frac{2\mu_1}{(1-\lambda_1\mu_1)}\Bigg[\Bigg(\frac{\lambda_2\mu_2}{\Gamma(2-\delta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}+\frac{\lambda_1\mu_2}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\\
&+ [\frac{\mu_2}{\Gamma(2-\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\Bigg)B_3\\
&- \Bigg(\frac{\lambda_2}{\Gamma(2-\delta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}+\frac{\lambda_1}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\\
&- [\frac{1}{\Gamma(2-\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\Bigg)A_3\\
&+ \Bigg(\frac{\lambda_2}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\frac{\mu_2\lambda_1\lambda_2}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\\
&+ \frac{\mu_2\lambda_2}{\Gamma(2-\delta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\Bigg)A_2\\
&- \Bigg(\frac{1}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\frac{\mu_2\lambda_1}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\\
&- \frac{\mu_2}{\Gamma(2-\delta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\Bigg)B_2\Bigg]\\
&+ \frac{\mu_1}{1-\mu_1\lambda_1}(\lambda_1A_1+B_1)+\frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha_2-1}y(s)ds.
\end{aligned}
$$

where

$B_1 = \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^b s^{\rho-1}(b^\rho-s^\rho)^{\alpha_1-1}x(s)ds$, $A_1 = \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^b s^{\rho-1}(b^\rho-s^\rho)^{\alpha_2-1}y(s)ds$,
$B_2 = \frac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)}\int_a^b s^{\rho-1}(b^\rho-s^\rho)^{\alpha_1-\gamma_1-1}x(s)ds$, $A_2 = \frac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\sum_{i=1}^{N}\int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho - s^\rho)^{\alpha_2-\delta_1-1}y(s)ds$,
$B_3 = \frac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)}\sum_{i=1}^{M}\int_a^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\alpha_2-\delta_2-1}x(s)ds$, $A_3 = \frac{\rho^{1-\alpha_1+\gamma_2}}{\Gamma(\alpha_1-\gamma_2)}\int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha_1-\gamma_2-1}y(s)ds$.

Proof. Solving the linear equation

$$({}^C D_a^{\alpha,\rho}u)(t) = x(t).$$

From Lemma (1.3.4), we find easily :

$$u(t) = C_0 + C_1\left(\frac{t^\rho - a^\rho}{\rho}\right) + \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1}x(s)ds. \tag{4.13}$$

$$v(t) = d_0 + d_1\left(\frac{t^\rho - a^\rho}{\rho}\right) + \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_2-1}y(s)ds. \tag{4.14}$$

where $c_i, d_i, i = 0, 1$, are arbitrary real constants. From (4.13) and (4.14) we have

$$ {}^cD_{a^+}^{\gamma_1,\rho}u(t) = \frac{2c_1}{\Gamma(2-\gamma_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} + \frac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-\gamma_1-1}x(s)ds, \tag{4.15} $$

$$ {}^cD_{a^+}^{\gamma_2,\rho}v(t) = \frac{2d_1}{\Gamma(2-\gamma_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} + \frac{\rho^{1-\alpha_2+\gamma_2}}{\Gamma(\alpha_2-\gamma_2)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_2-\gamma_2-1}y(s)ds, \tag{4.16} $$

$$ {}^cD_{a^+}^{\delta_1,\rho}v(t) = \frac{2d_1}{\Gamma(2-\delta_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\delta_1} + \frac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_2-\delta_1-1}y(s)ds, \tag{4.17} $$

$$ {}^cD_{a^+}^{\delta_2,\rho}u(t) = \frac{2c_1}{\Gamma(2-\delta_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\delta_2} + \frac{\rho^{1-\alpha_1+\delta_2}}{\Gamma(\alpha_1-\delta_2)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-\delta_2-1}x(s)ds. \tag{4.18} $$

From the boundary conditions $u(a) = \lambda_1 v(b)$ and $v(a) = \mu_1 u(b)$ and from (4.3)and(4.14), we have

$$ \Rightarrow c_0 = \lambda_1\left[d_0 + d_1\left(\frac{b^\rho - a^\rho}{\rho}\right) + A_1\right] \tag{4.19} $$

$$ \Rightarrow d_0 = \mu_1\left[c_0 + c_1\left(\frac{b^\rho - a^\rho}{\rho}\right) + B_1\right] \tag{4.20} $$

$$ \begin{aligned} c_0 &= \lambda_1[\mu_1\left[c_0 + c_1\left(\tfrac{b^\rho-a^\rho}{\rho}\right) + B_1\right] + d_1\left(\tfrac{b^\rho-a^\rho}{\rho}\right) + A_1\Big] \\ &= \lambda_1\left[\mu_1 c_0 + c_1\mu_1\left(\tfrac{b^\rho-a^\rho}{\rho}\right) + \mu_1 B_1 + d_1\left(\tfrac{b^\rho-a^\rho}{\rho}\right) + A_1\right] \\ c_0 &= \tfrac{\lambda_1}{1-\lambda_1\mu_1}\left[c_1\mu_1\left(\tfrac{b^\rho-a^\rho}{\rho}\right) + \mu_1 B_1 + d_1\left(\tfrac{b^\rho-a^\rho}{\rho}\right) + A_1\right], \end{aligned} $$

$$ d_0 = \frac{\mu_1}{1-\lambda_1\mu_1}\left[d_1\lambda_1\left(\frac{b^\rho - a^\rho}{\rho}\right) + c_1\left(\frac{b^\rho - a^\rho}{\rho}\right) + \lambda_1 A_1 + B_1\right]. $$

Using the boundary conditions ${}^cD_{a^+}^{\gamma_1,\rho}u(b) = \lambda_2\sum_{i=1}^N({}^cD_{a^+}^{\delta_1,\rho}v)(\eta_i)$ and ${}^cD_{a^+}^{\gamma_2,\rho}v(b) = \mu_2\sum_{i=1}^M({}^cD_{a^+}^{\delta_2,\rho}u)(\xi_i)$ from (4.15)to(4.18), we have

$$ \begin{cases} \Rightarrow c_1\frac{2}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1} - d_1\frac{2\lambda_2}{\Gamma(2-\delta_1)}\sum_{i=1}^N\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1} = A_2\lambda_2 - B_2 \\ \Rightarrow c_1\frac{-2\mu_2}{\Gamma(2-\delta_2)}\sum_{i=1}^M\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2} + d_1\frac{2}{\Gamma(2-\gamma_2)}\left(\frac{(b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2} = B_3\mu_2 - A_3 \end{cases} \tag{4.21} $$

Solving the resulting equations for c_1 and d_1, we find that

$$\begin{cases} c_1 = \frac{2}{\Delta}\left[\frac{(A_2\lambda_2-B_2)}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2}+\frac{\lambda_2(B_3\mu_2-A_3)}{\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\right] \\ d_1 = \frac{2}{\Delta}\left[\frac{(B_3\mu_2-A_3}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}+\frac{\mu_2(A_2\lambda_2-B_2)}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\right] \end{cases} \tag{4.22}$$

substituting c1 and d1 in (4.19)and(4.20), we have

$$\begin{aligned} c_0 &= \frac{2\lambda_1}{\Delta(1-\lambda_1\mu_1)}\left[\left(\frac{\mu_1\lambda_2\mu_2}{\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\left(\frac{b^\rho-a^\rho}{\rho}\right)+\frac{\mu_2}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_1}\right)B_3\right. \\ &- \left(\frac{\mu_1\lambda_2}{\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\left(\frac{b^\rho-a^\rho}{\rho}\right)]+\frac{1}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_1}\right)A_3 \\ &+ \left(\frac{\mu_1\lambda_2}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\frac{\mu_2\lambda_2}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\right)A_2 \\ &- \left.\left(\frac{\mu_1}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\frac{\mu_2}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\right)B_2\right] \\ &+ \frac{\lambda_1}{1-\lambda_1\mu_1}\left(\mu_1 B_1 + A_1\right), \end{aligned}$$

and

$$\begin{aligned} d_0 &= \frac{2\mu_1}{\Delta(1-\lambda_1\mu_1)}\left[\left(\frac{\lambda_2\mu_2}{\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\left(\frac{b^\rho-a^\rho}{\rho}\right)+\frac{\lambda_1\mu_2}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_1}\right)B_3\right. \\ &- \left(\frac{\lambda_2}{\Gamma(2-\delta_1)}\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\left(\frac{b^\rho-a^\rho}{\rho}\right)+\frac{\lambda_1}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_1}\right)A_3 \\ &+ \left(\frac{\lambda_2}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\frac{\mu_2\lambda_1\lambda_2}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\right)A_2 \\ &- \left.\left(\frac{1}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\frac{\mu_2\lambda_1}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\right)B_2\right] \\ &+ \frac{\mu_1}{1-\lambda_1\mu_1}\left(\lambda_1 A_1 + B_1\right). \end{aligned}$$

Substituting the values of c_1 and d_2 in (4.13), we get

$$
\begin{aligned}
u(t) &= \frac{2\lambda_1}{\Delta(1-\lambda_1\mu_1)}\Bigg[\Bigg(\frac{\mu_1\lambda_2\mu_2}{\Gamma(2-\delta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{N}(\frac{\eta_i^\rho-a^\rho}{\rho})^{1-\delta_1}+\frac{\mu_2}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\\
&+ \frac{\lambda_2\mu_2}{\Gamma(2-\delta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\Bigg)B_3\\
&- \Bigg(\frac{\mu_1\lambda_2}{\Gamma(2-\delta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}+\frac{1}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\\
&- \frac{\lambda_2}{\Gamma(2-\delta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{N}\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\Bigg)A_3\\
&+ \Bigg(\frac{\mu_1\lambda_2}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\frac{\mu_2\lambda_2}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\\
&+ \frac{\lambda_2}{\Gamma(2-\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2}\Bigg)A_2\\
&- \Bigg(\frac{\mu_1}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\frac{\mu_2}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\\
&- \frac{1}{\Gamma(2-\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2}\Bigg)B_2\Bigg]\\
&+ \frac{\lambda_1}{1-\lambda_1\mu_1}(\mu_1B_1+A_1)+\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha_1-1}x(s)ds.
\end{aligned}
$$

We concluded the following lemma.

Lemma 4.3.2. *Let $f_i : I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R};\ i=1,2$ such that $f_i(\cdot,u,v)\in C(I)$ for each $u,v\in C(I)$. Then the coupled system (4.10)-(4.11) is equivalent to the problem of* *obtaining the solution of the coupled system*

$$\begin{cases} u(t) = \frac{2\lambda_1}{\Delta(1-\lambda_1\mu_1)} \Bigg[\Bigg(\frac{\mu_1\lambda_2\mu_2}{\Gamma(2-\delta_1)} \left(\frac{b^\rho - a^\rho}{\rho}\right) \sum_{i=1}^{N} (\frac{\eta_i^\rho - a^\rho}{\rho})^{1-\delta_1} + \frac{\mu_2}{\Gamma(2-\gamma_1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \\ + \frac{\lambda_2\mu_2}{\Gamma(2-\delta_1)} \left(\frac{t^\rho - a^\rho}{\rho}\right) \sum_{i=1}^{N} \left(\frac{\eta_i^\rho - a^\rho}{\rho}\right)^{1-\delta_1} \Bigg) B_3 \\ - \Bigg(\frac{\mu_1\lambda_2}{\Gamma(2-\delta_1)} \left(\frac{b^\rho - a^\rho}{\rho}\right) \sum_{i=1}^{N} \left(\frac{\eta_i^\rho - a^\rho}{\rho}\right)^{1-\delta_1} + \frac{1}{\Gamma(2-\gamma_1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \\ - \frac{\lambda_2}{\Gamma(2-\delta_1)} \left(\frac{t^\rho - a^\rho}{\rho}\right) \sum_{i=1}^{N} \left(\frac{\eta_i^\rho - a^\rho}{\rho}\right)^{1-\delta_1} \Bigg) A_3 \\ + \Bigg(\frac{\mu_1\lambda_2}{\Gamma(2-\gamma_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{\mu_2\lambda_2}{\Gamma(2-\delta_2)} \sum_{i=1}^{M} \left(\frac{\xi_i^\rho - a^\rho}{\rho}\right)^{1-\delta_2} \left(\frac{b^\rho - a^\rho}{\rho}\right) \\ + \frac{\lambda_2}{\Gamma(2-\gamma_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right) \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} \Bigg) A_2 \\ - \Bigg(\frac{\mu_1}{\Gamma(2-\gamma_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{\mu_2}{\Gamma(2-\delta_2)} \sum_{i=1}^{M} \left(\frac{\xi_i^\rho - a^\rho}{\rho}\right)^{1-\delta_2} \left(\frac{b^\rho - a^\rho}{\rho}\right) \\ - \frac{1}{\Gamma(2-\gamma_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right) \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} \Bigg) B_2 \Bigg] \\ + \frac{\lambda_1}{1-\lambda_1\mu_1} (\mu_1 B_1 + A_1) + \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha_1 - 1} f_1(s, u(s), v(s)) ds \\ \\ v(t) = \frac{2\mu_1}{(1-\lambda_1\mu_1)} \Bigg[\Bigg(\frac{\lambda_2\mu_2}{\Gamma(2-\delta_1)} \left(\frac{b^\rho - a^\rho}{\rho}\right) \sum_{i=1}^{N} \left(\frac{\eta_i^\rho - a^\rho}{\rho}\right)^{1-\delta_1} + \frac{\lambda_1\mu_2}{\Gamma(2-\gamma_1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \\ + [\frac{\mu_2}{\Gamma(2-\gamma_1)} \left(\frac{t^\rho - a^\rho}{\rho}\right) \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \Bigg) B_3 \\ - \Bigg(\frac{\lambda_2}{\Gamma(2-\delta_1)} \left(\frac{b^\rho - a^\rho}{\rho}\right) \sum_{i=1}^{N} \left(\frac{\eta_i^\rho - a^\rho}{\rho}\right)^{1-\delta_1} + \frac{\lambda_1}{\Gamma(2-\gamma_1)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \\ - [\frac{1}{\Gamma(2-\gamma_1)} \left(\frac{t^\rho - a^\rho}{\rho}\right) \left(\frac{b^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \Bigg) A_3 \\ + \Bigg(\frac{\lambda_2}{\Gamma(2-\gamma_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{\mu_2\lambda_1\lambda_2}{\Gamma(2-\delta_2)} \sum_{i=1}^{M} \left(\frac{\xi_i^\rho - a^\rho}{\rho}\right)^{1-\delta_2} \left(\frac{b^\rho - a^\rho}{\rho}\right) \\ + \frac{\mu_2\lambda_2}{\Gamma(2-\delta_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right) \sum_{i=1}^{M} \left(\frac{\xi_i^\rho - a^\rho}{\rho}\right)^{1-\delta_2} \Bigg) A_2 \\ - \Bigg(\frac{1}{\Gamma(2-\gamma_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{\mu_2\lambda_1}{\Gamma(2-\delta_2)} \sum_{i=1}^{M} \left(\frac{\xi_i^\rho - a^\rho}{\rho}\right)^{1-\delta_2} \left(\frac{b^\rho - a^\rho}{\rho}\right) \\ - \frac{\mu_2}{\Gamma(2-\delta_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right) \sum_{i=1}^{M} \left(\frac{\xi_i^\rho - a^\rho}{\rho}\right)^{1-\delta_2} \Bigg) B_2 \Bigg] \\ + \frac{\mu_1}{1-\mu_1\lambda_1} (\lambda_1 A_1 + B_1) + \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha_2 - 1} f_2(s, u(s), v(s)) ds. \end{cases}$$

Definition 4.3.1. *By a solution of the problem (4.10)-(4.11) we mean a coupled continuous functions* $(u, v) \in C(I) \times C(I)$ *satisfying the boundary conditions (4.11),* *and the equations (4.10) on I.*

The following hypotheses will be used in the sequel.

(H_3') The function $f_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

(H_4') There exist constants m_i, n_i such that The functions $f_i;\ i = 1, 2$ satisfy the generalized Lipschitz condition :

$$|f_i(t, u_1, v_1) - f_i(t, u_2, v_2)| \leq m_i|u_1 - u_2| + n_i|v_1 - v_2|,$$

for $t \in I$ and $u_i, v_i \in \mathbb{R}$.

We are now in a position to state and prove our existence result for the problem(4.10)-(4.11) based on concept of measures of noncompactness and Darbo's fixed point theorem.

Remark 4.3.1. *Condition (H_4') is equivalent to the inequality*

$$\alpha(f_i(t, B_1, B_2)) \leq m_i\alpha(B_1) + n_i\alpha(B_2),$$

for any bounded sets $B_1, B_2 \subseteq \mathcal{C}$ and for each $t \in I$.

Theorem 4.3.1. *Assume (H_3'),(H_4') . If*

$$(K_1 + K_2)(m_1 + n_1) + (K_3 + K_4)(m_2 + n_2) < 1, \tag{4.23}$$

then the coupled system (4.10)- (4.11) has a last one solution defined on I.

Proof. Define the operator $N : \mathcal{C} \to \mathcal{C}$ by

$$(N(u, v))(t) = ((N_1u)(t), (N_2v)(t)), \tag{4.24}$$

where $N_1 : C \to C$ and$N_2 : C \to C$ with

$$\begin{aligned}
(N_1u)(t) &= \tfrac{2\lambda_1}{\Delta(1-\lambda_1\mu_1)}\Bigg[\Bigg(\tfrac{\mu_1\lambda_2\mu_2}{\Gamma(2-\delta_1)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)\textstyle\sum_{i=1}^{N}(\tfrac{\eta_i^\rho-a^\rho}{\rho})^{1-\delta_1}+\tfrac{\mu_2}{\Gamma(2-\gamma_1)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\\
&+ \tfrac{\lambda_2\mu_2}{\Gamma(2-\delta_1)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)\textstyle\sum_{i=1}^{N}\left(\tfrac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\Bigg)B_{3f_1}\\
&- \Bigg(\tfrac{\mu_1\lambda_2}{\Gamma(2-\delta_1)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)\textstyle\sum_{i=1}^{N}\left(\tfrac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}+\tfrac{1}{\Gamma(2-\gamma_1)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\\
&- \tfrac{\lambda_2}{\Gamma(2-\delta_1)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)\textstyle\sum_{i=1}^{N}\left(\tfrac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\Bigg)A_{3f_2}\\
&+ \Bigg(\tfrac{\mu_1\lambda_2}{\Gamma(2-\gamma_2)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\tfrac{\mu_2\lambda_2}{\Gamma(2-\delta_2)}\textstyle\sum_{i=1}^{M}\left(\tfrac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)\\
&+ \tfrac{\lambda_2}{\Gamma(2-\gamma_2)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2}\Bigg)A_{2f_2}\\
&- \Bigg(\tfrac{\mu_1}{\Gamma(2-\gamma_2)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\tfrac{\mu_2}{\Gamma(2-\delta_2)}\textstyle\sum_{i=1}^{M}\left(\tfrac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)\\
&- \tfrac{1}{\Gamma(2-\gamma_2)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2}\Bigg)B_{2f_1}\Bigg]\\
&+ \tfrac{\lambda_1}{1-\lambda_1\mu_1}\left(\mu_1B_{1f_1}+A_{1f_2}\right)+\tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha_1-1}f_1(s,u(s),v(s))ds,
\end{aligned}$$

and

$$\begin{aligned}
(N_1v)(t) &= \tfrac{2\mu_1}{(1-\lambda_1\mu_1)}\Bigg[\Bigg(\tfrac{\lambda_2\mu_2}{\Gamma(2-\delta_1)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)\textstyle\sum_{i=1}^{N}\left(\tfrac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}+\tfrac{\lambda_1\mu_2}{\Gamma(2-\gamma_1)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\\
&+ [\tfrac{\mu_2}{\Gamma(2-\gamma_1)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\Bigg)B_{3f_1}\\
&- \Bigg(\tfrac{\lambda_2}{\Gamma(2-\delta_1)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)\textstyle\sum_{i=1}^{N}\left(\tfrac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}+\tfrac{\lambda_1}{\Gamma(2-\gamma_1)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\\
&- [\tfrac{1}{\Gamma(2-\gamma_1)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\Bigg)A_{3f_2}\\
&+ \Bigg(\tfrac{\lambda_2}{\Gamma(2-\gamma_2)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\tfrac{\mu_2\lambda_1\lambda_2}{\Gamma(2-\delta_2)}\textstyle\sum_{i=1}^{M}\left(\tfrac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)\\
&+ \tfrac{\mu_2\lambda_2}{\Gamma(2-\delta_2)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)\textstyle\sum_{i=1}^{M}\left(\tfrac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\Bigg)A_{2f_2}\\
&- \Bigg(\tfrac{1}{\Gamma(2-\gamma_2)}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2}+\tfrac{\mu_2\lambda_1}{\Gamma(2-\delta_2)}\textstyle\sum_{i=1}^{M}\left(\tfrac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\tfrac{b^\rho-a^\rho}{\rho}\right)\\
&- \tfrac{\mu_2}{\Gamma(2-\delta_2)}\left(\tfrac{t^\rho-a^\rho}{\rho}\right)\textstyle\sum_{i=1}^{M}\left(\tfrac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\Bigg)B_{2f_1}\Bigg]\\
&+ \tfrac{\mu_1}{1-\mu_1\lambda_1}(\lambda_1A_{1f_2}+B_{1f_1})+\tfrac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha_2-1}f_2(s,u(s),v(s))ds.
\end{aligned}$$

Here,

$$B_{1f_1} = \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)} \int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha_1-1} f_1(s,u(s),v(s))ds,$$
$$A_{1f_2} = \frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)} \int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha_2-1} f_2(s,u(s),v(s))ds$$
$$B_{2f_1} = \frac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)} \int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha_1-\gamma_1-1} f_1(s,u(s),v(s))ds,$$
$$A_{2f_2} = \frac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)} \sum_{i=1}^N \int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho - s^\rho)^{\alpha_2-\delta_1-1} f_2(s,u(s),v(s))ds,$$
$$B_{3f_1} = \frac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)} \sum_{i=1}^M \int_a^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\alpha_2-\delta_2-1} f_1(s,u(s),v(s))ds,$$
$$A_{3f_2} = \frac{\rho^{1-\alpha_1+\gamma_2}}{\Gamma(\alpha_1-\gamma_2)} \int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha_1-\gamma_2-1} f_2(s,u(s),v(s))ds.$$

For computational convenience, we set

$$\begin{aligned}
\overline{A}_1 &= \frac{2|\lambda_1|}{\Delta|1-\lambda_1\mu_1|}\left(\frac{|\mu_1||\lambda_2|\mu_2|}{\Gamma(2-\delta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)\sum_{i=1}^N(\frac{\eta_i^\rho-a^\rho}{\rho})^{1-\delta_1} + \frac{|\mu_2|}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\right.\\
&+ \left.\frac{|\lambda_2||\mu_2|}{\Gamma(2-\delta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\sum_{i=1}^N\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\right).
\end{aligned}$$

$$\begin{aligned}
\overline{A}_2 &= \frac{2|\lambda_1|}{\Delta|1-\lambda_1\mu_1|}\left(\frac{|\mu_1||\lambda_2|}{\Gamma(2-\delta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)\sum_{i=1}^N\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1} + \frac{1}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\right.\\
&- \left.\frac{|\lambda_2|}{\Gamma(2-\delta_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\sum_{i=1}^N\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1}\right).
\end{aligned}$$

$$\begin{aligned}
\overline{A}_3 &= \frac{2|\lambda_1|}{\Delta|1-\lambda_1\mu_1|}\left(\frac{|\mu_1|}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{|\mu_2|}{\Gamma(2-\delta_2)}\sum_{i=1}^M\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\right.\\
&- \left.\frac{1}{\Gamma(2-\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2}\right)
\end{aligned}$$

$$\begin{aligned}
\overline{A}_4 &= \frac{2|\lambda_1|}{\Delta|1-\lambda_1\mu_1|}\left(\frac{|\mu_1||\lambda_2|}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{|\mu_2||\lambda_2|}{\Gamma(2-\delta_2)}\sum_{i=1}^M\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\right.\\
&+ \left.\frac{|\lambda_2|}{\Gamma(2-\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_2}\right).
\end{aligned}$$

$$\begin{aligned}
\overline{A}_5 &= \frac{2|\mu_1|}{|1-\lambda_1\mu_1|}\left(\frac{|\lambda_2||\mu_2|}{\Gamma(2-\delta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)\sum_{i=1}^N\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1} + \frac{|\lambda_1||\mu_2|}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\right.\\
&+ \left.[\frac{|\mu_2|}{\Gamma(2-\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\right).
\end{aligned}$$

$$\begin{aligned}
\overline{A}_6 &= \frac{2|\mu_1|}{|1-\lambda_1\mu_1|}\left(\frac{|\lambda_2|}{\Gamma(2-\delta_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)\sum_{i=1}^N\left(\frac{\eta_i^\rho-a^\rho}{\rho}\right)^{1-\delta_1} + \frac{|\lambda_1|}{\Gamma(2-\gamma_1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\right.\\
&- \left.[\frac{1}{\Gamma(2-\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\right).
\end{aligned}$$

$$\begin{aligned}\overline{A}_7 &= \frac{2|\mu_1|}{|1-\lambda_1\mu_1|}\left(\frac{|\lambda_2|}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{|\mu_2||\lambda_1||\lambda_2|}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\right.\\ &+ \left.\frac{|\mu_2||\lambda_2|}{\Gamma(2-\delta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\right).\end{aligned}$$

$$\begin{aligned}\overline{A}_8 &= \frac{2|\mu_1|}{|1-\lambda_1\mu_1|}\left(\frac{1}{\Gamma(2-\gamma_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2-\gamma_2} + \frac{|\mu_2||\lambda_1|}{\Gamma(2-\delta_2)}\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\left(\frac{b^\rho-a^\rho}{\rho}\right)\right.\\ &- \left.\frac{|\mu_2|}{\Gamma(2-\delta_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)\sum_{i=1}^{M}\left(\frac{\xi_i^\rho-a^\rho}{\rho}\right)^{1-\delta_2}\right).\end{aligned}$$

$$\begin{aligned}K_1 &= \overline{A}_1\frac{\rho^{-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2+1)}\sum_{i=1}^{M}(\xi_i^\rho - a^\rho)^{\alpha_2-\delta_2} + \overline{A}_4\frac{\rho^{-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1+1)}(b^\rho - a^\rho)^{\alpha_1-\gamma_1}\\ &+ \left(\frac{|\mu_1||\lambda_1|}{|1-\lambda_1\mu_1|}+1\right)\left(\frac{\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}(b^\rho - a^\rho)^{\alpha_1}\right).\end{aligned}$$

$$\begin{aligned}K_2 &= \overline{A}_5\frac{\rho^{-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2+1)}\sum_{i=1}^{M}(\xi_i^\rho - a^\rho)^{\alpha_2-\delta_2} + \overline{A}_8\frac{\rho^{-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1+1)}(b^\rho - a^\rho)^{\alpha_1-\gamma_1}\\ &+ \frac{|\mu_1|}{|1-\mu_1\lambda_1|}\left(\frac{\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}(b^\rho - a^\rho)^{\alpha_1}\right).\end{aligned}$$

$$\begin{aligned}K_3 &= \overline{A}_2\frac{\rho^{-\alpha_1+\gamma_2}}{\Gamma(\alpha_1-\gamma_2+1)}(b^\rho - a^\rho)^{\alpha_1-\gamma_2} + \overline{A}_3\frac{\rho^{-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1+1)}\sum_{i=1}^{N}(\eta_i^\rho - a^\rho)^{\alpha_2-\delta_1}\\ &+ \frac{|\lambda_1|}{|1-\lambda_1\mu_1|}\left(\frac{\rho^{-\alpha_2}}{\Gamma(\alpha_2+1)}(b^\rho - a^\rho)^{\alpha_2}\right).\end{aligned}$$

$$\begin{aligned}K_4 &= \overline{A}_6\frac{\rho^{-\alpha_1+\gamma_2}}{\Gamma(\alpha_1-\gamma_2+1)}(b^\rho - a^\rho)^{\alpha_1-\gamma_2} + \overline{A}_7\frac{\rho^{-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1+1)}\sum_{i=1}^{N}(\eta_i^\rho - a^\rho)^{\alpha_2-\delta_1}\\ &+ \left(\frac{|\lambda_1||\mu_1|}{|1-\mu_1\lambda_1|}+1\right)\frac{\rho^{-\alpha_2}}{\Gamma(\alpha_2+1)}(b^\rho - a^\rho)^{\alpha_2}.\end{aligned}$$

Clearly, the fixed points of the operator N are solutions of the coupled system (4.10)-(4.11). For each $u_i, v_i \in C;\ i = 1, 2$ and $t \in I$. Define

$$\sup_{t\in[a,b]} f_i(t, 0, 0) = \sigma_i \leq \infty.$$

such that

$$R \geq \frac{(K_1 + K_2)\sigma_1 + (K_3 + K_4)\sigma_2)}{1 - (K_1 + K_2)(m_1 + n_1) + (K_3 + K_4)(m_2 + n_2)} \tag{4.25}$$

and consider the closed and convex ball

$$B_R = \{(u, v) \in \mathcal{C} : \|(u, v)\| \leq R\}.$$

By assumption (H_4), for $(u,v) \in B_R, t \in [a,b]$, we have that

$$\begin{aligned} |(f_1(t,u(t),v(t))| &\leq |(f_1(t,u(t),v(t)) - (f_1(t,0,0)| + |(f_1(t,0,0)| \\ &\leq [m_1|u(t)| + n_1|v(t)|)] + \sigma_1 \\ &\leq [m_1\|u\| + n_1\|v\|)] + \sigma_1 \\ &\leq [(m_1+n_1)R + \sigma_1], \end{aligned}$$

$$|(f_2(t,u(t),v(t))| \leq [(m_2+n_2)R + \sigma_2].$$

Let $(u,v) \in B_R$. Then, for each $t \in I$, and any $i = 1,2$, we have

$$\begin{aligned} |(N_1u)(t)| &\leq K_1(m_1\|u\| + n_1\|v\| + \sigma_1) + K_3(m_2\|u\| + n_2\|v\| + \sigma_2) \\ &\leq [K_1(m_1+n_1) + K_3(m_2+n_2)]R + K_1\sigma_1 + K_3\sigma_2. \end{aligned}$$

Hence,

$$|(N_1u)(t)| \leq [K_1(m_1+n_1) + K_3(m_2+n_2)]R + K_1\sigma_1 + K_3\sigma_2.$$

In the same way, we can obtain that

$$|(N_2v)(t)| \leq [K_2(m_1+n_1) + K_4(m_2+n_2)]R + K_2\sigma_1 + K_4\sigma_2.$$

Consequently, it follows that

$$\begin{aligned} |N(u,v)(t)| &\leq [K_1(m_1+n_1) + K_3(m_2+n_2)]R + K_1\sigma_1 + K_3\sigma_2 \\ &+ [K_2(m_1+n_1) + K_4(m_2+n_2)]R + K_2\sigma_1 + K_4\sigma_2 \\ &\leq R. \end{aligned}$$

Thus

$$\|N(u,v)\|_{\mathcal{C}} \leq R. \tag{4.26}$$

Hence N maps the ball B_R into it self. We shalls how that N satisfies the assumption of Darbo's fixed point Theorem. The proof will be given in several steps.

Step 1 : We show that N is continuous. Let$\{(u_n,v_n)\}$ be a sequence such that $(u_n,v_n) \to (u,v)$ in B_R. Then, for each $t \in I$, we have

$$|(N_1u_n)(t) - (N_1u)(t)| \leq (K_1m_1 + K_3m_2)\|u_n - u\| + (K_1n_1 + K_3n_2)\|v_n - v\|).$$

Similarly,

$$|(N_2 v_n)(t) - (N_2 v)(t)| \leq (K_2 m_1 + K_4 m_2)\|u_n - u\| + (K_2 n_1 + K_4 n_2)\|v_n - v\|).$$

From inequalities 4.3.2 and 4.3.2, it yields

$$\begin{array}{rcl} |N(u_n,v_n)(t) - N(u,v)(t)| & \leq & [(K_1 m_1 + K_3 m_2) + (K_2 m_1 + K_4 m_2)]\|u_n - u\| \\ & + & [(K_1 n_1 + K_3 n_2) + (K_2 n_1 + K_4 n_2)]\|v_n - v\|). \end{array}$$

Since $u_n \to u$, $v_n \to v$ as $n \to \infty$ et f_1,f_2 are continuous , then by the Lebesgue dominated convergence theorem ;

$$\|N(u_n,v_n) - N(u,v)\|_{\mathcal{C}} \to 0 \;\; as \;\; n \to \infty.$$

Step 2 : We remark that $N(B_R)$ is bounded. This is clear since $N : B_R \to B_R$ and B_R is bounded.
Step 3 : We show that$N$ maps bounded sets into equicontinuous sets in B_R. Let $t_1, t_2 \in I$, such that $t_1 < t_2$ and let $(u,v) \in B_R$. Then, we have

$$\begin{array}{rcl} |(N_1 u)(t_2) - (N_1 u)(t_1)| & \leq & |\frac{2\lambda_1}{\Delta(1-\lambda_1\mu_1)}\left[\left(\frac{\lambda_2\mu_2}{\Gamma(2-\delta_1)}(\frac{t_2^\rho - t_1^\rho}{\rho})\sum_{i=1}^N(\frac{\eta_i^\rho - a^\rho}{\rho})^{1-\delta_1}\right) B_{3f_1}\right. \\ & - & \left(\frac{\lambda_2}{\Gamma(2-\delta_1)}(\frac{t_2^\rho - t_1^\rho}{\rho})\sum_{i=1}^N(\frac{\eta_i^\rho - a^\rho}{\rho})^{1-\delta_1}\right) A_{3f_2} \\ & - & \left(\frac{\lambda_2}{\Gamma(2-\gamma_2)}(\frac{t_2^\rho - t_1^\rho}{\rho})(\frac{b^\rho - a^\rho}{\rho})^{1-\gamma_2}\right) A_{2f_2} \\ & + & \left.\left(\frac{1}{\Gamma(2-\gamma_2)}(\frac{t_2^\rho - t_1^\rho}{\rho})(\frac{b^\rho - a^\rho}{\rho})^{1-\gamma_2}\right) B_{2f_1}\right] \\ & + & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^{t_2} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha_1-1} f_1(s,u(s),v(s))ds \\ & - & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^{t_1} s^{\rho-1}(t_1^\rho - s^\rho)^{\alpha_1-1} f_1(s,u(s),v(s))ds| \\ & \leq & |\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^{t_2} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha_1-1} f_1(s,u(s),v(s))ds \\ & - & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^{t_1} s^{\rho-1}(t_1^\rho - s^\rho)^{\alpha_1-1} f_1(s,u(s),v(s))ds| \\ & \leq & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_{t1}^{t_2} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha_1-1}|f_1(s,u(s),v(s))|ds \\ & + & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^{t_1} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha_1-1} - (t_1^\rho - s^\rho)^{\alpha_1-1}|f_1(s,u(s),v(s))|ds \\ & \leq & \frac{\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}(t_2^\rho - t_1^\rho)^{\alpha_1}[(m_1+n_1)R + \sigma_1] \\ & + & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}[(m_1+n_1)R + \sigma_1]\int_a^{t_1} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha_1-1} - (t_1^\rho - s^\rho)^{\alpha_1-1}ds, \end{array}$$

and

$$
\begin{aligned}
&\|(N_1v)(t_2)-(N_1v)(t_1)\| \\
\leq\ & \Big|\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^{t_2} s^{\rho-1}(t_2^\rho-s^\rho)^{\alpha_1-1}f_2(s,u(s),v(s))ds \\
-\ & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^{t_1} s^{\rho-1}(t_1^\rho-s^\rho)^{\alpha_1-1}f_2(s,u(s),v(s))ds\Big| \\
\leq\ & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_{t1}^{t_2} s^{\rho-1}(t_2^\rho-s^\rho)^{\alpha_1-1}|f_2(s,u(s),v(s))|ds \\
+\ & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^{t_1} s^{\rho-1}(t_2^\rho-s^\rho)^{\alpha_1-1}-(t_1^\rho-s^\rho)^{\alpha_1-1}|f_2(s,u(s),v(s))|ds \\
\leq\ & \frac{\rho^{-\alpha_1}}{\Gamma(\alpha_1+1)}(t_2^\rho-t_1^\rho)^{\alpha_1}[(m_2+n_2)R+\sigma_2] \\
+\ & \frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}[(m_2+n_2)R+\sigma_2]\int_a^{t_1} s^{\rho-1}(t_2^\rho-s^\rho)^{\alpha_1-1}-(t_1^\rho-s^\rho)^{\alpha_1-1}ds.
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Step 4 : The operator $N : B_R \to B_R$ is a strict set contraction.

Let $V \in B_R$ and $t \in I$, then we have

$$
\begin{aligned}
\alpha((NV)(t)) &= \alpha(\{((N_1u))(t),(N_2v))(t)) : (u,v)\in V\}) \\
&\leq \overline{A}_1\frac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)}\sum_{i=1}^M\int_a^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\alpha_2-\delta_2-1}\alpha(f_1(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \overline{A}_2\frac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\sum_{i=1}^N\int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho-s^\rho)^{\alpha_2-\delta_1-1}\alpha(f_2(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \overline{A}_3\frac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\sum_{i=1}^N\int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho-s^\rho)^{\alpha_2-\delta_1-1}\alpha(f_2(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \overline{A}_4\frac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)}\int_a^b s^{\rho-1}(t^\rho-s^\rho)^{\alpha_1-\gamma_1-1}\alpha(f_1(s,u(s),v(s))ds : (u,v)\in V\})ds \\
&+ \left[\frac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|}+1\right]\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^b s^{\rho-1}(t^\rho-s^\rho)^{\alpha_1-1}\alpha(f_1(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \frac{|\lambda_1|}{|1-\lambda_1\mu_1|}\frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^b s^{\rho-1}(t^\rho-s^\rho)^{\alpha_2-1}\alpha(f_2(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \overline{A}_5\frac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)}\sum_{i=1}^M\int_a^{\xi_i} s^{\rho-1}(\xi_i^\rho-s^\rho)^{\alpha_2-\delta_2-1}\alpha(f_1(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \overline{A}_6\frac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\sum_{i=1}^N\int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho-s^\rho)^{\alpha_2-\delta_1-1}\alpha(f_2(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \overline{A}_7\frac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\sum_{i=1}^N\int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho-s^\rho)^{\alpha_2-\delta_1-1}\alpha(f_2(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \overline{A}_8\frac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)}\int_a^b s^{\rho-1}(t^\rho-s^\rho)^{\alpha_1-\gamma_1-1}\alpha(f_1(s,u(s),v(s))ds : (u,v)\in V\})ds \\
&+ \left[\frac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|}+1\right]\frac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\int_a^b s^{\rho-1}(t^\rho-s^\rho)^{\alpha_2-1}\alpha(f_2(s,u(s),v(s)) : (u,v)\in V\})ds \\
&+ \frac{|\mu_1|}{|1-\lambda_1\mu_1|}\frac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\int_a^b s^{\rho-1}(t^\rho-s^\rho)^{\alpha_1-1}\alpha(f_1(s,u(s),v(s)) : (u,v)\in V\})ds.
\end{aligned}
$$

Then Remark 4.3.1 implies that, for each $s \in I$

$$\alpha(\{f_i(s,u(s);v(s)) : (u,v) \in V\}) \leq m_i\alpha(\{u(s); u \in V\}) + n_i\alpha(\{v(s); v \in V\}).$$

Then

$$\begin{aligned}
& \alpha(NV)(t) \\
\leq\ & \overline{A}_1 \tfrac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)} \textstyle\sum_{i=1}^{M} \int_a^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\alpha_2-\delta_2-1} m_1\alpha(\{u(s) : u \in V\}) + n_1\alpha(\{v(s) : v \in V\})ds \\
+\ & \overline{A}_2 \tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)} \textstyle\sum_{i=1}^{N} \int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho - s^\rho)^{\alpha_2-\delta_1-1} m_2\alpha(\{u(s) : u \in V\}) + n_2\alpha(\{v(s) : v \in V\})ds \\
+\ & \overline{A}_3 \tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)} \textstyle\sum_{i=1}^{N} \int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho - s^\rho)^{\alpha_2-\delta_1-1} m_2\alpha(\{u(s) : u \in V\}) + n_2\alpha(\{v(s) : v \in V\})ds \\
+\ & \overline{A}_4 \tfrac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)} \textstyle\int_a^{t} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-\gamma_1-1} m_1\alpha(\{u(s) : u \in V\}) + n_1\alpha(\{v(s) : v \in V\})ds \\
+\ & \left[\tfrac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|} + 1\right] \tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)} \textstyle\int_a^{t} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1} m_1\alpha(\{u(s) : u \in V\}) + n_1\alpha(\{v(s) : v \in V\})ds \\
+\ & \tfrac{|\lambda_1|}{|1-\lambda_1\mu_1|} \tfrac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)} \textstyle\int_a^{t} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_2-1} m_2\alpha(\{u(s) : u \in V\}) + n_2\alpha(\{v(s) : v \in V\})ds \\
+\ & \overline{A}_5 \tfrac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)} \textstyle\sum_{i=1}^{M} \int_a^{\xi_i} s^{\rho-1}(\xi_i^\rho - s^\rho)^{\alpha_2-\delta_2-1} m_1\alpha(\{u(s) : u \in V\}) + n_1\alpha(\{v(s) : v \in V\})ds \\
+\ & \overline{A}_6 \tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)} \textstyle\sum_{i=1}^{N} \int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho - s^\rho)^{\alpha_2-\delta_1-1} m_2\alpha(\{u(s) : u \in V\}) + n_2\alpha(\{v(s) : v \in V\})ds \\
+\ & \overline{A}_7 \tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)} \textstyle\sum_{i=1}^{N} \int_a^{\eta_i} s^{\rho-1}(\eta_i^\rho - s^\rho)^{\alpha_2-\delta_1-1} m_2\alpha(\{u(s) : u \in V\}) + n_2\alpha(\{v(s) : v \in V\})ds \\
+\ & \overline{A}_8 \tfrac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)} \textstyle\int_a^{b} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-\gamma_1-1} m_1\alpha(\{u(s) : u \in V\}) + n_1\alpha(\{v(s) : v \in V\})ds \\
+\ & \left[\tfrac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|} + 1\right] \tfrac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)} \textstyle\int_a^{b} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_2-1} m_2\alpha(\{u(s) : u \in V\}) + n_2\alpha(\{v(s) : v \in V\})ds \\
+\ & \tfrac{|\mu_1|}{|1-\lambda_1\mu_1|} \tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)} \textstyle\int_a^{b} s^{\rho-1}(t^\rho - s^\rho)^{\alpha_1-1} m_1\alpha(\{u(s) : u \in V\}) + n_1\alpha(\{v(s) : v \in V\})ds \\
\leq\ & (K_1 + K_2)(m_1 + n_1)\alpha(V) + (K_3 + K_4)(m_2 + n_2)\alpha(V).
\end{aligned}$$

Thus

$$\alpha_C(NV) \leq [(K_1 + K_2)(m_1 + n_1)\alpha(V) + (K_3 + K_4)(m_2 + n_2)]\,\alpha(V).$$

So, the operator N is a set contraction. As a consequence of Theorem (1.5.5),we deduce that N has a fixed point which is solution to the problem (4.10)-(4.11). This completes the proof.

Now, Our next existence result for the problem (4.10)-(4.11) is based on Mönch's fixed point theorem.

Theorem 4.3.2. *Assume that the hypothesis* (H_3')–(H_4')*, and and the condition (4.23) hold.*
Then the coupled system (4.10)- (4.11) has at least one solution.

Proof.$N : \mathcal{C} \to \mathcal{C}$ be the operator defined in (4.24). We shall show that N satisfies the assumption of Mönch's fixed point theorem. We know that $N : B_R \to B_R$ is bounded and continuous, we need to prove that the implication
$V = \overline{conv}N(V) \quad \text{or} \quad V = N(V) \cup \{(0,0)\} \Rightarrow \alpha(V) = 0$ holds for every subset V of B_R.
Now let V be a subset of B_R such that $V \subset N(V) \cup \{(0,0)\}$. V is bounded and equicontinuous and therefore the function $t \to \alpha(V(t))$ is continuous on I. By Remark (4.3.1.) and the properties of the measure α, we have for each $t \in I$.

$$\begin{aligned}
\alpha(V(t)) &\leq \alpha(NV)(t)\cup\{(0,0)\}) \leq \alpha((NV)(t)) \leq \alpha(\{(N_1u))(t),(N_2u))(t):(u,v)\in V\})\\
&\leq \overline{A}_1\tfrac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)}\textstyle\sum_{i=1}^{M}\int_a^{\xi_i}s^{\rho-1}(\xi_i^{\rho}-s^{\rho})^{\alpha_2-\delta_2-1}m_1\alpha(\{u(s);u\in V\})+n_1\alpha(\{v(s);v\in V\})ds\\
&+ \overline{A}_2\tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\textstyle\sum_{i=1}^{N}\int_a^{\eta_i}s^{\rho-1}(\eta_i^{\rho}-s^{\rho})^{\alpha_2-\delta_1-1}m_2\alpha(\{u(s);u\in V\})+n_2\alpha(\{v(s);v\in V\})ds\\
&+ \overline{A}_3\tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\textstyle\sum_{i=1}^{N}\int_a^{\eta_i}s^{\rho-1}(\eta_i^{\rho}-s^{\rho})^{\alpha_2-\delta_1-1}m_2\alpha(\{u(s);u\in V\})+n_2\alpha(\{v(s);v\in V\})ds\\
&+ \overline{A}_4\tfrac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)}\textstyle\int_a^{t}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-\gamma_1-1}m_1\alpha(\{u(s);u\in V\})+n_1\alpha(\{v(s);v\in V\})ds\\
&+ \left[\tfrac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|}+1\right]\tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\textstyle\int_a^{t}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-1}m_1\alpha(\{u(s);u\in V\})+n_1\alpha(\{v(s);v\in V\})ds\\
&+ \tfrac{|\lambda_1|}{|1-\lambda_1\mu_1|}\tfrac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\textstyle\int_a^{t}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_2-1}m_2\alpha(\{u(s);u\in V\})+n_2\alpha(\{v(s);v\in V\})ds\\
&+ \overline{A}_5\tfrac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)}\textstyle\sum_{i=1}^{M}\int_a^{\xi_i}s^{\rho-1}(\xi_i^{\rho}-s^{\rho})^{\alpha_2-\delta_2-1}m_1\alpha(\{u(s):u\in V\})+n_1\alpha(\{v(s):v\in V\})ds\\
&+ \overline{A}_6\tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\textstyle\sum_{i=1}^{N}\int_a^{\eta_i}s^{\rho-1}(\eta_i^{\rho}-s^{\rho})^{\alpha_2-\delta_1-1}m_2\alpha(\{u(s):u\in V\})+n_2\alpha(\{v(s):v\in V\})ds\\
&+ \overline{A}_7\tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\textstyle\sum_{i=1}^{N}\int_a^{\eta_i}s^{\rho-1}(\eta_i^{\rho}-s^{\rho})^{\alpha_2-\delta_1-1}m_2\alpha(\{u(s):u\in V\})+n_2\alpha(\{v(s):v\in V\})ds\\
&+ \overline{A}_8\tfrac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)}\textstyle\int_a^{b}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-\gamma_1-1}m_1\alpha(\{u(s):u\in V\})+n_1\alpha(\{v(s):v\in V\})ds\\
&+ \left[\tfrac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|}+1\right]\tfrac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\textstyle\int_a^{b}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_2-1}m_2\alpha(\{u(s):u\in V\})+n_2\alpha(\{v(s):v\in V\})ds\\
&+ \tfrac{|\mu_1|}{|1-\lambda_1\mu_1|}\tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\textstyle\int_a^{b}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-1}m_1\alpha(\{u(s):u\in V\})+n_1\alpha(\{v(s):v\in V\})ds\\
&\leq \overline{A}_1\tfrac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)}\textstyle\sum_{i=1}^{M}\int_a^{\xi_i}s^{\rho-1}(\xi_i^{\rho}-s^{\rho})^{\alpha_2-\delta_2-1}(m_1+n_1)\alpha(V(s))ds\\
&+ \overline{A}_2\tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\textstyle\sum_{i=1}^{N}\int_a^{\eta_i}s^{\rho-1}(\eta_i^{\rho}-s^{\rho})^{\alpha_2-\delta_1-1}(m_2+n_2)\alpha(\{V(s))ds\\
&+ \overline{A}_3\tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\textstyle\sum_{i=1}^{N}\int_a^{\eta_i}s^{\rho-1}(\eta_i^{\rho}-s^{\rho})^{\alpha_2-\delta_1-1}(m_2+n_2)\alpha(V(s))ds\\
&+ \overline{A}_4\tfrac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)}\textstyle\int_a^{t}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-\gamma_1-1}(m_1+n_1)\alpha(V(s))ds\\
&+ \left[\tfrac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|}+1\right]\tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\textstyle\int_a^{t}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-1}(m_1+n_1)\alpha(V(s))ds\\
&+ \tfrac{|\lambda_1|}{|1-\lambda_1\mu_1|}\tfrac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\textstyle\int_a^{t}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_2-1}(m_2+n_2)\alpha(V(s))ds\\
&+ \overline{A}_5\tfrac{\rho^{1-\alpha_2+\delta_2}}{\Gamma(\alpha_2-\delta_2)}\textstyle\sum_{i=1}^{M}\int_a^{\xi_i}s^{\rho-1}(\xi_i^{\rho}-s^{\rho})^{\alpha_2-\delta_2-1}(m_1+n_1)\alpha(V(s))ds\\
&+ \overline{A}_6\tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\textstyle\sum_{i=1}^{N}\int_a^{\eta_i}s^{\rho-1}(\eta_i^{\rho}-s^{\rho})^{\alpha_2-\delta_1-1}(m_2+n_2)\alpha(V(s))ds\\
&+ \overline{A}_7\tfrac{\rho^{1-\alpha_2+\delta_1}}{\Gamma(\alpha_2-\delta_1)}\textstyle\sum_{i=1}^{N}\int_a^{\eta_i}s^{\rho-1}(\eta_i^{\rho}-s^{\rho})^{\alpha_2-\delta_1-1}(m_2+n_2)\alpha(V(s))ds\\
&+ \overline{A}_8\tfrac{\rho^{1-\alpha_1+\gamma_1}}{\Gamma(\alpha_1-\gamma_1)}\textstyle\int_a^{b}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-\gamma_1-1}(m_1+n_1)\alpha(V(s))ds\\
&+ \left[\tfrac{|\lambda_1||\mu_1|}{|1-\lambda_1\mu_1|}+1\right]\tfrac{\rho^{1-\alpha_2}}{\Gamma(\alpha_2)}\textstyle\int_a^{b}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_2-1}(m_2+n_2)\alpha(V(s))ds\\
&+ \tfrac{|\mu_1|}{|1-\lambda_1\mu_1|}\tfrac{\rho^{1-\alpha_1}}{\Gamma(\alpha_1)}\textstyle\int_a^{b}s^{\rho-1}(t^{\rho}-s^{\rho})^{\alpha_1-1}(m_1+n_1)\alpha(V(s))ds.
\end{aligned}$$

Thus,

$$\sup_{t\in I}\alpha(V(t)) \leq [(K_1+K_2)(m_1+n_1)+(K_3+K_4)(m_2+n_2)]\sup_{t\in I}\alpha(V(t)).$$

implies that $\sup_{t\in I}\alpha(V(t)) = 0$, that is $\alpha(V(t)) = 0$, for each $t \in I$, and then

$V(t)$ is relatively compact in $\mathcal{C}$. In view of the Ascoli–Arzel'a theorem,Vis relatively compact in B_R. Applying now Theorem (1.5.4) we conclude that N has a fixed point $(u,v) \in B_R$.. Hence N has a fixed point which is solution to the problem (4.10)-(4.11).This completes the proof.

4.4 Examples

Consider the coupled system of Caputo–Katugampola fractional differential equations

$$\begin{cases} ({}^cD_{0^+}^{3/2,\rho}u)(t) = f_1(t,u(t),v(t)) \\ ({}^cD_{0^+}^{3/2,\rho}v)(t) = f_2(t,u(t),v(t)) \end{cases} ;\ t \in I := [0,1], \tag{4.27}$$

with the boundary conditions

$$\begin{cases} u(0) = v(1);{}^c D_{a^+}^{1/2,\rho}u(1) = 1/3({}^cD_{0^+}^{1/3,\rho}v)(3/2) + 1/3({}^cD_{0^+}^{1/3,\rho}v)(4/3) \\ \\ v(0) = 2u(1);{}^c D_{a^+}^{1/4,\rho}v(1) = 1/5({}^cD_{0^+}^{1/5,\rho}u)(3/5) + 1/5({}^cD_{0^+}^{1/5,\rho}u)(4/5) \end{cases} ; \tag{4.28}$$

Here $a = 0, b = 1,\quad , \alpha_1 = \alpha_2 = 3/2, \gamma_1 = 1/2, \gamma_2 = 1/4, \delta_1 = 1/3, \delta_2 = 1/5, N = M = 2, \eta_1 = 3/2, \eta_1 = 4/3, \xi_1 = 3/5, \xi_1 = 4/5, \lambda_1 = 1, \lambda_2 = 1/3, \mu_1 = 2, \mu_2 = 1/5$. By simple calculation, we found that $\Delta = 0.265381$,
where

$$f_1(t,u,v) = \frac{1}{15\sqrt{25+t^2}}\frac{|u(t)|}{1+|u(t)|} + \frac{\sin v(t)}{65+t^2} + \frac{1}{2}, \quad t \in [0,1],$$
$$f_2(t,u,v) = \frac{\sin|u(t)|}{125+t^2} + \frac{\tan^{-1}(v)}{120+2t^2} + \frac{3}{2}, \quad t \in [0,1].$$

Note that

$$\|f(t,u_1,v_1) - f(t,u_2,v_2) \le \frac{1}{75}\|u_1 - u_2\| + \frac{1}{65}\|v_1 - v_2\|, \quad t \in [0,1],$$
$$\|g(t,u_1,v_1) - g(t,u_2,v_2) \le \frac{1}{125}\|u_1 - u_2\| + \frac{1}{120}\|v_1 - v_2\|, \quad t \in [0,1].$$

The hypothesis [(H_2)]is satisfied with

$$m_1^* = \frac{1}{75}, n_1^* = \frac{1}{65}.$$

$$m_2^* = \frac{1}{125}, n_2^* = \frac{1}{120}.$$

In addition, with good choice of the constants $d_i; i = 1, 2$, a simple computation show that all conditions of Theorem 4.3.1 are satisfied. Hence, the problem (4.27)-(4.28) has at least one solution defined on $[0, 1]$.

Conclusion and Perspectives

In this thesis ; we have considered the following of Caputo-Hadamard fractional differential system

$$\begin{cases} ({}^{HC}D^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ ({}^{HC}D^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \; t \in I := [1, T], \tag{4.29}$$

The Implicit Coupled Caputo-Hadamard Fractional Differential Systems

$$\begin{cases} ({}^{Hc}D_1^{\alpha_1}u_1)(t) = f_1(t, u_1(t), u_2(t), ({}^{Hc}D_1^{\alpha_1}u_1)(t)) \\ ({}^{Hc}D_1^{\alpha_2}u_2)(t) = f_2(t, u_1(t), u_2(t), ({}^{Hc}D_1^{\alpha_2}u_2)(t)) \end{cases} ; \; t \in I := [1, T], \tag{4.30}$$

Here ${}^{Hc}D_1^{\alpha}$ is the **Caputo-Hadamard** fractional derivative.

After that, The existence of solutions for the following coupled conformable fractional differential system

$$\begin{cases} (\mathcal{T}_{0^+}^{\alpha_1}u)(t) = f_1(t, u(t), v(t)) \\ (\mathcal{T}_{0^+}^{\alpha_2}v)(t) = f_2(t, u(t), v(t)) \end{cases} ; \; t \in I, \tag{4.31}$$

Here $\mathcal{T}_0^{\alpha_i}$ is the **conformable** fractional derivative.
We discussed and established the existence, the uniqueness, the stability and the attractivity .

We consider the problem of the existence and uniqueness of solutions and ulam-type stability and the attractivity of system differential with fractional derivatives of Caputo, Hadamard and conformable in b-metric space.

Bibliographie

[1] S. Abbas, B. Ahmad, M. Benchohra and A. Salim, Fractional Differential Equations and Inclusions : Analysis and Stability, *Computer Science Elsevier,* Morgan Kaufmann Publishers In (16 janvier 2024)

[2] S. Abbas, N. Al Arifi, M. Benchohra and J. R. Graef, Random coupled systems of implicit Caputo-Hadamard fractional differential equations with multi-point boundary conditions in generalized Banach spaces, *Dynam. Syst. Appl.* **28** (2) (2019), 229-350.

[3] S. Abbas, N. Al Arifi, M. Benchohra and G. M. N'Guérékata, Random coupled Caputo-Hadamard fractional differential systems with four-point boundary conditions in generalized Banach spaces, *Annals Commun. Math.* **2** (1) (2019), 1-15.

[4] S. Abbas and M. Benchohra, On the generalized Ulam-Hyers-Rassias stability for Darboux problem for partial fractional implicit differential equations. *Applied Mathematics E-Notes,* **14** (2014), 20-28.

[5] S. Abbas, M. Benchohra, and T. Diagana, Existence and attractivity results for some fractional order partial integrodifferential equations with delay, *Afr. Diaspora J. Math.* vol. 15, pp 87 100, 2013.

[6] Saïd Abbas, Mouffak Benchohra, Farida Berhoun, and Johnny Henderson. Caputo-Hadamard fractional differential Cauchy problem in Fréchet spaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 113(3) :2335–2344, 2019.

[7] S. Abbas, M. Benchohra, J. R. Graef and J. Henderson, Implicit Fractional Differential and Integral Equations :Existence and Stability, *De Gruyter, Berlin,* 2018.

[8] S.Abbas,M. Benchohra,G.M. N'Guérékata, Topics in Fractional Differential Equations. *Springer, New York* (2012)

[9] S. Abbas, M. Benchohra and G. M. N'Guérékata, Advanced Fractional Differential and Integral Equations, *Nova Science Publishers,* New York, 2015.

[10] Saïd Abbas, Mouffak Benchohra, Naima Hamidi, and Johnny Henderson. Caputo-Hadamard fractional differential equations in Banach spaces. *Fract. Calc. Appl. Anal.*, 21(4) :1027–1045, 2018.

[11] S. Abbas, M. Benchohra, and J. Henderson, Existence and attractivity results for Hilfer fractional differential equations, *J. Math. Sci., vol.* 243, 347 357, 2019.

[12] S. Abbas, M. Benchohra, J. E. Lazreg, J. J. Nieto and Y. Zhou, Fractional Differential Equations and Inclusions : Classical and Advanced Topics, *World Scientific, Singapore,* 2023.

[13] S. Abbas, M. Benchohra, and J. J. Nieto, Global attractivity of solutions for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes partial integral equations, *Electron. J. Qual. Theory Differ. Equat,* vol. 81, pp. 1 15, 2012.

[14] S. Abbas, M. Benchohra and A. N. Vityuk, On fractional order derivatives and Darboux problem for implicit differential equations, *Frac. Calc. Appl. Anal.* **15** (2) (2012), 168-182.

[15] B. Ahmad and J.J. Nieto. Existence results for a coupled system of nonlinear fractional differential equations with three point boundary conditions. *Comput.Math. Appl.*, 58 :1838-1843, 2009.

[16] T. Abdeljawad, On conformable fractional calculus. *J. Comput. Appl. Math.* **279** (2015), 57-66.

[17] T. Abdeljawad, Q. M. Al-Mdallal, F. Jarad, Fractional logistic models in the frame of fractional operators generated by conformable derivatives. *Chaos Solitons Fractals* **119** (2019), 94-101.

[18] P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, Fuzzy fractional integral equations under compactness type condition. *Fract. Calc. Appl. Anal.* **15** (2012), 572-590.

[19] P. Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Adv. Stud. Contemp. Math.* **16** (2) (2008), 181-196.

[20] P. Agarwal, M. Benchohra and B.A. Slimani, Existence results for differential equations with fractional order impulses, *Mem. Differential Equations Math. Phys.* **44** (1) (2008), 1-21.

[21] P. Agarwal, M. Meehan and D. O'Regan, Fixed Point Theory and Applications, *Cambridge University Press, Cambridge,* 2001.

[22] B. Ahmad and J.J. Nieto, Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. *Abstrac. Appl. Anal.* Volume 2009 (2009), Article ID 494720, 9 pages.

[23] B. Ahmad and S.K. Ntouyas. Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Appl. Math. Comput.*, 266 :615-622, 2015.

[24] B. Ahmad, S.K. Ntouyas, and A. Alsaedi. On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. *ChaosSolitons Fractals.* 83 :234-241, 2016.

[25] B. Ahmad and S. Sivasundaram, Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations. *Nonlinear Anal. Hybrid Syst.* **3** (3) (2009), 251–258.

[26] B. Ahmad, and S. Sivasundaram, Theory of fractional differential equations with three-point boundary conditions, *Commun. Appl. Anal.* **12** (2008), 479-484.

[27] S. Aibout, S. Abbas, M. Benchohra and M. Bohner, A coupled Caputo-Hadamard fractional differential system with multipoint boundary conditions, *Dynamics Con. Discrete Impul. Sys. Series A : Math. Anal.* **29** (2022) 191-209.

[28] K.K. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina and B.N. Sadovskii, Measures of noncompactness and condensing operators, *Birkhäuser Verlag, Basel, Bostn, Berlin,* 1992.

[29] S. Alfaqeih, I. Kayijuka, Solving system of conformable fractional differential equations by conformable double Laplace decomposition method. *J. Partial Differ. Equ.* **33** (2020), no. 3, 275-290.

[30] C. Alsina and R. Ger, On some inequalities and stability results related to the exponential function. *J. Inequal. Appl.* **2**, (1998), 373-380.

[31] J.C. Alvàrez, Measure of noncompactness and fixed points of nonexpansive condensing mappings in locally convex spaces, *Rev. Real. Acad. Cienc. Exact. Fis. Natur.* Madrid **79** (1985), 53-66.

[32] R.Ameen,F. Jarad ,T. Abdeljawad, Ulam stability for delay fractional differential equations with a generalized Caputo derivative. *Filomat* **32**(15), 5265-5274 (2018).

[33] J. Andres and L. Górniewicz, Topological Fixed Point Principles for Boundary Value Problems, *Kluwer Academin Publishers,* Dordrecht (2003).

[34] T. Aoki, On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan* **2**, (1950), 64-66.

[35] J. Appell, Implicit Functions, Nonlinear Integral Equations, and the Measure of Noncompactness of the superposition Operator. *J. Math. Anal. Appl.*83, (1981),251-263

[36] A. Babakhani and V. Daftardar-Gejji, Existence of positive solutions for multi-term non-autonomous fractional differential equations with polynomial coefficients. *Electron. J. Differential Equations* **2006**, No. 129, 12 pp.

[37] A. Babakhani and V. Daftardar-Gejji, Existence of positive solutions for N-term non-autonomous fractional differential equations, *Positivity* **9** (2) (2005), 193-206.

[38] Z. Bai, On positive solutions of a nonlocal fractional boundary value problem, *Nonlinear Anal.* **72** (2) (2010), 916-924, 2010.

[39] D. Baleanu, K. Diethelm, E. Scalas, and J.J. Trujillo, Fractional Calculus Models and Numerical Methods, *World Scientific Publishing, New York,* 2012.

[40] D. Baleanu, Z.B. Güvenç and J.A.T. Machado, New Trends in Nanotechnology and Fractional Calculus Applications, *Springer, New York,* 2010.

[41] D. Baleanu, J.A.T. Machado and A.C.-J. Luo Fractional Dynamics and Control, Springer, 2012.

[42] J. Banaŝ and K. Goebel, Measures of Noncompactness in Banach Spaces, *Marcel Dekker, New York,* 1980.

[43] J. Banaȁs and L. Olszowy, Measures of noncompactness related to monotonicity, *Comment. Math.* (Prace Mat.) **41** (2001), 13-23.

[44] Banaś, J. Zając, A new approach to the theory of functional integral equations of fractional order. *J. Math. Anal. Appl.* **375**, 375-387 (2011)

[45] H. Batarfi, J. Losada, J.J. Nieto, W. Shammakh, Three-point boundary value problems for conformable fractional differential equations, *J. Funct. Spaces* **70** (2015), 63-83.

[46] M. Belmekki and M. Benchohra, Existence results for fractional order semilinear functional differential equations, *Proc. A. Razmadze Math. Inst.* **146** (2008), 9-20.

[47] M. Benchohra,S. Bouriah and J. Henderson, Existence and stability results for nonlinear implicit neutral fractional differential equations with finite delay and impulses, *Communications on Applied Nonlinear Analysis,* Vol. 22 No. 1(2015), 46-67.

[48] M. Benchohra,S. Bouriah, J.J.Nieto, Existence of periodic solutions for nonlinear implicit Hadamard's fractional differential equations. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **112**(1), 25-35 (2019).

[49] M. Benchohra, A. Cabada and D. Seba, An existence result for non-linear fractional differential equations on Banach spaces, Boundary Value Problems. *Volume* 2009 (2009), Article ID 628916, 11 pages.

[50] M. Benchohra, J.R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.* **87** (7) (2008), 851-863.

[51] M. Benchohra, and S. Hamani, Boundary Value Problems for Differential Inclusions with Fractional Order, *Discus. Mathem. Diff. Incl., Contr. and Optim.* **28** (2008), 147-164.

[52] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[53] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2) (2008), 1340-1350.

[54] M. Benchohra and J. E. Lazreg, Nonlinear fractional implicit differential equations. *Commun. Appl. Anal.* **17** (2013), 471-482.

[55] M. Benchohra and J. E. Lazreg, Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. *Romanian J. Math. Comput.* Sc.**4** (1) (2014), 60-72.

[56] M. Benchohra and J.E. Lazreg, Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative, *Stud.Univ.Babeş-Bolyai Math.*62(2017),No.1, 27-38.

[57] M. Benchohra and D. Seba, Impulsive fractional differential equations in Banach Spaces, *Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I* (2009), No. 8, pp. 1-14.

[58] Martin Bohner and Veysel Fuat Hatipoğlu. Dynamic cobweb models with conformable fractional derivatives. *Nonlinear Anal. Hybrid Syst.*, 32 :157–167, 2019.

[59] Martin Bohner, Gauhar Rahman, Shahid Mubeen, and Kottakkaran Sooppy Nisar. A further extension of the extended Riemann-Liouville fractional derivative operator. *Turkish J. Math.*, 42(5) :2631–2642, 2018.

[60] Martin Bohner, Sabrina Streipert, and Delfim F. M. Torres. Exact solution to a dynamic SIR model. *Nonlinear Anal. Hybrid Syst.*, 32 :228–238, 2019.

[61] Felix E. Browder. On the convergence of successive approximations for nonlinear functional equations. *Nederl. Akad. Wetensch. Proc. Ser. A 71=Indag. Math.*, 30 :27–35, 1968.

[62] L. Byszewski, Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* **162** (1991), 494-505.

[63] M. Chohri, S. Bouriah, A. Salim and M. Benchohra, On nonlinear periodic problems with Caputo's exponential fractional derivative, *ATNAA.* **7** (2023), 103-120. https ://doi.org/10.31197/atnaa.1130743

[64] Y.J. Cho , T.M. Rassias and R. Saadati, Stability of Functional Equations in Random Normed Spaces, *Science-Business Media,* Springer 52, 2013.

[65] C. Corduneanu, Integral Equations and Stability of Feedback Systems, *Acad. Press, New York* (1973).

[66] Amar Deep, Deepmala and Cemil Tunç. On the existence of solutions of some non-linear functional integral equations in Banach algebra with applications. *Arab Journal of Basic and Applied Sciences* **27** (1) (2020) 279-286.

[67] C. Derbazi, H. Hammouche, A. Salim and M. Benchohra, Weak solutions for fractional Langevin equations involving two fractional orders in Banach spaces. *Afr. Mat.* **34** (2023), 10 pages. https ://doi.org/10.1007/s13370-022-01035-3

[68] K. Diethelm. The Analysis of Fractional Differential Equations. *Springer, Berlin,*2010.

[69] S. Dudek, Fixed point theorems in Fréchet Algebras and Fréchet spaces and applications to nonlinear integral equations, *Appl. Anal. Discrete Math,* **11** (2017), 340-357.

[70] S. Dudek and L. Olszowy, Continuous dependence of the solutions of nonlinear integral quadratic Volterra equation on the parameter, *J. Funct. Spaces,* V. 2015, Article ID 471235, 9 pages.

[71] A. El-Ajou, A modification to the conformable fractional calculus with some applications. *Alexandria Engineering J.* **59** (2020), 2239-2249.

[72] N. Engheta, Fractional Curl Operator in Electromagnetics. *Microwave Opt. Tech. Lett.* **17** (1) (1998), 86-91.

[73] M. Faryad and Q.A. Naqvi, Fractional Rectangular Waveguide. *Progress In Electromagnetics Research PIER* **75** (2007), 383-396.

[74] P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431-436.

[75] L. Górniewicz, Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications, 495, *Kluwer Academic Publishers, Dordrecht,* 1999.

[76] Haide Gou and Baolin Li. Existence of weak solutions for fractional integrodifferential equations with multipoint boundary conditions. *Int. J. Differ. Equ.*, pages Art. ID 1203031, 8, 2018.

[77] J. R.Graef, J. Henderson and A. Ouahab, Impulsive Diferential Inclusions. A Fixed Point Approch, *De Gruyter, Berlin/Boston,* 2013.

[78] A.Granas and J.Dugundji, Fixed Point Theory, *Springer-Verlag, New York,* 2003.

[79] T.H. Gronwall. Note on the derivatives with respect to a parameter of the solutionsof a system of differential equations. *Ann. Math.* 20 :292-296, 1918-1919.

[80] D.J. Guo, V. Lakshmikantham, X. Liu, Nonlinear Integral Equations in Abstract Spaces, *Kluwer Academic Publishers, Dordrecht,* 1996.

[81] J. Hale and S.M. Verduyn Lunel, Introduction to Functional Differential Equations, *Applied Mathematicals Sciences,* 99, Springer-Verlag, New York, 1993.

[82] M.A. Hammad, R. Khalil, Abels formula and Wronskian for conformable fractional differential equations. *Int. J. Differ. Equ. Appl.* **13**(3) (2014), 177-183.

[83] A. Harir, S. Melliani, L.S. Chadli, Fuzzy Conformable Fractional Differential Equations. *Int. J. Differ. Equ.* **2021,** Art. ID 6655450, 6 pp

[84] R. Hermann, Fractional Calculus : An Introduction For Physicists, *World Scientific Publishing Co. Pte. Ltd.* 2011.

[85] N. Heymans and I. Podlubny, Pysical interpretation of initial conditions for fractional differential equations with Riemnn-Liouville fractional derivatives, *Rheol Acta* **45** (2006), 765-771.

[86] R. Hilfer, Applications of Fractional Calculus in Physics, *World Scientific, Singapore,* 2000.

[87] C. Horvath, Measure of Non-compactness and multivalued mappings in complete metric topological spaces, *J. Math. Anal. Appl.* **108** (1985), 403-408.

[88] M. Hu and L. Wang, Existence of solutions for a nonlinear fractional differential equation with integral boundary condition, *Int. J. Math. Comp. Sc.*, **7**(1) (2011).

[89] D. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad.Sci. U.S.A.* 27 (1941), 222-224.

[90] F. Jarad., T. Abdeljawad, and D. Baleanu. Caputo-type modification of the Hadamard fractional derivatives. *Adv. Difference Equ.* **2012** (2012), Art. ID 142.

[91] N. Jin and S. Sun. On a coupled system of fractional compartmental models for abiological system. *Adv. Difference Equ.*, 146, 2017.

[92] S.M. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.* **222** (1998), 126-137

[93] S.M. Jung, Hyers-Ulam stability of linear differential equations of first order, *II Appl. Math. Lett.* **19** (2006), 854-858.

[94] S.M.Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis. *Springer, New York* (2011).

[95] S.M. Jung, K.S. Lee, Hyers-Ulam stability of first order linear partial differential equations with constant coefficients *Math. Inequal. Appl.***10** (2007), 261-266.

[96] K.W. Jun and H.M. Kim, On the stability of an n-dimensional quadratic and additive functional equation, *Math. Inequal. Appl.* **19** (9) (2006), 854-858.

[97] N. Kadkhoda, H. Jafari, An analytical approach to obtain exact solutions of some space-time conformable fractional differential equations. *Adv. Difference Equ.***2019** Paper No. 428, 10 pp.

[98] K. Karthikeyan and J.J. Trujillo, Existence and uniqueness results for fractional integrodifferential equations with boundary value conditions, *Commun. Nonlinear Sci. Numer. Simulat.* **17** (2012) 4037-4043.

[99] U.N. Katugampola. New approach to a generalized fractional integral. *Appl. Math.Comput.* 218(3) :860-865, 2011.

[100] U.N. Katugampola. A new approach to generalized fractional derivativesl integral. *Bull. Math. Anal. Appl.* 6(4) :1-15, 2014.

[101] R. Khalil, M.A. AL Horani, M. Yousef, Sababheh, A new definition of fractional derivative. *J. Comput. Appl. Math.* **264** (2014), 65-70.

[102] H.Khan, T.Abdeljawad, M.Aslam, R.A.Khan, A.Khan, Existence of positive solution and Hyers-Ulam stability for a nonlinear singular-delay-fractional differential equation. *Adv. Differ. Equ.* **2019**, Article ID 104 (2019).

[103] Hasib Khan, Cemil Tunç and Aziz Khan. Stability results and existence theorems for nonlinear delay-fractional differential equations with Φ_p-operator. *J. Appl. Anal. Comput.* **10** (2) (2020), 584-597.

[104] Anatoly A. Kilbas. Hadamard-type fractional calculus. *J. Korean Math. Soc.*, **38** (6) :1191–1204, 2001.

[105] A.A. Kilbas and S.A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Diff. Equat.* **41** (2005), 84-89

[106] A.A. Kilbas, H.H. Srivastava and J.J. Trujillo. Theory and Applications of Fractional Differential Equations. *Elsevier Science B.V, Amsterdam,* 2006.

[107] G.H. Kim, On the stability of functional equations with square-symmetric operation, *Math. Inequal. Appl* **17** (4) (2001), 257-266.

[108] W.A. Kirk and B. Sims, Handbook of Metric Fixed Point Theory, *Springer-Science, Business Media, B.V, Dordrecht,* 2001.

[109] S. Krim, A. Salim, S. Abbas and M. Benchohra, On implicit impulsive conformable fractional differential equations with infinite delay in b-metric spaces. *Rend. Circ. Mat. Palermo (2).* (2022), 1-14. https ://doi.org/10.1007/s12215-022-00818-8

[110] S. Krim, A. Salim, S. Abbas and M. Benchohra, Functional k-generalized ψ-Hilfer fractional differential equations in b-metric spaces. *Pan-Amer. J. Math.* **2** (2023), 10 pages. https ://doi.org/10.28919/cpr-pajm/2-5

[111] V. Lakshmikantham and A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.* **69** (2008), 2677-2682.

[112] V. Lakshmikantham and A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations *Appl. Math. Lett,* **21** (2008) 828-834.

[113] V. Lakshmikantham and A.S. Vatsala, Theory of fractional differential inequalities and applications, *Commun. Appl. Anal,* **11**, No. 3-4 (2007), 395-402.

[114] V. Lakshmikantham, S. Leela and J. Vasundhara, Theory of Fractional Dynamic Systems, *Cambridge Academic Publishers, Cambridge,* 2009.

[115] V. Lakshmikantham, and J. Vasundhara Devi. Theory of fractional differential equations in a Banach space. *Eur. J. Pure Appl. Math.* **1** (2008), 38-45.

[116] Q. Liu, F. Liu, I. Turner and V. Anh, Numerical simulation for the 3D seep age flow with fractional derivatives in porous media, *IMA Journal of Applied Mathematics* 74 (2) (2009), 201-229.

[117] C. Li, F. Zeng and F. Liu, Spectral approximations to the fractional integral and derivative, *Fractional Calculus and Applied Analysis* 15 (3) (2012), 383-406.

[118] A.J. Luo and V. Afraimovich, Long-range Interactions, Stochasticity and Fractional Dynamics, *Springer, New York, Dordrecht, Heidelberg, London,* 2010.

[119] Janusz Matkowski. Integrable solutions of functional equations. *Dissertationes Math. (Rozprawy Mat.)*, 127 :68, 1975.

[120] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, *John Wiley, New York,* 1993.

[121] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[122] Q.A. Naqvi and M. Abbas, Complex and Higher Order Fractional Curl Operator in Electromagnetics, *Optics Communications* **241**, (2004), 349-355.

[123] K.B. Oldham and J. Spanier, The Fractional Calculus : theory and application of differentiation and integration to arbitrary order, *Academic Press, New York, London,* 1974.

[124] L. Olszowy, Existence of mild solutions for the semilinear nonlocal problem in Banach spaces, *Nonlinear Anal.* **81** (2013) 211-223.

[125] M.D. Ortigueira, Fractional Calculus for Scientists and Engineers. *Lecture Notes in Electrical Engineering,* 84. Springer, Dordrecht, 2011.

[126] I. Podlubny. Fractional Differential Equations, *Mathematics in Science and Engineering.Academic Press,* New York, 1999.

[127] Rabha W. Ibrahim, Stability for Univalent Solutions of Complex Fractional Differential Equations, *Proceedings of the Pakistan Academy of Sciences*, Vol. 49 **3**, 2012, 227-232.

[128] W. Rahou, A. Salim, J. E. Lazreg and M. Benchohra, Existence and stability results for impulsive implicit fractional differential equations with delay and Riesz-Caputo derivative. *Mediterr. J. Math.* **20** (2023), 143. https ://doi.org/10.1007/s00009-023-02356-8

[129] T.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc.Amer. Math. Soc.* **72** , (1978), 297-300.

[130] J.M. Rassias, Functional Equations, Difference Inequalities and Ulam Stability Notions (F.U.N.), *Nova Science Publishers, Inc. New York,* 2010.

[131] T.M. Rassias and J. Brzdek, Functional Equations in Mathematical Analysis, Springer 86, *New York Dordrecht Heidelberg London* 2012.

[132] M. Rehman and R.A. Khan. A note on boundary value problems for a coupled system of fractional differential equations. *Compu. Math. Appl.* 61 :2630-2637,2011.

[133] B. Ross, Fractional Calculus and Its Applications, *Proceedings of the International Conference,* New Haven, Springer-Verlag, New York (1974).

[134] I.A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.* **26** (2010), 103-107.

[135] I.A.Rus, Ulam stability of ordinary differential equations. *Stud. Univ. Babe¸s-Bolyai, Math.* LIV(4), 125-133 (2009)

[136] I.A.Rus, Remarks on Ulam stability of the operatorial equations. *Fixed Point Theory* **10**, 305-320 (2009).

[137] Ioan A. Rus, Adrian Petruşel, and Gabriela Petruşel. Fixed point theory. *Cluj University Press, Cluj-Napoca,* 2008.

[138] P. Sahoo, T. Barman and J.P. Davim, Fractal Analysis in Machining, *Springer, New York, Dordrecht, Heidelberg, London,* 2011.

[139] A. Salim, M. Benchohra and J. E. Lazreg, On implicit k-generalized ψ-Hilfer fractional differential coupled systems with periodic conditions, *Qual. Theory Dyn. Syst.* **22** (2023), 46 pages. https ://doi.org/10.1007/s12346-023-00776-1

[140] S.G.Samko, A.A.Kilbas, O.I.Marichev, Fractional Integrals and Derivatives. *Theory and Applications.* Gordon Breach, Amsterdam (1987). English translation from the Russian.

[141] K.S. Shah,R.A. Khan, D. Baleanu, Study of implicit type coupled system of non-integer order differential equations with anti-periodic boundary conditions. *Math. Methods Appl. Sci.* **42**(6), 2033-2042 (2019).

[142] X. Su. Boundary value problem for a coupled system of nonlinear fractional dif-ferential equations. *Compu. Math. Appl.* 22 :64-69, 2009.

[143] X. Su, Q. Li and Y. Li. Existence and uniqueness of solutions for a coupled system of multiterm nonlinear fractional differential equations. *Compu. Math.Appl.*, 64 :3310-3320, 2012.

[144] X. Su and L. Liu, Existence of solution for boundary value problem of nonlinear fractional differential equation, *Appl. Math.* **22** (3) (2007) 291-298.

[145] Vasily E. Tarasov. *Fractional dynamics.* Nonlinear Physical Science. Springer, Heidelberg ; Higher Education Press, Beijing, 2010. Applications of fractional calculus to dynamics of particles, fields and media.

[146] J. M. Ayerbe Toledano, T. Domínguez Benavides, and G. López Acedo. *Measures of noncompactness in metric fixed point theory*, volume 99 of *Operator Theory : Advances and Applications.* Birkhäuser Verlag, Basel, 1997.

[147] Cemil Tunç and Osman Tunç. On the boundedness and integration of non-oscillatory solutions of certain linear differential equations of second order. *J. Advanced Research.* **7** (1)(2016), 165-168.

[148] S.M. Ulam, Problems in Modern Mathematics, *Chapter 6, JohnWiley and Sons, New York, USA,*1940.

[149] S.M. Ulam, A Collection of Mathematical Problems, *Interscience, New York,*1960.

[150] J. Wang, C. Bai, Antiperiodic boundary value problems for impulsive fractional functional differential equations via conformable derivative. *J. Funct. Spaces* **2018,** Art. ID 7643123, 11 pp.

[151] J. Wang, M. Fečkan and Y. Zhou, Ulam type stability of impulsive ordinary differential equations, *J. Math. Anal. Appl.* **395** (2012), 258-264.

[152] J. Wang, X. Li, A uniformed method to Ulam-Hyers stability for some linear fractional equations. *Mediterr. J. Math.* **13**, 625-635 (2016).

[153] J. Wang, L. Lv and Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theory Differ. Equat.* **63** (2011), 1-10.

[154] J. Wang and Y. Zhang, Existence and stability of solutions to nonlinear impulsive differential equations in β-normed spaces, *Electron. J. Differential Equations,* (2014), No. 83, 1-10.

[155] G. Xiao, J. Wang, Representation of solutions of linear conformable delay differential equations. *Appl. Math. Lett.* **117** (2021), 107088.

[156] K. Yosida, Fonctional Analysis, *6th edn. Springer-Verlag, Berlin,* 1980.

[157] S. Zhai, X. Feng and Z. Weng, New high-order compact adi algorithms for 3D nonlinear time-fractional convection-diffusion equation, *Math. Prob. Eng* **2013** (2013), 11 pages.

[158] C. Zhai and R. Jiang. Unique solutions for a new coupled system of fractional differential equations. *Adv. Difference Equ.*, 1, 2018.

[159] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional diffrential equations, *Electron. J. Differential Equations* 2006, No. 36, pp. 1-12.

[160] X. Zhang, J. Liu, L. Wei and C. Ma, Finite element method for Grwnwald-Letnikov time-fractional partial differential equation, *Appl. Anal.* **92** (10) 2013, pp. 1-12.

[161] Yong Zhou. *Basic theory of fractional differential equations.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.

الملخص:
في هذه الرسالة، ندرس وجود، وحدانية الحلول و استقرار من نوع اولام لبعض أصناف الأنظمة التفاضلية ذات الرتب الكسرية مع مشتقات كابيتو،هدامارد و كاتيكمبولة . المشاكل التي تمت دراستها هي مع الشروط الأولية و الحدية. تستند النتائج التي تم الحصول عليها إلى بعض نظريات النقطة الثابتة و قياس عدم التراص في فضاءات باناخ و فريشي.

الكلمات المفتاحية: معادلة تفاضلية، رتبة كسرية، حل، استقرار، ضمني، فضاء فريشي، فضاء باناخ.

Résumé :
Dans cette thèse, nous étudions l'existence et l'unicité de solutions et la stabilité de type Ulam de quelques systèmes différentiels couplés d'ordre fractionnaires avec la dérivée de Caputo, Hadamard, Katugampola et Conformable. Les problèmes étudiés sont à conditions initiales et aux limites. Les résultats obtenus sont basés sur quelques théorèmes de points fixes et la mesure de non-compacité dans les espaces de Banach, Fréchet.

Mots clés : équation différentielle, ordre fractionnaire, solution, stabilité, implicite, fixe, mesure de non-compacité, espace de Fréchet, espace de Banach.

Abstract :
In this thesis, we study the existence and uniqueness of solutions and the Ulam-type stability of some coupled differential systems fractional order derivatives of Caputo, Hadamard, Katugampola and Conformable. The problems studied are with boundary conditions. The results obtained are based on some fixed point theorems and the measure of non-compactness in the space Banach, Fréchet.

Key words: differential equation, fractional order, solution, stability, implicit, fixed measure of non-compactness, Fréchet space, Banach space.